

**A Sixth-Order Plate Theory—Derivation and Error Estimates<sup>3</sup>**

**R. Schmidt.**<sup>5</sup> The theory of plates to which the author refers to as “Levinson’s theory” and cites Levinson (1980) as his reference is nothing else but a linearized version of the nonlinear theory published by Schmidt (1977). A statement to that effect can be found in *Applied Mechanics Reviews*, Vol. 34, No. 7, July 1981, Review 6280, p. 952. Moreover, Dr. Levinson has acknowledged the identicalness of the two theories in a private letter to the discussor.

The theory of Schmidt (1977) has not been completely unknown. It has been referred to in several journals, including this one (e.g., Sathyamoorthy and Chia, 1980).

In the name of fairness, this oversight should be acknowledged and corrected.

**References**

Levinson, M., 1980, “An Accurate, Simple Theory of the Statics and Dynamics of Elastic Plates,” *Mechanics Research Communications*, Vol. 7, No. 6, pp. 343-350.  
 Sathyamoorthy, M., and Chia, C. Y., 1980, “Effect of Transverse Shear and Rotatory Inertia on Large Amplitude Vibration of Anisotropic Skew Plates; Part 1—Theory,” *ASME JOURNAL OF APPLIED MECHANICS*, Vol. 47, No. 1, pp. 128-132.  
 Schmidt, R., 1977, “A Refined Nonlinear Theory of Plates with Transverse Shear Deformation,” *Industrial Mathematics*, Vol. 27, Part 1, pp. 23-38.

**A Sixth-Order Plate Theory—Derivation and Error Estimates<sup>3</sup>**

**E. Reissner.**<sup>6</sup> The unqualified acceptance of results in Speare and Kemp (1977), in conjunction with the author’s derivation of analogous results based on an analysis in Levinson (1980), suggests the following observation.

The essence of the results in Speare and Kemp (1977) is that it should be appropriate to reduce the sixth order theory of shear deformable plates in Reissner (1945) to one equivalent sixth order equation

$$D \left\{ 1 + \frac{2-\nu}{1-\nu} \frac{h^2}{10} \right\} \nabla^4 w = q, \tag{1}$$

for the deflection  $w$  of a homogeneous isotropic plate, with  $h$  designating plate thickness and  $\nabla^2$  for the Laplace operator, in place of the author’s  $2h$  and  $\Delta$ . Unfortunately, the derivation of equation (1) depended on a fundamental oversight which invalidates this equation as well as its consequences.

To describe what is involved in this matter we depart, as in Spear and Kemp (1977), from the following three equations for shear stress resultants  $Q_x$  and  $Q_y$  and a load intensity function  $q$ ,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q, \tag{2}$$

$$\left[ 1 - \frac{h^2}{10} \nabla^2 \right] [Q_x, Q_y] = - \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] \left[ D \nabla^2 w + \frac{h^2 q}{10(1-\nu)} \right]. \tag{3}$$

<sup>3</sup>See footnote 3, p. 249.

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In order to deduce equation (1) from equations (2) and (3), the assumption was made that the terms  $h^2 \nabla^2(Q_x, Q_y)$  and  $h^2 q$  in equation (3) were small compared with the terms  $(Q_x, Q_y)$  and  $D \nabla^2 w$ , respectively, in such a way that it was appropriate to set, in these terms,  $(Q_x, Q_y) = -D(\partial \nabla^2 w / \partial x, \partial \nabla^2 w / \partial y)$ , with  $q$  as in equation (2), so as to have, in place of equation (3)

$$(Q_x, Q_y) = -D \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] \left[ \nabla^2 w + \frac{h^2}{10} \left[ 1 + \frac{1}{1-\nu} \right] \nabla^4 w \right], \tag{4}$$

with the introduction of equation (4) into (2) leading to equation (1).

