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Analytical solution for soil water redistribution during evaporation process

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ABSTRACT

Simulating the dynamics of soil water content and modeling soil water evaporation are critical for many environmental and agricultural strategies. The present study aims to develop an analytical solution to simulate soil water redistribution during the evaporation process. This analytical solution was derived utilizing an exponential function to describe the relation of hydraulic conductivity and water content on pressure head. The solution was obtained based on the initial condition of saturation and an exponential function to model the change of surface water content. Also, the evaporation experiments were conducted under a climate control apparatus to validate the theoretical development. Comparisons between the proposed analytical solution and experimental result are presented from the aspects of soil water redistribution, evaporative rate and cumulative evaporation. Their good agreement indicates that this analytical solution provides a reliable way to investigate the interaction of evaporation and soil water profile.

Key words | climate control apparatus, evaporation, Richards' equation, soil water redistribution

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INTRODUCTION

Evaporation is an important process in earth's hydrological cycle whereby liquid water undergoes a change of state and is converted into vapor. It is critical for various engineering, environmental and hydrological applications such as water evaporation from land surface and its application in hydrological modeling (Bittelli *et al.* 2008; Shokri & Or 2011), solution accumulation near earth's surface and performance of soil cover (Yanful & Mousavi 2003), and so on. Evaporation and water redistribution in soil can exist simultaneously and detract from each other. Moreover, the dynamic of soil water content during evaporation is required for the assessment of soil water management practices such as irrigation scheduling (Chanzy & Bruckler 1993; Suleiman & Ritchie 2003).

The evaporation process consists of two distinct stages: (1) constant rate stage, which occurs when the soil surface is at or near saturation and is determined by atmospheric conditions; and (2) falling-rate stage in which the water movement is controlled by soil water status and the hydraulic properties (Hillel 1980; Brutsaert 1982). Brutsaert & Chen (1995) reported that most of soil evaporation occurs during falling-rate stage evaporation because first-stage evaporation does not usually last long after rainfall or irrigation events. This is especially prevailing for the arid and semi-arid region where the rainfall events are sparse. Although the evaporation rate is relatively lower during falling-rate evaporation stage, the cumulative evaporation can be significant for an extended period such as several months. Therefore, the change of soil water content would also be profound during evaporation process (Suleiman & Ritchie 2003).

The space and time evolution of the soil water content in an unsaturated media is represented by the Richards' equation, which is highly nonlinear due to dependence of both hydraulic conductivity and the soil water potential on the soil water content. With respect to models of water content redistribution, many numerical solutions of Richards' equation are available nowadays employing different finite difference and finite element approximations. However, analytical solutions are relatively easy to be implemented and thus allow insight into the physics of the process. During the past few decades, several analytical solutions have been developed to describe the water content distribution in unsaturated zone. However, most of them describe the downward water movement that is induced by infiltration (Philip 1969; Srivastava & Yeh 1991; Warrick

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et al. 1991; Basha 2002; Chen et al. 2003; Yuan & Lu 2005). Mechanism of evaporation is quite different from that of infiltration, and evaporation rate estimated from climate variables would not always be the actual surface flux. Therefore, the surface flux is not suitable to define the boundary condition or it could only simulate the partial stage of the evaporation process. The analytical solutions for infiltration cannot be directly applied to handle with the evaporative flux. Also, analytical solutions capable of simultaneously modeling the evaporation process are scarce.

In this paper, on the basis of the exponential hydraulic parameter model proposed by Gardner (1958), an analytical solution is proposed by linearizing Richards' equation to model soil water dynamic during the evaporation process. In addition, an exponential function scarcely capable of modeling the evaporative curve of whole stages without a switch between atmosphere-controlled and soil-controlled phases was put forward to formulate the surface water content during evaporation. In order to validate the proposed analytical solution, a series of experiments were conducted under different evaporative demand.

THEORY

The Richards' equation generalizing isothermal vertical flow in unsaturated soil can be written as:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial \theta}{\partial z}\right) - \frac{\partial k}{\partial \theta}\frac{\partial \theta}{\partial z} \tag{1}$$

where θ is volumetric water content, t represents time, z is the vertical coordinate (positive downward), k is hydraulic conductivity, and D is the soil water diffusivity.

