Integral Representations for Scattering Amplitudes in Perturbation Theory

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(Received May 2, 1961)

Some integral representations for scattering amplitudes are proposed, which are more general than the Mandelstam representation. Their support properties are investigated for some practical cases and for the general case in every order of perturbation theory. In the Appendices Nambu-Symanzik's formula is proved in terms of the Feynman parametric integral, and an example (for the two-particle scattering) is given in which stability conditions are satisfied but no dispersion relation holds.

§ 1. Introduction

Mandelstam\textsuperscript{1} proposed a conjecture on the possibility of a double dispersion representation for the two-particle scattering amplitude. Its validity has been investigated by many authors,\textsuperscript{2} and it seems that optimistic standpoints are generally accepted. But until now no complete proof is given even in perturbation theory\textsuperscript{3} in spite of many efforts. So it will be also possible to take a more pessimistic standpoint. Even if the Mandelstam representation is indeed correct, its proof will be too difficult to be given within a few years ahead and moreover this representation cannot be generalized to production processes as is easily seen.

In these circumstances, it will probably be worthwhile to investigate some other integral representations for scattering amplitudes. The present author already proved integral representations for the vertex function with two or three variables.\textsuperscript{4} The purpose of this paper is to propose such representations for scattering amplitudes which are valid in every order of perturbation theory. This is not a trivial generalization.

In § 2, as preliminaries Nambu-Symanzik's formula\textsuperscript{5} for the general term of perturbation theory is given in terms of the Feynman parametric integral, and corresponding integral representations are proposed. In § 3, the support properties of weight functions are investigated in the cases of the nucleon-nucleon, pion-nucleon and kaon-nucleon scatterings. The subsequent two sections are devoted to the consideration of support properties in the general case.

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In this connection we introduce a new concept which is convenient for indicating mass-spectral conditions in each Feynman graph. This concept seems to be very essential in graphical considerations.

In the final section the relations between the proposed integral representations and the Mandelstam representation and the dispersion relation are discussed.

Throughout this paper we use the terminology defined in § 2 of N.*

§ 2. Integral representations

The general term of perturbation theory (corresponding to a Feynman graph G) is given by

\[
\text{const.} \prod_{i=1}^{N} \frac{d(1-\sum_{i=1}^{N} x_i) \prod_{i=1}^{N} dx_i}{U^2 (V-i\varepsilon)^{N-2n}},
\]

where we have assumed for simplicity that all particles are scalar and all couplings are direct. The notations used in (2·1) are as follows:

- \(N\): number of internal lines belonging to G,
- \(n\): number of independent circuits of G,
- \(\varepsilon\): infinitesimal positive quantity,
- \(x_i\): Feynman parameter,
- \(U = \sum_{i=1}^{N} x_i x_2 \cdots x_n\), where the summation goes over all possible sets such that any circuit in G contains at least one line among \(\{v_1, v_2, \ldots, v_n\}\);

and

\[
V = \sum_{i=1}^{N} x_i m_i^2 + \frac{1}{U} \sum_{S} W \sum_{S} k_S^2,
\]

where

- \(m_i\): internal mass,
- \(S\): intermediate state (the summation \(\sum_{S}\) goes over all possible intermediate states),
- \(k_S\): effective external momentum in the reduced graph which is obtained by shrinking each line of \(G-S\) to one point (metric: \(k^2 = k^2 - k_S^2\)),
- \(W\): effective external momentum in the reduced graph which is obtained by shrinking each line of \(G-S\) to one point (metric: \(k^2 = k^2 - k_S^2\)),

A proof of (2·2) will be presented in Appendix A. The formula (2·1) with (2·2) is just the Feynman-parametric form of Nambu-Symanzik’s formula, in which integration parameters are inversely proportional to the Feynman parameters.

