On the Generation and Propagation of Shock Waves From Apollo Rockets at Orbital Altitudes

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Summary

Acoustic signals from Apollo rockets at orbital altitude (188 km) appear to be explainable with the assumption that the exhaust plume serves as a conical body of large cross-section moving supersonically with the rocket. The presence of the surface signal (1.3 Hz and higher) implies that propagation in the upper atmosphere occurred as an N-wave shock cone without the strong attenuation to which an acoustic wave or even a saw-toothed (shocked) wave of similar frequency would be subjected. The shock cone does not attenuate because energy is continually re-supplied along the shock cone from the vehicle and its plume acting as a piston. Calculated overpressures do not reduce to acoustic amplitudes until the wave is below 40 km where acoustic attenuation becomes negligible.

Introduction

Tripartite arrays of infrasonic sensors at Bermuda have detected sound signals from the Apollo 12 and Apollo 13 spacecrafts (Cotten & Donn 1971), passing almost overhead. Coherent acoustic waves with strong impulsive beginning and periods of one second and less (Fig. 1) were recorded at the appropriate arrival times for the rockets' Mach cones at Bermuda. These infrasonic acoustic signals have been interpreted (Cotten & Donn 1971) as having been initiated by the shock waves from the Saturn rockets passing eastbound at an average elevation of 188 km, 55 km south of the sensor array. In this paper we examine the application of existing theory to the problem of propagation and attenuation of infrasonic, shock and acoustic waves from an elevation of 188 km, where the air density is only \(4.6 \times 10^{-10}\) kg m\(^{-3}\), compared to \(1.22\) kg m\(^{-3}\) at the ground, and the molecular mean free path length is 158 metres, compared with \(6.63 \times 10^{-8}\) m at the ground.

The analysis will first apply steady-state sonic boom theory to determine the relevant source parameters necessary for the signal to reach the surface. We then apply the existing theories of attenuation of acoustic waves and of saw-toothed (shocked) acoustic waves to show that any disturbance generated at 188 km will be severely attenuated unless its energy is continuously replenished from the source. A discussion of existing steady-state shock wave theory will suggest that this resupply of energy actually does occur. This gives confidence in the interpretation of Cotten & Donn (1971).

The duration or period \(\tau\) of the first N-wave in the signals is \(3/4\) s. Thus the N-wavelength at the ground is \(\lambda = c_0 \tau = 255\) m, where \(c_0 = 340\) m s\(^{-1}\), the ground
level sound speed. The wave will be shown to have propagated most of the way down as a shock N-wave. The wavelength, or separation of the front and rear shocks, in such a wave spreads as the wave leaves the source (Whitham 1952, Carlson, Mack & Morris 1966). The N-wavelength at 188 km altitude is therefore less than 250 m.

The mean free path length and the N-wave length are thus about equal at the source. The Apollo-Saturn rocket length is 68.5 m, with the first stage detached; its diameter is 10 m. The Knudsen number is thus $158 \text{ m}/68.5 \text{ m} = 2.3$, so that the air flow past the rocket is almost in the range of free molecule flow. Taking the sound speed $c$ to be $c = \sqrt{\gamma p/\rho}$ at 188 km, where the ratio of specific heats $\gamma = 1.5$, pressure $p = 1.77 \times 10^{-6} \text{ mb}$, and density $\rho = 4.62 \times 10^{-10} \text{ kg m}^{-3}$, we get $c = 758 \text{ m s}^{-1}$. The rocket speeds for Apollo 12 and 13 were 6400 and 6000 m s$^{-1}$ respectively so that a representative Mach number $M$ is 8.

Although one might question whether the inviscid Euler equations on which Whitham’s (1952) sonic boom analysis is based would be applicable in the nearly free molecule flow regime, Grad’s (1959) kinetic theory analysis indicates that the simplest non-dissipative continuum equations will give the correct behaviour sufficiently far from the source.