In order to linearize the above differential equation, the hydraulic conductivity model (Gardner 1958) and the dependence of the water content on the pressure head (Chen et al. 2001) is assumed as the following constitutive relations:

$$k(\psi) = k_s e^{\alpha\psi} \tag{2a}$$

$$\theta(\psi) = \theta_r + (\theta_s - \theta_r)e^{\alpha\psi} \tag{2b}$$

where k_s is the saturated hydraulic conductivity, α is a soil pore-size distribution parameter, ψ is soil water pressure, and θ_s and θ_r are saturated and residual water content, respectively. Since the parameter D is defined as $D = k\, d\psi/d\theta$, into which Equations (2a) and (2b) are substituted, its expression yields:

$$D = \frac{k_s}{\alpha(\theta_s - \theta_r)} \tag{3}$$

here, normalized soil water content Θ is defined as:

$$\Theta = \frac{\theta - \theta_r}{\theta_s - \theta_r} \tag{4}$$

Substituting Equations (2)–(4) into Equation (1), Equation (5) is achieved:

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial \Theta}{\partial z}\right) - D\alpha\frac{\partial \Theta}{\partial z} \tag{5}$$

In this study, we treat the soil as a semi-infinite porous media, in other words, the water table is deep enough to neglect the water supply from the lower boundary. In analytical solutions, the initial condition is always the soil water content profile while the boundary conditions can be the soil water content at the surface or the surface fluxes. However, it is relatively difficult to model the whole two stages of evaporative flux. The following initial and boundary conditions are chosen when the surface water content varies with the time for arbitrary water content profile:

$$\Theta(z, 0) = f(z) \tag{6a}$$

$$\Theta(0, t) = g(t) \tag{6b}$$

where $f(z)$ and $g(t)$ represent the vertical water content profile and the change of surface water content, respectively. Following solutions of Carslaw & Jaeger (1959) and Menziani et al. (2005, 2007), the solution of Equation (5) subjected to Equation (6) can be written as:

$$\begin{aligned}\Theta(z, t) = {} & \frac{e^{\frac{2\alpha z - D\alpha^2 t}{4}}}{\sqrt{\pi}\sqrt{4Dt}}\int_0^\infty f(z')\, e^{\frac{-\alpha z'}{2}}\left[e^{-\left(\frac{z-z'}{\sqrt{4Dt}}\right)^2} - e^{-\left(\frac{z+z'}{\sqrt{4Dt}}\right)^2}\right] dz' \\ & + \frac{1}{2\sqrt{\pi}}\int_0^t g(t')\, e^{\frac{-[D\alpha(t-t')-z]^2}{4D(t-t')}}\frac{zD}{[D(t-t')]^{3/2}}\, dt'\end{aligned} \tag{7}$$

Since $f(z)$ and $g(t)$ are arbitrary functions of depth z and time t, respectively, various soil water redistribution can be produced by selecting appropriate functions for $f(z)$ and $g(t)$. But it is noted that the integrals in the aforementioned equations can be difficult to solve analytically. In this paper, a specific condition that the evaporation occurs in homogenous saturated soil is chosen to be analyzed, and the change of water content at soil surface is assumed to decrease exponentially with time (Menziani *et al.* 2005; 2007). The initial and boundary conditions are formulated in Equation (8),

$$\Theta(z, 0) = 1, \text{ and } \Theta(0, t) = e^{-\beta t} \quad (8)$$

where β is a positive constant with the unit of one over time, and it could be written as $\beta = 4Db^2$ (Menziani *et al.* 2005), where b is a fitting parameter. According to the value of b, the solution has two different expressions because of the presence of a square root with a radicand. Therefore, the solution of Equation (7) corresponding to the above conditions can be expressed as:

for $b \leq \alpha/4$,

$$\Theta(z, t) = \frac{1}{2}\left[\mathrm{erfc}\left(\frac{D\alpha t - z}{2\sqrt{Dt}}\right) - e^{\alpha z}\mathrm{erfc}\left(\frac{D\alpha t + z}{2\sqrt{Dt}}\right)\right] + \frac{1}{2}e^{(\frac{\alpha z}{2} - 4b^2 Dt)}\left[e^{2zc}\mathrm{erfc}\left(\frac{z}{2\sqrt{Dt}} + 2c\sqrt{Dt}\right) + e^{-2zc}\mathrm{erfc}\left(\frac{z}{2\sqrt{Dt}} - 2c\sqrt{Dt}\right)\right] \quad (9)$$