* N stands for reference 4).
We denote the set of (effective) external lines of $G$ by $g$ and a subset of $g$ by $h$. The complement of $h$ is denoted by $h^*$ ($=g-h$). Then it is obvious that

$$(h^*)^* = h,$$

$$(h_1 \cap h_2)^* = h_1^* \cup h_2^*,$$

$$(h_1 \cup h_2)^* = h_1^* \cap h_2^*.$$  

(2·3)

A Schnitt $S$ divides $g$ into two parts $h$ and $h^*$. Then the momentum $k_S$ is expressed as

$$k_S = \sum_{J \neq h} k_J = -\sum_{h^*} k_J.$$  

(2·4)

So we may denote it by $k_h$. Then (2·2) is rewritten as

$$V = \sum_{i=1}^N x_i m_i^2 - \sum_h \zeta_h s_h.$$  

(2·5)

with

$$s_h = -k_h^2 = -k_{h^*}^2,$$

$$\zeta_h = -\frac{1}{U} \sum_{\tilde{S}(S) \neq 0} W_{S} \geq 0.$$  

(2·6)

(2·7)

The summation $\sum_h$ in (2·5) means to take the sum over all possible divisions $(h|h^*)$ of $g$. The summation in (2·7) means to take the sum over all intermediate states corresponding to the division $(h|h^*)$.

We denote the number of the elements belonging to a set $\Upsilon$ by $\nu(\Upsilon)$. Then the number of the squares $s_h$ is equal to $2^{\nu(g)-1} - 1$. So when $\nu(g) \geq 4$, all squares are not linearly independent.*

Now, we consider integral representations. We multiply (2·1) by

$$1 = \int_0^\infty \int \frac{d\kappa \delta(\kappa - \sum_{i=1}^N x_i m_i^2/\tilde{\xi})}{\Pi_h} \prod_{h} \delta(z_h - \zeta_h/\tilde{\xi}) dz_h.$$  

(2·8)

with

$$\tilde{\xi} = \sum_h \zeta_h \geq 0.$$  

(2·9)

After integrations by parts, we obtain an integral representation

$$\int_0^\infty \int \cdots \int \prod_{h} \delta(1 - \sum_h z_h) \prod_{h} \frac{\varphi(\kappa, z_h)}{z_h - \sum_h \zeta_h s_h - i\epsilon} dz_h.$$  

(2·10)

The most important problem is to find the support of the weight function $\varphi(\kappa, z_h)$.

* The number of independent squares is $4\nu(g) - 10$ for $\nu(g) \geq 4$. 
In particular, when $\nu(y) = 2$, (2.10) reduces to
\[ \int_{M^2}^\infty d\kappa \frac{\varphi(\kappa)}{\kappa - s - i\epsilon} \] (2.11)
where $M$ stands for the lowest intermediate-state mass. This is the well-known Umezawa-Kamefuchi-Källén-Lehmann representation for the one-body propagator. When $\nu(y) = 3$, (2.10) becomes
\[ \int_0^\infty d\kappa \int_0^1 dz_A \int_0^1 dz_B \int_0^1 dz_C (1 - z_A - z_B - z_C) \frac{\varphi(\kappa, z_A, z_B, z_C)}{\kappa - z_A s_A - z_B s_B - z_C s_C - i\epsilon} \] (2.12)
where $A$, $B$, and $C$ are the three external lines of the vertex part. The support property of $\varphi(\kappa, z_A, z_B, z_C)$ was investigated in §16 of N, that is, $\varphi$ vanishes unless
\[ \kappa \geq \text{Max}[M_A^2 z_A + M_B^2 z_B + (M_A - M_B)^2 z_C, M_A^2 z_A + (M_A - M_C)^2 z_B + M_C^2 z_C, (M_B - M_C)^2 z_A + M_B^2 z_B + M_C^2 z_C], \] (2.13)
where $M_A$, $M_B$ and $M_C$ are the respective lowest intermediate-state masses, provided that they satisfy triangular inequalities.*