**Application of sonic boom analysis**

The pressure amplitudes of shock waves produced by supersonic projectiles are calculated (Hubbard 1966; Carlson et al. 1966; Kane 1966) by some variant of the formula:

$$\Delta p = K_0 \alpha^4 \delta r^{-4} + o(r^{-5})$$ (1)
in atmospheres, as given for a uniform atmosphere by Whitham (1950, 1952). Here \( K_s \) is a shape factor, about 0.5, \( \alpha = (M^2 - 1)^{1/2} \), where \( M \) is the Mach number, \( \delta \) is the slenderness ratio \( D/L \), and \( r \) the radial distance away from the axisymmetric projectile. All the variables here are normalized to ambient pressure and vehicle length.

The sonic boom analysis based on the original work of Whitham assumes a stationary front and rear shock pattern in the co-ordinate system fixed in the rocket. In the co-ordinate system fixed on the Earth a disturbed region between the shocks moves with the rocket. The energy and momentum contained within this region was supplied when the stationary pattern was being established.

Common usage (Hayes 1967; Carlson 1966; McLean 1967) of Whitham's equation, given in the form

\[
\Delta p = p_h K_s (M^2 - 1)^{1/2} \left( K_s \frac{D}{L} \right) \left( \frac{L}{h} \right)^{1/2}
\]

by Carlson (1966), involves area or diameter enlargement by the flow around the lifting surface (wing) for aircraft. In equation (2), \( \Delta p \) is the initial pressure jump and \( p_h \) is the ambient pressure at the rocket, both in the same units (mb). \( M \) is the Mach number, \( K_s \) is a shape factor, \( D \) is the (effective) vehicle diameter, \( L \) is the vehicle length, \( h \) is the height interval from source to observer, and \( K_r \) is a reflection coefficient which is nearly 2 when the observer is near the ground, and 1 when the observer is far from the ground.

The enlargement of the effective \( D \) by the lifting flow around supersonic aircraft is analogous to the spreading of the exhaust gases from the rockets. This spreading has been plainly visible in televised pictures of the stage I rocket, and amounts to many rocket diameters just before stage I and II separation, at \( h = 60 \text{ km} \). Also, recent telephotos of Apollo 8 in orbit (ca 190 km) clearly show that the exhaust plume broadens to a width hundreds of times the rocket diameter. We shall see that \( K_s D/L \) of the order of 10 is sufficient to give the observed order of magnitude of \( \Delta p \) at the ground. Thus with \( K_s = 0.5 \), the effective exhaust diameter is 1370 m or 137 rocket diameters.

Kane (1966) and Pierce & Thomas (1969) modify equation (2) by multiplying the result by a correction factor \( K_A \) to account for propagation downward through the non-uniform atmosphere. Kane gives

\[
K_A = N \sqrt{\left( \frac{p_0}{p_h} \right)}
\]

Here \( p_0 \) is the ambient pressure at the observing height. The subscript \( g \) will be used at the ground. \( N \) is a non-linear correction term given by Kane, similar to the term \( N_c \) given by Pierce & Thomas.

Multiplying the result of equation (2) by this \( K_A \) gives

\[
\Delta p = p_h K_s K_A (M^2 - 1)^{1/2} \left( K_s \frac{D}{L} \right) \left( \frac{L}{h} \right)^{1/2}
\]

or

\[
\Delta p = (p_h p_0)^{1/2} N K_s (M^2 - 1)^{1/2} \left( K_s \frac{D}{L} \right) \left( \frac{L}{h} \right)^{1/2}.
\]

The factor \( \sqrt{(p_h p_0)} \) is the geometric mean pressure, which Carlson (1966) mentions is often used to correct roughly for non-uniform atmosphere, in place of \( p \) on equation (2). Evaluating equation (4) with \( p_h = 1.778 \times 10^{-6} \text{ mb} \), \( p_0 = p_g = 1 \times 10^3 \text{ mb} \), \( N = 2 \), \( K_r = 2 \), \( M = 8 \), \( K_s D/L = 10 \), \( h = 188000 \text{ m} \), \( L = 68.5 \text{ m} \), we get \( K_A = 4.76 \times 10^4 \) and \( \Delta p = 7.5 \mu \text{bar} \). This is approximately the observed order of magnitude, 10 \( \mu \text{bar} \).
If we were to use Whitham’s value $K_s = 0.5$ for a slender projectile, we would have $K_s = 1.073$ without any ‘spreading’ of the effective diameter. The predicted $\Delta p$ would then be 0.55 µbar, more than an order of magnitude less than observed. It thus appears that the enlargement of the source projectile to its plume diameter is essential to explain the observations.