where $c = \sqrt{(\alpha/4)^2 - b^2}$; and for $b > \alpha/4$,

$$\Theta(z, t) = \frac{1}{2}\left[\mathrm{erfc}\left(\frac{D\alpha t - z}{2\sqrt{Dt}}\right) - e^{\alpha z}\mathrm{erfc}\left(\frac{D\alpha t + z}{2\sqrt{Dt}}\right)\right] + e^{-\frac{(z - Dt\alpha)^2}{4Dt}} R\left\{W\left(2\sqrt{\left(b^2 - \frac{\alpha^2}{16}\right)Dt} + i\frac{z}{2\sqrt{Dt}}\right)\right\} \quad (10)$$

where $W(x + iy) \equiv W(z) = e^{-z^2}\mathrm{erfc}(-iz)$ is an error function of complex variable whose real and imaginary parts are reported in the Appendix II, Table I, p. 485 (Carslaw & Jaeger 1959) and in Table 7.9, p. 326 (Abramowitz & Stegun 1965).

Consider the surface flux $E(t)$ can be given by (neglecting gravity):

$$-E(t) = -D\frac{\partial \theta}{\partial z}\bigg|_{z=0} \quad (11)$$

As it is quite complicated and difficult to analytically calculate the derivative of error function of complex variable, this study only reports the surface flux solution in the case of $b \leq \alpha/4$. Substituting Equations (2), (4), (8) and (9) into Equation (11), yields:

$$E(t) = D(\theta_s - \theta_r)e^{-4b^2 Dt}\left[\frac{\alpha}{2} - 2\mathrm{erfc}(2c\sqrt{Dt}) - \frac{\alpha}{2}\mathrm{erfc}\left(\frac{\alpha}{2}\sqrt{Dt}\right)\right] \quad (12)$$

For the case that complicated forms of $f(z)$ and $g(t)$ are chosen, the solutions for Equations (5) and (11) have to be expressed as integral solutions and numerical integration. The solutions obtained here is for a specific condition when the evaporation is from an initially saturated soil.

EXPERIMENTAL PROCEDURES

Laboratory evaporation tests have been conducted to investigate soil water dynamics during evaporation. These tests were performed inside a climate control apparatus whose temperature, relative humidity and wind speed can be maintained constant to obtain different potential evaporation rate (Teng *et al.* 2012). The K-7 sand, a kind of standard fine sand in Japan, was adopted as the material, of which the relevant physical and hydraulic properties are shown in Table 1. The sample was packed into a cylinder with diameter 10 cm and height 20 cm. Five water content probes (EC-5, Decagon Devices) were inserted into the cylinder at depths of 1, 5, 10, 15 and 19 cm, respectively. Enough water was supplied to saturate the specimen virtually. The soil columns were then subjected to different environmental conditions from time to time that were controlled in three cases as shown in Table 2. The potential evaporation rate for each case was measured throughout the experiment by means of a free water surface of the same dimensions taken in advance; they are 0.31, 0.36 and 0.22 mm/h for cases 1, 2, and 3, respectively. The amount of the evaporation was monitored by weighing the entire cylinder for each 15 min. The water content

Table 1 | Summary of the properties of the soil sample

Specimen	Specific gravity (g/cm³)	Bulk density (g/cm³)	Sand (%)	Silt (%)	Clay (%)	Uniformity coefficient	Curvature coefficient	Median diameter (mm)	θ_r	θ_s	k_s (m/s)	α (m^{-1})
K-7 sand	2.67	1.45	85.8	14.2	0	3.57	1.20	0.214	0	0.4	3.9×10^{-6}	4.8

Table 2 | Experimental conditions. The number in parentheses is the mean error for each item

Condition no.	Relative humidity (%)	Wind speed (m/s)	Temperature (°C)	Duration (h)
Case 1	60 (±3.2)	2.0 (±0.2)	30 (±1.0)	574
Case 2	40 (±2.3)	2.0 (±0.2)	30 (±1.0)	343
Case 3	40 (±3.2)	1.0 (±0.1)	30 (±1.0)	868

and temperature profile along the column were also measured at the same interval. The terminal condition for each case was achieved when the weight change was lower than 0.3 g in one hour. The duration of the experiment was 77 days.