The case $\nu(y) = 4$ is the two-particle scattering. We denote the four external lines by $A$, $B$, $C$, $D$. The possible divisions of $y$ are expressed by
\[ h = \{A\}, \{B\}, \{C\}, \{D\}, \{AB\}, \{AC\}, \{AD\}. \] (2.14)
The corresponding squares $s_h$ are denoted by $M_A^2$, $M_B^2$, $M_C^2$, $M_D^2$, $s$, $t$, $u$, respectively. Between them the well-known identity
\[ M_a^2 + M_b^2 + M_c^2 + M_d^2 = s + t + u \] (2.15)
holds. The first four squares are usually fixed on mass shells. Then putting
\[ \zeta_{AB}^2 + \zeta_{AC}^2 + \zeta_{AD}^2 - 3\zeta_0^2 \]
(2.16)
and
\[ \zeta_1 = \frac{\zeta_{AB} - \zeta_0}{\zeta_{AB} + \zeta_{AC} + \zeta_{AD} - 3\zeta_0}, \]
\[ \zeta_2 = \frac{\zeta_{AC} - \zeta_0}{\zeta_{AB} + \zeta_{AC} + \zeta_{AD} - 3\zeta_0}, \]
(2.17)
* If triangular inequalities are not satisfied, instead of (2.13) we get
\[ \kappa \geq \text{Max} [M_A^2 z_A + M_B^2 z_B, M_A^2 z_A + M_C^2 z_C, M_B^2 z_B + M_C^2 z_C] \]
by the method which will be stated in §4 and §5.
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\[ z_3 = \frac{\zeta_{AB} - \zeta_0}{\zeta_{AB} + \zeta_{AC} + \zeta_{AB} - 3\zeta_0} \]

instead of (2.8), where \( \zeta_0 \) is an arbitrary function satisfying

\[ \zeta_0 \leq \text{Min}(\zeta_{AB}, \zeta_{AC}, \zeta_{AD}) , \quad (2.18) \]

we get

\[
\int_{-\infty}^{\infty} d\lambda \int_{0}^{1} dx_1 \int_{0}^{1} dx_2 \int_{0}^{1} dx_3 \delta(1 - \sum_{j=1}^{3} x_j) \frac{\phi(\lambda, z_1, z_2, z_3)}{\lambda - z_1 s - z_2 t - z_3 u - i\varepsilon} . \quad (2.19)
\]

The support in this integral representation is ambiguous on account of the identity (2.15). For example, if \( \zeta_0 \) tends to minus infinity, the support shrinks to a point

\[ \lambda = (M_a^2 + M_b^2 + M_c^2 + M_d^2)/3 , \]

\[ z_j = 1/3 \quad (j=1, 2, 3) . \quad (2.20) \]

Thus (2.19) is not interesting.

In order to avoid such an ambiguity, we put

\[ \zeta_0 = \text{Min}(\zeta_{AB}, \zeta_{AC}, \zeta_{AD}) . \quad (2.21) \]

Then (2.19) reduces to

\[
\int_{-\infty}^{\infty} d\alpha \int_{0}^{1} dx \frac{f_{12}(\alpha, x)}{\alpha - x s - (1 - x) t - i\varepsilon} + \int_{-\infty}^{\infty} d\beta \int_{0}^{1} dy \frac{f_{23}(\beta, y)}{\beta - y t - (1 - y) u - i\varepsilon}
\]

\[ + \int_{-\infty}^{\infty} d\gamma \int_{0}^{1} dz \frac{f_{31}(\gamma, z)}{\gamma - z u - (1 - z) s - i\varepsilon} \quad (2.22) \]

plus three possible single dispersion terms (and a constant term). Here the first, second and third terms in (2.22) correspond to the cases \( \zeta_0 = \zeta_{AB}, \zeta_0 = \zeta_{AB}, \) and \( \zeta_0 = \zeta_{AC} \) respectively. In the next section the support properties of (2.22) will be investigated for some practical cases.