Pierce & Thomas (1969) give a more suitable atmospheric correction, which for large $M$, $(M > 3)$, becomes

$$K_A = N_c \left( \frac{\rho_o}{\rho_h} \right)^{1/2} \frac{c_0}{\bar{c}}$$

(6)

where $\rho_o$ and $\rho_h$ are the atmospheric densities at the observing point and at the vehicle height, respectively,

$$\bar{c} = \frac{1}{h-z_0} \int_{z_0}^{h} c dz$$

(7)

is the average sound speed between the source height $h$ and the observation height, $z_0$, while $c_0$ is the sound speed at the observation point. Their $K_A$ was obtained by tracing an acoustic ray tube, and then correcting for non-linear propagation effects due to the steep shock front by the non-linear correction factor $N_c$, given, when $M \gg 1$, as

$$N_c = \left( \frac{4}{\pi} \right)^{1/2} \left[ \frac{\zeta}{\phi(\zeta)} \right]^{1/2}$$

(8)

Here $\zeta = [(h-z)/H]^1$, (9)

where $H$ is the pressure scale height and $\phi(\zeta)$ is the cumulative error function

$$\phi(\zeta) = \frac{2}{\sqrt{\pi}} \int_{0}^{\zeta} e^{-t^2} dt.$$  (10)

Standard data are taken from the U.S. Standard Atmosphere (1962). The pressures tabulated there give $H$ as 10.0 km for the effective scale height between the ground and 188 km. Thus equation (9) gives $\zeta = 4.34$, equation (10) gives $\phi(\zeta) \approx 1.000$, and equation (8) gives $N_c = 2.21$ near the ground. The average sound speed estimated from the observations was 412 m s$^{-1}$, (Cotten & Donn 1971), and for ground observations this value for $\bar{c}$ will be used in equation (6), rather than the almost identical value obtained from a crude evaluation of equation (7). Equation (6) gives $K_A = 0.95 \times 10^5$, which is about twice the value given by equation (3). Accordingly, equation (4) then gives $\Delta p = 15.0$ µbar, if $K_s = 2$ is retained. Actually, because the sensors were placed in deep woods, $1 < K_s < 2$ is closer to one. Thus the use of an appropriate correction for atmospheric propagation has increased the predicted $\Delta p$, by a factor of two, to the observed order of magnitude, 10 µbar.

If we substitute the Pierre-Thomas correction (equation (6)) into equation (4) we get:

$$\Delta p = \rho_h N_c \left( \frac{c_0}{\bar{c}} \right) \left( \frac{\rho_o}{\rho_h} \right)^{1/2} K_s (M^2 - 1) K_A \frac{D}{L} \left( \frac{L}{h} \right)^{1/2}.$$  (11)

Confidence in this substitution is gained if we compare equation (6) with equation (12) which gives the atmospheric correction for a simple acoustic plane wave, propagating downward through an isothermal atmosphere:

$$\sqrt{\left( \frac{\rho_o}{\rho_h} \right)}.$$  (12)
Equations (6) and (12) differ only by the non-linear correction factor $N_e$ in (6), since the $c_0/\sqrt{\gamma}$ term in (6) is unity for the isothermal assumption in equation (12). It is seen from equation (12) that the pressure amplitude of a plane wave must grow as $p_0/p_h = 5.68 \times 10^4$, as used in equation (6) and not as $p_0/p_h = 2.58 \times 10^4$ used in equation (5). Although this introduces a significant and helpful factor of two for the orbital altitudes of interest here, it is not particularly significant when dealing with aircraft altitudes, so that the results of Kane (1966) are essentially correct.