RESULTS AND DISCUSSION

Figure 1 illustrates the variations of soil water content at the top 1 cm versus the elapsed time for the three cases. It indicates that it takes a shorter time for the soil surface to get dry under higher potential evaporation rate conditions. Since water content gradient is not clearly performed yet at the first 24 h, water transfer in soil is not activated, thus the normalized water content Θ sharply reduces at the beginning. Then, the value of Θ gradually decreases with no obvious inflection point observed, which is somehow distinct from the transformation of first stage and falling-rate stage in the evaporation curve.

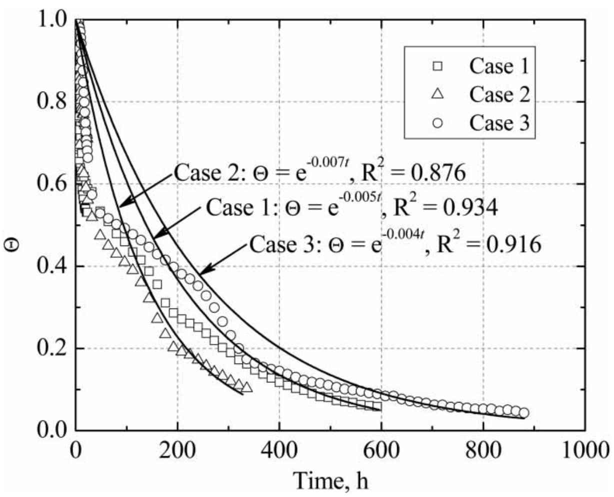

Figure 1 | Measured and simulated soil water content at a depth of 1 cm versus the elapsed time. The solid line represents simulated results, and the symbols are measured ones.

When its value is lower than about 0.2, the value of Θ smoothly changes and can last for a long period coincidently corresponding to the residual falling-rate stage. It is deduced that the transformation of surface water content is relatively easier to be simulated than that of evaporative fluxes. The experimental data are also modeled by exponential function following Equation (8), which indicates a relatively satisfying agreement. It should be pointed out that the exponential function provides rough simulation of water content variation at the first couple of days. The values of fitting parameter b for cases 1, 2 and 3 can be calculated from that of β, which are 0.413, 0.489, and 0.370, respectively.

In order to evaluate the application of the aforementioned theory, the water content distributions computed at the required time are displayed in Figure 2 compared with the experiment result. The experiment data marked as scatters show that the surface water content (top 5 cm) is much lower than that of deeper soil at the beginning stage of the evaporation process, and the water content gradually changes to be uniform for the latter stage. It can be observed in all three figures that the higher drying rate results in lower water content profile at any given time, although a lower drying rate is maintained for a longer time. The computed result shown as solid line in Figure 2 are obtained using Equation (9) since all three cases coincide with the condition of $b \leq \alpha/4$. Good agreements between theory and experiments suggest that the theory proposed in this paper can competently simulate the water content redistribution during the evaporation process. However, the nearly straight solid line simulated by this model also indicates that it cannot fit the variation of water content at soil surface very well, which may be due to the limitation of the soil depth and the simplified assumption for surface water content versus elapsed time.

Figure 3 shows the resulting analytical evaporation rates compared to those obtained from experiments under the three cases. The experimental results show two distinct evaporation stages for each case, the constant rate stage lasts about 90, 60 and 180 h for cases 1, 2, and 3, respectively. Then it comes to the falling rate stage and finally a residual falling-rate stage starts from 250, 200 and 370 h, respectively. Figure 3 also shows that the evaporation rates computed from Equation (12) sharply increase from initial zero at the beginning of 25 h, and then they decrease similar to the negative exponential function curve. After all, the comparisons of the results suggest that the analytical solution is capable of capturing trends of the evaporation curve.