When \( M_a^2, M_b^2, M_c^2, M_d^2 \) are considered to be variables, we can derive an integral representation, which consists of seven terms such as

\[
\int_{-\infty}^{\infty} d\kappa \int_{0}^{1} dz_A \ldots \int_{0}^{1} dz_{AC} \delta(1 - z_A - \cdots - z_{AC}) \]

\[ \times \frac{\varphi(\kappa, z_A, z_B, z_C, z_D, z_{AB}, z_{AC})}{\kappa - z_A M_a^2 - z_B M_b^2 - z_C M_c^2 - z_D M_d^2 - z_{AB} s - z_{AC} t - i\varepsilon} + \ldots \quad (2.23) \]

with some possible subtraction terms, by a procedure similar to the above (\( \zeta_0 = \text{Min}(\zeta_A, \zeta_B, \zeta_C, \zeta_D, \zeta_{AB}, \zeta_{AC}, \zeta_{AD}) \) in the present case). The support properties will be investigated in \( \S \, 5. \)

We can likewise deal with the cases \( \nu(g) \geq 5. \) But the number of terms will become enormously large.
§ 3. Support properties in some practical cases

In this section the support properties of the weight functions in (2.22) are investigated for the nucleon-nucleon, pion-nucleon and kaon-nucleon scatterings. For this purpose, it is convenient to make use of the method used in the proof of dispersion relations (see §14 and §15 of N).

We denote the four external momenta by \( k_A, k_B, k_C, k_D \). The V-function for the two-particle scattering is

\[
V = \sum_{i=1}^{N} x_i m_i^2 + \sum_{j=A,B,C} \zeta_j k_j^2 + \zeta_{AB}(k_A + k_B)^2 + \zeta_{AC}(k_A + k_C)^2 + \zeta_{AD}(k_A + k_D)^2 + \zeta_{BC}(k_B + k_C)^2 + \zeta_{BD}(k_B + k_D)^2 + \zeta_{CD}(k_C + k_D)^2.
\]

(3·1)

We assume that the graph \( G \) contains no one-particle line, otherwise it reduces to simpler graphs.

We first consider the equal-mass case. Let \( k_1 \) and \( k_2 \) be two vectors having the following properties:

\[
-k_1^2 = -k_2^2 = \mu^2, \quad k_1 k_2 = 0.
\]

(3·2)

We abbreviate \( a_1 k_1 + a_2 k_2 \) as \( (a_1, a_2) \). We then have proved in §15 of N that if the four external momenta are given by \((1, 1), (1, -1), (-1, 1) \) and \((-1, -1)\) then the V-function is non-negative definite. For example, when we put

\[
k_A = (1, 1), \quad k_B = (1, -1),
\]

(3·3)

\[
k_C = (-1, 1), \quad k_D = (-1, -1),
\]

we get

\[
\sum x_i \mu^2 = \sum (\zeta_j + \zeta_{AB}) \cdot 2\mu^2 - (\zeta_{AB} - \zeta_{AD}) (2\mu)^2 - (\zeta_{AC} - \zeta_{AD}) (2\mu)^2 \geq 0.
\]

(3·4)

Since

\[
\alpha = \frac{\sum x_i \mu^2 - \sum (\zeta_j + \zeta_{AB}) \mu^2}{\zeta_{AB} + \zeta_{AC} - 2\zeta_{AD}}
\]

and

\[
x = \frac{\zeta_{AB} - \zeta_{AD}}{\zeta_{AB} + \zeta_{AC} - 2\zeta_{AD}},
\]

(3·5)

from (3·4) we obtain

\[
\alpha \geq (2\mu)^2
\]

(3·6)

as a restriction for the support of \( \rho_{12}(\alpha, x) \). Those of \( \rho_{23} \) and \( \rho_{34} \) are the same.