The greatest height $z$ at which the equations (4) can be applied is about 185 km, 3 km below the rocket. Since the mean free path for atmospheric molecules at 188 km is 158 m $\approx$ 0.16 km, the 185-km level is 20 mean free path lengths away. The kinetic theory (Boltzmann equation) treatment by Grad (1959) indicates that the (Whitham) results of the continuum mechanics treatment become correct when the observer is many mean free path lengths away from the rocket. At this height $\Delta p/p \approx 1$.

It is interesting to note that between 188 and 150 km, the amplitude $\Delta p$ as given by (4) diminishes due to distance as $h^{-4}$, and that $K_A \approx 1$ or 2 at those heights. However, below 150 km $K_A$ grows rapidly, being 8 by 115 km and 50 by 90 km, and causes growth of the amplitude $\Delta p$ as the signal comes down from 150 km. Between 188 km and the ground, $K_A$ increases from 1 to $10^5$, while the shock amplitude $\Delta p$ increases by a factor of $4 \times 10^3$.

Acoustic attenuation

At the ground, $\Delta p/p_0 = 1.5 \times 10^{-5}$. This perturbation is small enough to be called ‘acoustic’ rather than a shock wave. An $N$-wave of this amplitude can reach the ground, since its energy is supplied by the rocket acting like a piston. This is the picture described by sonic boom theory. If the wave were of sufficiently small amplitude to satisfy the acoustic approximation at all heights, and if energy were not being replenished, then it would be severely attenuated at the upper levels, and would not reach the ground. This will be seen by examination of the acoustic attenuation lengths which we shall now calculate and tabulate in Table 1.

We must ask what perturbation amplitude is small enough to say that $\Delta p/p_0 \ll 1$ so that propagation is linear, or acoustic? At 1000 Hz, a condensation $S_m = (p - p_0)/p_0 = 10^{-4}$ is a sound so intense as to be painful to the human ear, and ordinary conversation involves $S_m \approx 10^{-7}$ (Towne 1967). More important, what really matters is not $\Delta p/p_0$ or $\Delta p/p_0$ but that the displacement gradient $\partial \xi / \partial x \gg 1$ at the front. The $N$-wave is therefore never truly acoustic, even when $\Delta p/p = 10^{-5}$. The nonlinearity will continuously steepen the wave front and cause amplitude growth, as given by $N_e$, (equation (8)).

A sinusoidal waveform distorts by 10 per cent within a distance (Towne 1967)

$$d = \frac{\lambda}{20 (\gamma + 1) S_m} = \frac{ct}{34.3 (\Delta p/p)}$$

(13)

where $\gamma = C_p/C_v$, $\lambda =$ wavelength, and $c$ is the local acoustic speed. Thus an intense sound wave with $S_m = 10^{-4}$ distorts by 10 per cent in 200 wavelengths. We may with some confidence call a wave acoustic when $\Delta p/p_0 = S_m/\gamma$ is small enough to make the distortion distance (equation (13)) greater than the distance to be traversed.

We tabulate in Table 1 some values of $\Delta p/p_0$ as given by equation (11) at various heights $z$, and the 10 per cent distortion distances $d$ there. It is thus seen that at 115 km, the top of the ‘sound channel’ where $c$ roughly equals its value at the ground, the $N$-wave is not propagating acoustically though $\Delta p/p_0 = 0.03 \ll 1$, because a sine wave of that amplitude will distort its waveform, due to non-linearity, by 10 per cent every wavelength or 0.25 km. Our wave cannot by this criterion be called ‘acoustic’ until
Table 1