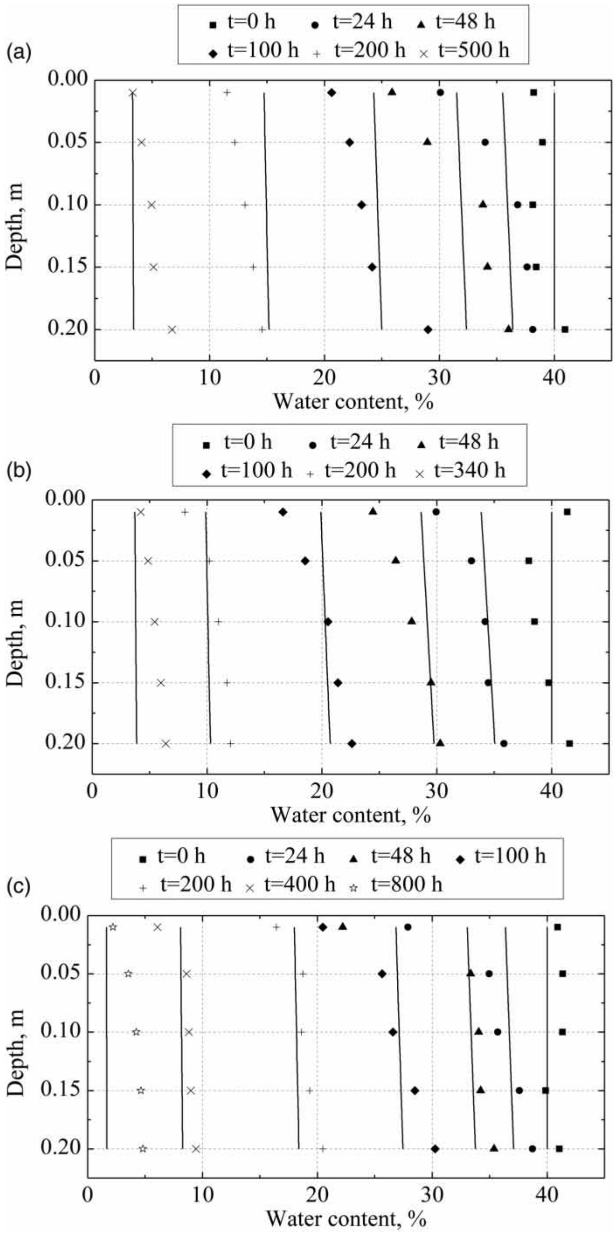

Figure 2 | Measured and computed water content profile for: (a) case 1, (b) case 2, and (c) case 3. Symbols present the experimental profile while the solid lines are theoretical trends.