In practical cases we put \( \mu = m_a \). The nucleon number conservation law means that there are some nucleon paths (i.e. nucleon open-polygons) in \( G \).
Then we can add either \((f, 0)\) and \((-f, 0)\) or \((0, f)\) and \((0, -f)\) to both ends of each nucleon path, where

\[
f = \frac{m_N - m_\pi}{m_\pi}.
\]  

(3·7)

In the nucleon-nucleon scattering there are two nucleon paths (either \(P(AC)\) and \(P(BD)\) or \(P(AD)\) and \(P(BC)\), where \(P(AC)\) stands for a path connecting \(A\) and \(C\)). We take external momenta as follows:

\[
k_A = (f+1, 1), \quad k_B = (f+1, -1),
\]
\[
k_C = (-f-1, 1), \quad k_D = (-f-1, -1).
\]  

(3·8)

We then obtain

\[
-k_A^2 = -k_B^2 = -k_C^2 = -k_D^2 = m_N^2 + m_\pi^2 > m_N^2
\]  

(3·9)

and

\[
k_A + k_B = (2m_N/m_\pi, 0), \quad -(k_A + k_B)^2 = (2m_N)^2,
\]
\[
k_A + k_C = (0, 2), \quad -(k_A + k_C)^2 = (2m_\pi)^2.
\]  

(3·10)

Therefore the support of \(\rho_{12}\) is included in the domain

\[
\alpha \geq (2m_N)^2 x + (2m_\pi)^2 (1-x).
\]  

(3·11a)

Likewise we get

\[
\beta \geq \min[(2m_\pi)^2 y + (2m_N)^2 (1-y), (2m_N)^2 y + (2m_\pi)^2 (1-y)],
\]  

(3·11b)

\[
\gamma \geq (2m_\pi)^2 z + (2m_N)^2 (1-z).
\]  

(3·11c)

Next we consider the pion-nucleon scattering. Putting

\[
k_A = (f+1, 1), \quad k_B = (1, -1),
\]
\[
k_C = (-f-1, 1), \quad k_D = (-1, -1),
\]  

(3·12)

so that

\[
-k_A^2 = -k_C^2 = m_N^2 + m_\pi^2 > m_N^2,
\]  

(3·13)

and

\[
-(k_A + k_B)^2 = (m_N + m_\pi)^2,
\]  

(3·14)

we obtain

\[
\alpha \geq (m_N + m_\pi)^2 x + (2m_\pi)^2 (1-x).
\]  

(3·15a)

Likewise we have
\[ \beta \geq (2m_x)^2 y + (m_N + m_x)^2 (1 - y), \quad (3\cdot15b) \]
\[ \gamma \geq \text{Max} \left[ (m_N + m_x)^2 z + \{ (m_x - m_x)^2 + (2m_x)^2 \} (1 - z), \right. \]
\[ \left. \{ (m_N - m_x)^2 + (2m_x)^2 \} z + (m_N + m_x)^2 (1 - z) \right], \quad (3\cdot15c) \]

where the latter has been obtained by putting
\[ k_A = -k_c = (f + 1, 1), \quad k_B = -k_D = \pm (-1, 1). \quad (3\cdot16) \]

Finally, we consider the kaon-nucleon scattering. In this case, we have not succeeded in proving the non-forward scattering dispersion relation for experimental mass values. So we must content ourselves with making use of the proof of the forward scattering dispersion relation. We obtain
\[ a \geq (m_N + m_K)^2 x, \quad (3\cdot17a) \]
\[ \beta \geq (m_A + m_x)^2 (1 - y), \quad (3\cdot17b) \]
\[ \gamma \geq \text{Max} \left[ (m_A + m_x)^2 z + \{ 2m_N^2 + 2m_K^2 - (m_A + m_x)^2 \} (1 - z), \right. \]
\[ \left. (m_N - m_K)^2 z + (m_N + m_K)^2 (1 - z) \right]. \quad (3\cdot17c) \]

§ 4. Generalized intermediate state

Before entering into the consideration on the support properties in the general case, we will introduce some new concepts and investigate their properties.

Let \( Z \) be a set of internal lines such that any external line belonging to \( h \) does not connect with any one of \( h^* \) without passing through \( Z \). We denote the set of \( Z \) by \( Z(h) \). Then it is evident that
\[ Z(h^*) = Z(h), \quad (4\cdot1) \]
\[ Z(h) \neq \emptyset, \quad (4\cdot2) \]

where \( \emptyset \) means "empty set". The latter is because \( Z(h) \) always contains the set of all internal lines of \( G \), provided that all elements of \( y \) are effective external lines. If \( Z' \supset Z \) for \( Z \in Z(h) \), then one has \( Z' \in Z(h) \).