While it should be evident that the validity of equation (1) should be suspect in view of the facts that (i) the indicated reduction procedure necessarily ceases to be valid for plates acted upon by edge loads only, and (ii) the positiveness of the coefficient of the term  $h^2 \nabla^2$  is associated with physically unreasonable behavior of portions of the solution function  $w$ , one can, going beyond this, state the reason for these unacceptable results as follows. It is the essence of shear-deformable plate theory that portions of the terms  $h^2 \nabla^2(Q_x, Q_y)$  in equation (3) are *not* small compared to the corresponding portions of the terms  $(Q_x, Q_y)$ , consistent with the fact that the solution of the sixth order theory of shear deformable plates involves a boundary layer solution contribution which is absent in the classical fourth order Kirchhoff theory.

**References**

Levinson, M., 1980, “An Accurate Simple Theory of the Statics and Dynamics of Elastic Plates,” *Mechanics Research Communications*, Vol. 7, pp. 343-350.  
 Reissner, E., 1945, “The Effect of Transverse Shear Deformations on the Bending of Elastic Plates,” *ASME JOURNAL OF APPLIED MECHANICS*, Vol. 12, pp. A69-A77.  
 Speare, P. R. S., and Kemp, K. G., 1977, “A Simplified Reissner Theory for Plate Bending,” *International Journal of Solids and Structures*, Vol. 13, pp. 1073-1079.

**A Method of Eliminating Lagrangian Multipliers from the Equation of Motion of Interconnected Mechanical Systems<sup>7</sup>**

**J. G. Papastavridis.**<sup>8</sup> Although its final result, i.e., equations (14) are correct, the Note contains a number of erroneous statements:

(i) The imposition of the  $N$  *catatastic* nonholonomic constraints (1) on the system does *not* affect the number of its (independent or unconstrained or minimal) generalized coordinates; the latter, as we infer from equations (10), are still  $M$  in number and not  $M-N$  as the Note states.

(ii) The constraint reactions, in an ideal system, produce no work for any *virtual* displacement, i.e., displacement compatible with the instantaneous/frozen constraints (in the form (1), but with  $\dot{q}$  replaced with  $\delta q$ ), and not just “. . . any displacement compatible with the constraints,” as that author states (near the top of p. 236). Thus the following “zero reaction power” condition (between his equations (7’) and (8)) holds *only for catatastic* (i.e., homogeneous) nonholonomic constraints *such as* (1), but *not for general acatastic* (i.e., nonhomogeneous) ones; for the relevant definitions see Rosenberg (1977).

<sup>7</sup>By S. Vlase, published in the March, 1987, issue of *ASME JOURNAL OF APPLIED MECHANICS*, Vol. 54, pp. 235-237.

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(iii) The method proposed by the author, in particular his equations (12)–(14), are neither “new” nor “natural”. Instead they constitute a special application of the general reaction-free equations of Maggi (presented by him first in 1896, and then elaborated in 1901—see, e.g., Neimark and Fufaev, 1967), *when one chooses the last  $M-N$  generalized velocities (author's  $\dot{q}_c$ ) as the independent parameters or quasivelocities*. This particular choice of quasivelocities (“elimination” or “embedding” of constraints) dates back from Chaplygin (1895, 1897) and Voronets (1901); these two also formulated reaction-free equations of motion that are special cases of the general nonholonomic Boltzmann (1902)/Hamel (1903, 1094) equations. That special case of Maggi's equations has been derived and discussed by such mechanicians as Hamel (1924), and Lur'e (1961) in his monumental monograph (see also Appell 1953, or earlier editions, and Rosenberg, 1977). The same special case was also independently rediscovered by Passerello and Huston (1973) in a rather ad hoc and unmotivated fashion, as an isolated result; on the contrary, both Hamel and Lur'e discuss the relation of Maggi's equations with other sets of nonholonomic systems equations such as those by Appell, Chaplygin, Voronets, Tzenov. It should be added here that these special Maggi equations hold for general rheonomic systems (i.e., nonstationary in their holonomic and nonholonomic constraints), and not only for scleronomic (i.e., stationary constraint) ones, as the author's equations (9) and (1) imply.