<table>
<thead>
<tr>
<th>( z ) (km)</th>
<th>( C ) (m s(^{-1}))</th>
<th>( K_A )</th>
<th>( \Delta p ) (mb)</th>
<th>( \Delta p / p_2 ) (km)</th>
<th>( L_m ) (m)</th>
<th>( 1/x ) (km)</th>
<th>( 1/B ) (km)</th>
<th>( B/A )</th>
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<td>188</td>
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<td>0.98</td>
<td>0.017</td>
<td>158</td>
<td>0.43</td>
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<tr>
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<td>1.1</td>
<td>0.75(−6)</td>
<td>0.30</td>
<td>0.05</td>
<td>108</td>
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<td>1.9(−6)</td>
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<tr>
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<td>0.65(−6)</td>
<td>0.24</td>
<td>0.07</td>
<td>92</td>
<td>0.80</td>
<td>1.8(−6)</td>
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<td>721</td>
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<td>0.56(−6)</td>
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<td>0.10</td>
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<td>1.6(−6)</td>
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<td>687</td>
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<td>0.11</td>
<td>0.12</td>
<td>41</td>
<td>2.5</td>
<td>1.4(−6)</td>
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<td>638</td>
<td>2.1</td>
<td>0.50(−6)</td>
<td>0.21</td>
<td>3.2</td>
<td>8.3</td>
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<td>1.1(−6)</td>
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<td>0.025</td>
<td>0.29</td>
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<td>15</td>
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<td>130</td>
<td>485</td>
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<td>0.70(−6)</td>
<td>0.013</td>
<td>0.49</td>
<td>16</td>
<td>60</td>
<td>0.9(−6)</td>
</tr>
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<td>120</td>
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<td>1.0(−6)</td>
<td>0.040</td>
<td>0.99</td>
<td>0.026</td>
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<tr>
<td>110</td>
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<td>1.2(1)</td>
<td>1.8(−6)</td>
<td>0.095</td>
<td>0.60(−2)</td>
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<td>1.05(−4)</td>
<td>4.65(−4)</td>
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<td>2.7(−4)</td>
<td>8.6(3)</td>
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<tr>
<td>50</td>
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<td>2.1(3)</td>
<td>2.05(−4)</td>
<td>2.55(−4)</td>
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<tr>
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<td>317</td>
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<td>1.35(−4)</td>
<td>52</td>
<td>2.0(−5)</td>
<td>1.1(5)</td>
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<tr>
<td>30</td>
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<td>8.8(3)</td>
<td>0.80(−3)</td>
<td>0.65(−4)</td>
<td>100</td>
<td>4.4(−6)</td>
<td>4.6(5)</td>
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<tr>
<td>20</td>
<td>295</td>
<td>2.0(4)</td>
<td>1.6(−3)</td>
<td>2.9(−3)</td>
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<td>10</td>
<td>300</td>
<td>4.6(4)</td>
<td>3.75(−3)</td>
<td>1.4(−5)</td>
<td>470</td>
<td>2.0(−7)</td>
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<tr>
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<td>7.5(−3)</td>
<td>0.75(−5)</td>
<td>990</td>
<td>6.6(−8)</td>
<td>3.9(7)</td>
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</table>

Numbers in parenthesis ( ) are powers of 10.
it gets down to about 40 km, where further non-linear steepening is small, amounting to about 2-4 per cent. Because of the steep displacement gradient $\partial x/\partial z$ at the N-front we cannot say that the wave actually is acoustic at 35 km. However what can be said is that if it somehow were acoustic above some 80 km it could not remain so, but would steepen into a shocked wave.

Below about 60 km the ordinary attenuation formulas of linear acoustic theory may perhaps be applied to a wave of the amplitude we are dealing with. Various authors (Landau & Lifshitz 1959; Lighthill 1956; Morse & Ingard 1968) derive the Stokes–Kirchhoff attenuation constant as

$$\alpha = \frac{\omega}{2\pi k^3} \left[ \left( \frac{4}{3} \mu + \zeta \right) + \kappa \left( \frac{1}{C_v} - \frac{1}{C_p} \right) \right]$$