Let \( Z \) be the union of \( Z(h) \) where \( h \) moves over all subsets of \( y \) except for \( \emptyset \) and \( y \). We call an element of \( Z \) a "generalized intermediate state".

When \( Z_1 \in Z(h_1) \) and \( Z_2 \in Z(h_2) \), \( Z_1 \cup Z_2 \) separates \( y \) into four parts \( h_1 \cap h_2 \), \( h_1 \cap h_2^* \), \( h_1^* \cap h_2 \) and \( h_1^* \cap h_2^* \). We therefore get
\[ Z_1 \cup Z_2 \in Z(h_1 \cap h_2), \quad (4\cdot3a) \]
\[ Z_1 \cup Z_2 \in Z(h_1 \cap h_2^*), \quad (4\cdot3b) \]
\[ Z_1 \cup Z_2 \in Z(h_1^* \cap h_2^*), \quad (4\cdot3c) \]
\[ Z_1 \cup Z_2 \in Z(h_1 \cup h_2), \quad (4\cdot3d) \]
by using (2·3) and (4·1).

An intermediate state $S$ is defined by

$$S \in Z$$

and

$$\forall R < S \quad R \not\in Z.$$  \tag{4·4} \ast

Let $S$ be the set of intermediate states. We write

$$S(h) = Z(h) \cap S.$$  \tag{4·5}

Then $S(h^*) = S(h)$ holds, but $S(h) \not\approx \phi$ is not always assured. If $h_1 \not\approx h_2$ and $h_2^*$, $S(h_1) \cap S(h_2^*) = \phi$.

It is an important property that if $Z \in Z$, then $\exists S \subseteq Z \subseteq S$, because if $Z \in S$ then $\exists Z_i < Z$ $Z_i \in Z$ on account of (4·4), and if $Z_i \in S$ then $Z_i < Z_1 \subseteq Z \in Z$; since the series $Z > Z_1 > Z_2 > \ldots \in Z$ is of course a finite one, there exists the final element $Z_r : \forall R < Z$, $R \not\in Z$; namely, $Z_r \in S$. But it should be noticed that $\exists S \subset Z \subseteq S(h)$ does not follow from $Z \subseteq Z(h)$.

When $H$ is a set of internal lines in $G$, we define the “mass measure” of $H$ by

$$m(H) = \sum \frac{1}{m_i} m_i.$$  \tag{4·6}

This evidently has the following properties:

i) $m(\emptyset) = 0,$ \hspace{1cm} \tag{4·7a}

ii) $m(H) \geq 0,$ \hspace{1cm} \tag{4·7b}

iii) $m(H_1 \cup H_2) + m(H_1 \cap H_2) = m(H_1) + m(H_2),$ especially, $m(H_1 \pm H_2) = m(H_1) \pm m(H_2).$ \hspace{1cm} \tag{4·7c\textsuperscript{d}}

We define the “lowest intermediate-state mass of $G$” and the “lowest generalized-intermediate-state mass of $G$” as follows:

$$M_0(h) = \min_{S(h)} \{ m(S) \} \quad \text{if} \quad S(h) \not\approx \emptyset,$$

$$= \infty \quad \text{if} \quad S(h) = \emptyset.$$  \tag{4·8}

$$M_0(h) = \min_{S(h)} \{ m(Z) \}. \tag{4·9}

Then it is obvious that

$$M_0(h^*) = M_0(h), \quad M_0(h^*) = M_0(h)$$  \tag{4·10}

and

$$M_0(h) \geq M_0(h) \geq 0.$$  \tag{4·11}

$M_0(h)$ has some remarkable properties. When $h_1 \cap h_2 = \emptyset$, the triangular inequalities

\* $\not\subset \not\supset$ means $\not\subset \not\subset$ and $\not\not\subset$.\*
always holds.