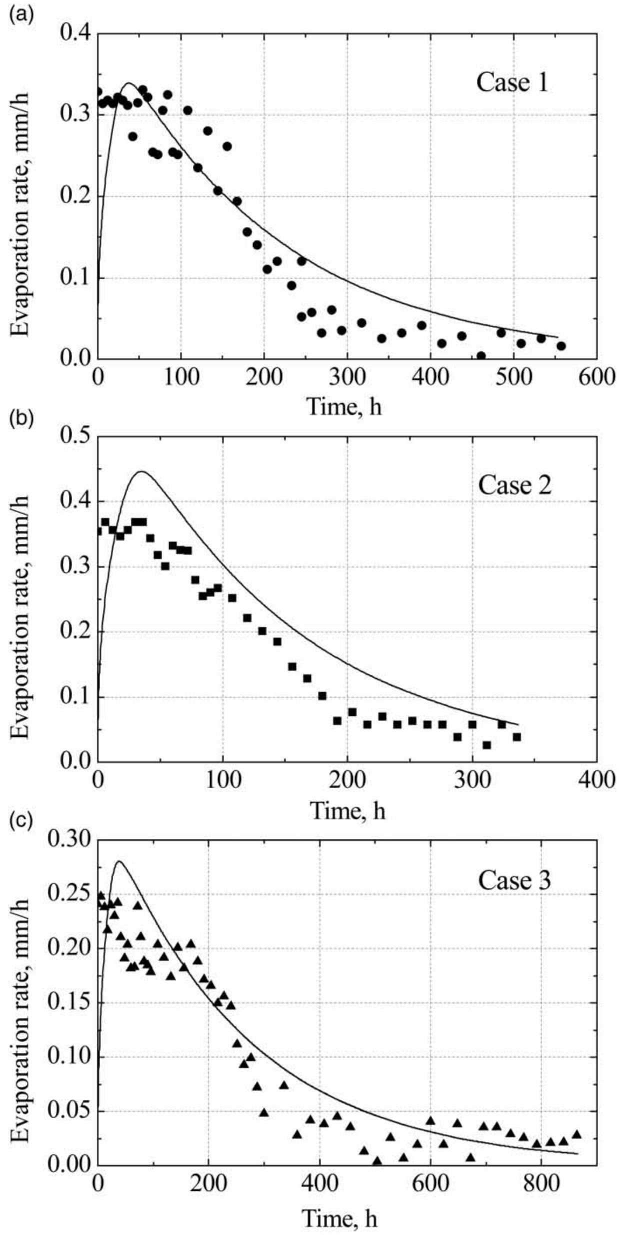

Figure 3 | Measured and computed evaporative rate for: (a) case 1, (b) case 2, and (c) case 3.

The cumulative evaporation is also obtained by integrating the analytical evaporation rate as presented in Figure 4, in which solid line representing analytical results compared with experimental symbols. It could be observed that the value of analytical cumulative evaporation is quite close to the experimental measurements at the first stage mentioned above, and then the gap starts to be greater near the residual falling stage. The accumulated differences between analytical and experimental results are satisfied, about 4.4, 10.9 and 6.0 mm respectively for the three cases. In addition, it shows that the water content distributions analytically predicted are lower than the measured data at nearly end stage of 500, 340 and 800 h, respectively, which is consistent with Figure 2. The reason may be that the exponential function of the hydraulic conductivity and the water content on the pressure head somehow overestimates the value of hydraulic conductivity and the diffusivity.

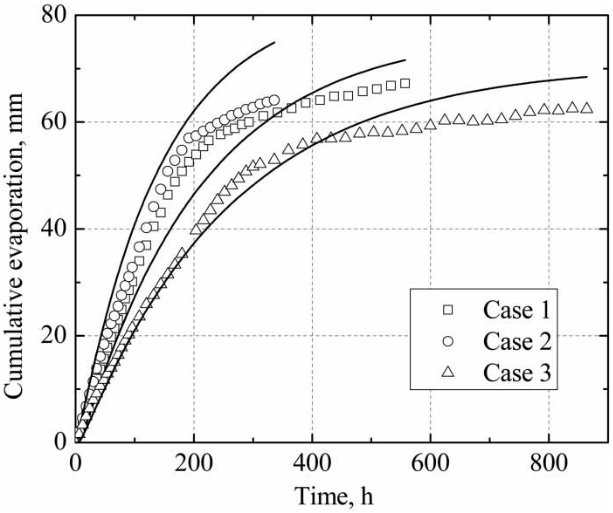

Figure 4 | Measurement and computed cumulative evaporation versus elapsed time for the three cases.

CONCLUSIONS

This paper focuses on an analytical solution of linearized Richards' equation for the evaporation process. The solution is obtained based on two assumptions: (1) the hydraulic conductivity and water content are exponential functions of the pressure head; (2) the surface water content as boundary condition changes exponentially.

To validate the analytical solution, laboratory evaporation tests are carried out in three cases with the climate conditions controlled. By comparing the analytical results and experimental data, it is found that the proposed model can provide good estimation for soil water profile during evaporation expected for the soil at top 5 cm. It is capable of closely capturing trend of the evaporation curve, and the accumulated differences between analytical and experimental result are about 4.4, 10.9 and 6.0 mm respectively for three cases, which are also in the reasonable range. Overall, the proposed method succeeded in simulating the soil water dynamic during evaporation. Further studies should be conducted to demonstrate the effectiveness of the analytical solution for different soil types and the region with greater depth.

From another aspect, there are still some imperfections in this analytical solution. The integrals of general analytical form (Equation (7)) somehow are difficult to be solved, which would limit the application of this approach to handle with other more complex problems, for example, the surface water content variation is not able to be, or is hard to be, determined by a simple formula. In addition, the solution for analyzing the precipitation problem would be available by changing the boundary formulation.

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