where $\omega = 2\pi/\tau$ is the circular frequency, $\mu$ is the ordinary viscosity coefficient as tabulated in the Handbook of Geophysics, $\zeta$ is the second or bulk viscosity coefficient which Landau and Lifshitz state is of the same order of magnitude as $\mu$, and $\kappa$ is the thermal conductivity, and $C_v$ and $C_p$ are the specific heats. Equation (14) was derived for small-amplitude sinusoidal waves of form $p = p_0 \cos(kx-\omega t) \exp(-\xi x)$, for which $\alpha \ll k$, the propagation constant, so that the attenuation length, $(1/\alpha)^{1/2}$, has the wavelength. For $\tau = 1$ s and taking $\zeta = \mu$ as tabulated in the U. S. Standard Atmosphere and Supplements (1966) we find the values of attenuation length tabulated in Table 1. At 100 km, the attenuation length of 12 km is surely valid, since it satisfies the $\alpha > \lambda$ condition under which equation (14) was derived. However, it applies only to small-amplitude acoustic waves, a condition certainly not met by these N-waves above about 40 km. If applicable below about 40 km, the Stokes–Kirchhoff attenuation coefficient (equation (14)) gives negligible attenuation there.

If the wave did attenuate acoustically from about 80 km downward, it would lose its N-shape by greater attenuation of the higher frequencies, since the attenuation length $1/\alpha$ is inversely proportional to frequency squared. This rounding of the N into a sine wave does not happen, because the large $\Delta p/p$ and short distortion length (2.5 km) there causes the wave to steepen, thus sharpening the N. The damping at elevations above some 40 km must be handled by the equations of shock waves, and not by the acoustic attenuation coefficient, equation (14).

**Saw-toothed (shocked) wave attenuation.**

Above 40 km or so we must deal with the attenuation or decay of shock fronts, rather than of small-amplitude sinusoidal waves. The shock waves treated by Whitham and later authors are not damped, since the dissipated energy and momentum is continuously being supplied by the vehicle producing the sonic boom, as discussed in the following section. The imposition of the appropriate jump conditions for pressure, velocity, temperature and entropy takes account of the irreversible viscous and heat conduction processes which occur in the narrow shock transition zone. This zone has a thickness, of the order of the mean free path length, which depends on the viscosity and heat conductivity. The viscosity and heat conductivity alter only the thickness of this zone by counteracting the tendency of the shock to steepen (Lighthill 1956). Viscous effects in the shock zone are thus implicitly accounted for in Whitham’s equations.

We shall now consider the attenuation of shock waves which are not continuously supplied with energy, so that dissipation will reduce the amplitude. If any attenuation considerations were appropriate at elevations above some 40 km, then the following analysis would apply.
Morse & Ingard (1968) give the rate of attenuation of a saw-toothed sound wave, which they call a 'shocked' wave, of amplitude $p_s$, wavelength $\lambda$, and average ambient pressure $p_0$, density $\rho$, and sound speed $c$ by

$$\frac{dp_s}{dx} = -\frac{\gamma + 1}{\gamma^2} \frac{\rho c^2}{p_0^2} p_s^2 - \frac{3\delta}{2\rho_0 c \lambda^2} p_s$$  \hspace{1cm} (15)$$

where

$$\delta = 4 \left[ \left( \frac{4}{3} \mu + \zeta \right) + \kappa (\gamma - 1)/C_p \right].$$  \hspace{1cm} (16)$$

The first term in (15) is related to both viscous and heat conduction losses involved in the entropy jumps at the shock fronts. The second term in (15) is caused by viscous and heat conduction losses on the pressure slope between the shock fronts. It differs from the similar loss involved in acoustic sine waves only through the difference in temperature and velocity gradients, involving a factor of $\pi^2$. Equation (15) is of form

$$dp_s = -(A p_s^2 + B p_s) \, dx$$

with

$$A = (\gamma + 1)/\gamma \lambda p_0$$  \hspace{1cm} (17)$$

as given by Morse and Ingard. It follows easily that

$$B = \frac{3L'}{2\lambda^2}$$  \hspace{1cm} (18)$$

with

$$L' = \delta/\rho_0 c.$$  \hspace{1cm} (19)$$

The solution to (15) is given as

$$\frac{\Delta p}{\Delta p_0} \frac{[\Delta p_0 + (B/A)]}{[\Delta p + (B/A)]} = \exp \left[ -B(\Delta z) \right]$$  \hspace{1cm} (20)$$

where $\Delta p$ has replaced the amplitude $p_s$, and $\Delta z$ will be taken positive down. Substituting (16) into (19) into (18) we see that

$$B = 6 \left( \frac{4}{3} \mu + \zeta + \kappa (\gamma - 1)/C_p \right)/\rho_0 c \lambda^2.$$  \hspace{1cm} (21)$$