Proof. From (4·7c) and (4·7b) we have

\[ M_c(h_1) + M_c(h_2) = \min_{Z_1 \in Z(h_1)} \min_{Z_2 \in Z(h_2)} m(\alpha(Z_1, Z_2)) \geq \min_{Z \in Z(h_1 + h_2)} m(Z). \]  

(4·13)

Since for any \( Z_1 \subseteq Z(h_1) \) and \( Z_2 \subseteq Z(h_2) \)

\[ Z_1 \cup Z_2 \subseteq Z(h_1 + h_2) \]  

(4·14)

on account of (4·3d), we obtain

\[ \min_{Z \subseteq Z(h_1 + h_2)} m(Z) = \min_{Z_1 \in Z(h_1)} \min_{Z_2 \in Z(h_2)} m(\alpha(Z_1, Z_2)). \]  

(4·15)

The second inequality of (4·11) follows from (4·13) and (4·15). In order to prove the first one, instead of (4·14) we have only to use that for any \( Z \in Z(h_1 + h_2) \) and \( Z' \in Z(h_1) \) (or \( \in Z(h_2) \))

\[ Z \cup Z' \subseteq Z(h_1) \quad (or \in Z(h_i)) \]  

(4·16)

on account of (4·3b) and \( h_1 \cap h_2 = \emptyset \). (Q.E.D.)

Next we consider the relation between \( M_c(h) \) and \( M_c(h) \). Let \( G[h] \) be the self-energy graph which is obtained from \( G \) by joining the external lines of \( h \) and those of \( h^* \) respectively. If we consider \( Z \subseteq Z(h_1 + h_2) \) in \( G[h] \), we get

\[ \exists S \subseteq Z \quad S \subseteq S(h) \quad (4·17) \]

because \( S(h) = S \) in \( G[h] \). In \( G \), therefore, we have

\[ S \subseteq Z(h) \]  

(4·18)

and

\[ S = \sum_{i=1}^{r} S_i \quad S_i \subseteq S(h_i) \]

with

\[ 1 \leq r \leq \nu(g) - 1. \]  

(4·19)

We call \( S \) defined by (4·18) a “primitive generalized intermediate state with respect to \( h \)”, and denote the set of primitive generalized intermediate states with respect to \( h \) by \( P(h) \). We call \( \{S(h_i)\} \) the “type of \( S \)”.

\( P(h) \) is of course a subset of \( Z(h) \). Since \( m(S) \leq m(Z) \) on account of \( S \subseteq Z \), one has

\[ M_c(h) = \min_{\alpha \in P(h)} m(\alpha) = \min_{\sum_{\alpha \in P(h)} m(\alpha)} \sum_{\alpha \in P(h)} m(\alpha). \]  

(4·20)

Calculating the minimum of \( m(S_i) \) within \( S(h_i) \) first, we obtain

\[ M_c(h) = \min_{h \in \operatorname{prim}} \left[ \sum_{i} M_c(h_i) \right] \]  

(4·21)
where Min means to take the minimum for all types, \( \{S(h_i)\} \), of primitive generalized intermediate states with respect to \( h \). In particular, if \( h \) consists of only one external line, (4.21) becomes

\[
M_0(h) = \text{Min} \left[ \sum_{i: h \rightarrow h^0} M_0(h_i) \right].
\]  

(4.22)

The mass spectral condition is usually indicated by

\[
M_h = \text{Min}_{\tilde{\alpha}} M_0(h),
\]

(4.23)

where Min means to take the minimum for all graphs having the same \( g \) which are permissive under given interactions. Then

\[
M_h = \text{Min}_{\alpha} M_0(h).
\]

(4.24)

**Proof.** Because of (4.11), it suffices to show that

\[
\forall G \exists G' \quad M_0(h) \leq M_{0'}(h).
\]

(4.25)

Such a graph \( G' \) is constructed by joining three \( G \)'s: The first \( G \) is joined to the second in \( h^* \) and the second is joined to the third in \( h \). (Q.E.D.)