As Prange (1935) puts it, *Maggi's equations are the projection of the general Lagrange/Routh/Voss equations (10) onto the  $M-N$  dimensional “nonholonomic manifold” or “virtual hyperplane” of the system at the “point”  $(q, t)$  in configuration space* (that plane is defined by the Note's constraint equations (1) with  $\dot{q}$  replaced by the virtual  $\delta q$ ). Since the system constraint-reaction vector is perpendicular to that hyperplane, Maggi's equations are reaction-free! Several other authors have presented their nonholonomic system discussions in simple and fruitful geometrical/tensorial language (see, e.g., Dobronravov, 1970).

## References

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- Hamel, G., 1924, “Über Nichtholonome Systeme,” *Mathematische Annalen*, Vol. 92, pp. 33–41.
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- Rosenberg, R. M., 1977, *Analytical Dynamics of Discrete Systems*, Plenum, New York, pp. 43–45, pp. 243–245.

## Author's Closure

The method is the result of the author's concerns toward conceiving a computer program for the elastodynamic analysis of mechanisms. In this respect, the computing algorithm must determine the constraint reactions.

Many difficulties have been encountered in making a program that first builds the reactionless equations of motion, and then computes the reaction forces concerning its efficiency. It is therefore more convenient to generate equations (10) only once and then operate with them.

Although the basic idea of our method has *implicitly* been found in classical analytical mechanics, as documented by Papastravridis, its *explicit* form has not been used yet. The method can therefore be considered new. Dynamic systems analysis programs eliminate multipliers in equations (10) by using a coefficient matrix inversion technique. The method allows the replacement of all the abovementioned procedures by a simple matrix multiplication.

## A Method of Eliminating Lagrangian Multipliers from the Equations of Motion of Interconnected Mechanical Systems<sup>7</sup>

**L. Y. Bahar.**<sup>9</sup> The author should be commended for an interesting approach of first adjoining constraints via Lagrange multipliers, then eliminating them.

A simpler and more natural method consists of not introducing Lagrange multipliers at all, but utilizing the well established approach of “constraint embedding,” most elegantly described in a recent text by Rosenberg [1], and generally attributed to Woronetz. This deviation is elementary and straightforward, as it consists of the variational counterpart of finding constrained extrema in elementary calculus through the process of elimination of dependent differentials. It is recognized that this approach is particularly useful when implicit differentiation is involved.

The basic idea in [1] is to begin with the fundamental equation of dynamics (or the principle of virtual work) stated through Lagrange's form of d'Alembert's principle, followed by the elimination of the excess or dependent virtual displacements by expressing them in terms of the independent (minimal set) of virtual displacements through the use of nonholonomic constraints, or the variation of holonomic constraints. The vanishing of the coefficients then yields the equations governing the motion. For algebraic details, as well as an illustrative example, [1] may be consulted.

It should be pointed out that the above method is in keeping with the spirit of analytical dynamics, where the concept of virtual work is utilized in order to eliminate the total virtual work performed by all the reaction forces undergoing virtual displacements compatible with the constraints. It is thus possible to obtain equations of motion that reflect only the impressed forces. Once these equations of motion have been obtained, it is a simple procedure to determine the constraint forces at a later stage, through the introduction of Lagrange multipliers. In contrast with the Newtonian approach in which all the reactions and external forces appear in the equations of motion (through the free-body diagram method, for example), in Lagrangian dynamics the problem is divided into two logically separate steps. In the first step, the reactionless equations of motion are formulated and solved in terms of the known impressed forces (i.e., generalized coordinates and their time derivatives are obtained in terms of the impressed forces by analytical and/or numerical means). The dynamical quantities thus arrived at as outputs of the first stage of the problem are now used as inputs to determine the constraint reactions that arise in the second stage of the problem.

## Reference

- 1 Rosenberg, R. M., *Analytical Dynamics of Discrete Systems*, Plenum Press, New York, pp. 243–246.

<sup>7</sup>See footnote 7, p. 250.

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