This is, with $\lambda = 2\pi c/\omega$, of the same form as the Stokes–Kirchoff attenuation coefficient $\alpha$ (equation (14)) derived by these authors and others for sinusoidal acoustic waves. In fact

$$B = 3\alpha/\pi^2 \approx 0.3\alpha.$$  \hspace{1cm} (22)$$

The numbers tabulated for $B$ and $B/A$ at elevations above 90 km must be based upon the tabulated mean free paths, $L_m$, since $\mu$ and $\kappa$ are not tabulated above 90 km in the U. S. Standard Atmosphere. Morse and Ingard state that $L'$ in (19) is of the order of $L_m$, and that

$$\kappa = 5\mu C_v/3,$$

$$\mu \approx L_m \rho c/\sqrt{\gamma}.$$
Equation (19) however, taking $\zeta = 0$, as is probably correct above say 100 km, gives $L' = 6.1 L_m$. Therefore $B/A$ and $1/B$ have been tabulated using $L' = 6L_m$ in (17) and (18), as

$$\frac{B}{A} = \frac{9L_m p_z}{c}$$

$$\frac{1}{B} = \frac{c^2 \tau^2}{9L_m}$$

for $\tau = 1$ s.

Note that $(1/B) \approx 3.3(1/\alpha)$ is not the $e$-folding length for $\Delta p$, except where $(B/A \gg \Delta p)$. The values tabulated in Table 1 show this is the situation above $z = 120$ km, so that a saw-toothed wave there would attenuate exponentially with an attenuation length 3 times that appropriate for acoustic waves at the same location. Since the coefficient $A$ thus plays no role there, the entropy jump at the shock front involves negligible attenuation at these heights, as given by equation (20).

At about $z \approx 120$ km, $B/A \approx \Delta p$, and so the attenuation given by equation (23) is faster than $e^{-B\alpha z}$, with $\alpha = 0.3B$.

Below 100 km, $B/A \approx \Delta p$, so that $\Delta p$ attenuates much faster than $e^{-B\alpha z}$, but over much of the range still not as fast as $e^{-\alpha z}$. Yet the attenuation as given by equation (23) would be severe, even for this shocked wave, at elevations above about 110 km.

**Attached shock cone: lack of attenuation**

As stated above, the Whitham (1950, 1952) description of the shock fronts is a steady-state solution. He assumes a stationary shock pattern in a co-ordinate system fixed to the rocket. Whitham’s analysis is based on the inviscid Euler equations. Both the front and rear shocks appear as discontinuities giving the characteristic $N$ shape to the pressure signature. Lighthill (1956) shows that the only effect of adding viscosity and heat conductivity is to give thickness to each shock discontinuity. The entropy jumps across the shocks remain the same to lowest order. The energy and momentum contained within the region between the shock fronts is understood to have been supplied earlier, when the stationary pattern was established. In this steady-state solution any losses due to heat conductivity and viscosity have been implicitly treated by the entropy jump conditions at the shock fronts. In this picture the rocket acts like a piston compressing the air in front of it and forming a wake behind it. To maintain the steady pattern the rocket must supply energy and momentum lost in the shocks and the wake. That energy is supplied at the rocket, where a part $F_D$ of the rocket thrust force is transmitted to the fluid moving past, thus supplying power at the rate $F_D \cdot v$, where $v$ is the relative speed of the fluid, equal to the vehicle velocity in earth-fixed co-ordinates. $F_D$ is not the total thrust, but only that part required to overcome the drag, that is, to hold the rocket from being forced back by the moving fluid. This energy input at rate $F_D \cdot v$ must flow away from the rocket, otherwise energy will accumulate, a situation not allowed in the steady state. The only direction in which mechanical energy can be transported away is along or behind the shock cone. The energy can not get ahead of the shock cone, for that would simply move the shock cone forward into the undisturbed region. The energy must therefore flow aft along the shock cone and perhaps into the wake behind the shock cone, as seen in rocket-fixed co-ordinates. The flow of an element of mechanical energy along the cone terminates at the point at which the viscous and thermal (heat conduction) dissipation related to the entropy jump occurs.

The direction of flow of energy in earth-fixed co-ordinates cannot be the same as the direction in rocket-fixed co-ordinates fixed to the rocket, because of the difference in velocity $v$ between the two co-ordinate systems, amounting to $8c$, horizontally. This adds an energy flux vector $\delta v = 8c \varepsilon$ in the forward direction to the rocket fixed energy flux, where $\varepsilon$ is the energy density. The energy flux in earth-fixed co-ordinates
FIG. 2. Energy flux diagram in which $e_r$ is the energy flux in rocket fixed co-
ordinates, $e_e$ is the energy flux in earth fixed co-ordinates, $c$ is sound velocity, $v$ is 
rocket velocity.

co-ordinates will therefore be downward and forward, as seen in Fig. 2. Since the 
shock cone travels away from the rocket at a supersonic speed and is attached to the 
moving source, energy propagation will be at angle less than 90° to the shock front. 
Hence there is a small energy flux component toward the rear along the shock cone. 
Thus the energy flow description both in earth-fixed co-ordinates and in rocket-fixed 
co-ordinates is consistent with the resupply of energy needed to maintain the steady 
state solution. That solution thus implicitly accounts for the major dissipative effects 
in the propagation of the shock cone from a projectile.

We do not claim that this is a rigorous solution to the problem, only that the main 
features of Whitham’s approach modified by the correction factors $K_A$ described 
above adequately account for dissipative effects, so that no further attenuation factors 
are appropriate.

Discussion

Where the ordinary acoustic exponential attenuation coefficient, equation (14), 
may be applied, below some 40 km, it predicts little attenuation. Equation (14) is 
invalid whenever it gives severe attenuation, as it would at some 130 km. However, 
above some 40 km the wave is certainly non-linear, and if any attenuation coefficient 
is appropriate it is that for shocked waves, equation (18). That attenuation turns out 
to be severe also above 120 km, for a saw-toothed wave of period one second.

Why then is a signal observed? The major reason is that the N-wave is attached 
to a ‘piston’, the rocket and its plume, which continually supplies energy to the wave. 
Thus there is no attenuation apparent other than the $h^{-4}$ factor in Whitham’s equation. 
The dissipation terms merely serve to affect the shape of the shock fronts, not their 
amplitude (Lighthill 1956). The derivation of shocked-wave attenuation (equations 
(15)–(20)) by Morse & Ingard applies roughly to a continuous saw-toothed plane 
wave remote from its source, and not to an N-wave or individual shock front attached 
to a source.

We hypothesize that the plume serves as a conical piston penetrating the ambient 
air and producing a shock, because the plume itself is moving forward at the same 
speed as the vehicle, although individual molecules of exhaust gas move at that speed 
minus the nozzle exhaust velocity. By virtue of replenishment, the plume is connected 
to the nozzle and moves forward at the same speed. Better knowledge of the tempera-
ture and velocity of the rocket exhaust plume at orbital altitude is required for further 
study of this problem.

The overriding conclusion is that attached shock cones differ from explosive 
shocks and do not attenuate significantly. Their amplitudes are given essentially 
correctly by the steady state theories, which have implicit within them a resupply of 
energy at the ‘piston’ (vehicle) to make up for that which is dissipated in the entropy 
jump at the shock front. The observations and conclusions of Cotten & Donn (1971) 
are consistent with this conclusion.
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