Finally, we notice that the following interesting inequality always holds in case of the two-particle scattering \( (g = \{A, B, C, D\}) \):

\[
M_c(A) + M_c(B) + M_c(C) + M_c(D) \leq M_0(AB) + M_0(AC) + M_0(AD).
\]

(4.26)

It should be remarked that a corresponding inequality for \( M_h \) does not hold in general.

**Proof.** It suffices to prove that for any \( Z_1 \in Z(AB), Z_2 \in Z(AC) \) and \( Z_3 \in Z(AD) \)

\[
Z_1 + Z_2 + Z_3 \subset Z_1 + Z_2 + Z_3.
\]

(4.27)

where we have assumed \( Z_1 \cap Z_2 = Z_2 \cap Z_1 = Z_1 \cap Z_2 = \phi \). If not so, we have only to introduce some 2-vertices appropriately (then \( G \) is topologically unchanged). We denote a path connecting \( A \) and \( B \) by \( P(AB) \). By definition, \( Z_1 \) intersects any one of \( P(AC), P(AD), P(BC) \) and \( P(BD) \), \( Z_2 \) does any one of \( P(AB), P(AD), P(BC) \) and \( P(CD) \), and \( Z_3 \) does any one of \( P(AB), P(AC), P(BD) \) and \( P(CD) \). Therefore \( Z_1 + Z_2 + Z_3 \) intersects any path at least at two lines. Now, we consider a reduced graph \( G_R \) that is obtained from \( G \) by shrinking all the lines which do not belong to \( Z_1 + Z_2 + Z_3 \). Then the four external stars (i.e. the intermediate states nearest to external lines) of \( G_R \) (they are denoted by \( S^A, S^B, S^C, S^D \)) are mutually non-overlapping, and of course

\[
S^A + S^B + S^C + S^D \subset Z_1 + Z_2 + Z_3.
\]

(4.28)

We put \( Z_d = S^J (J = A, B, C, D) \), then \( Z_d \in Z(J) \) in \( G \) is obvious. Thus \( Z_d, Z_{\bar{d}}, Z_{\bar{a}}, Z_{\bar{b}} \) satisfy (4.27). (Q.E.D.)
§ 5. Support properties in the general case

For the two-particle scattering the triangular inequalities (4·12) are

\[ |M_0(A) - M_0(B)| \leq M_0(AB) \leq M_0(A) + M_0(B), \]
\[ |M_0(C) - M_0(D)| \leq M_0(AB) \leq M_0(C) + M_0(D), \]
\[ |M_0(A) - M_0(C)| \leq M_0(AC) \leq M_0(A) + M_0(C), \]
\[ |M_0(B) - M_0(D)| \leq M_0(AD) \leq M_0(B) + M_0(D), \]
\[ |M_0(A) - M_0(D)| \leq M_0(AC) \leq M_0(A) + M_0(D), \]
\[ |M_0(B) - M_0(C)| \leq M_0(AD) \leq M_0(B) + M_0(C). \] (5·1)

Combining them and (4·26), we get

\[ M_0(C) + M_0(D) \leq M_0(AC) + M_0(AD), \]
\[ M_0(A) + M_0(B) \leq M_0(AC) + M_0(AD), \]
\[ M_0(B) + M_0(D) \leq M_0(AB) + M_0(AD), \]
\[ M_0(C) + M_0(C) \leq M_0(AB) + M_0(AD), \]
\[ M_0(B) + M_0(C) \leq M_0(AB) + M_0(AC), \]
\[ M_0(A) + M_0(D) \leq M_0(AD) + M_0(AC). \] (5·2)

Now, the following property follows from Theorems 13-1 and 13-3 in N:

If

\[ k_J = a_J k \quad (J = A, B, C, D), \]
\[ \sum_I a_J = 0, \]
\[ k^2 = 1, \] (5·3)

and if

\[ |a_J| \leq M_0(J) \quad (J = A, B, C, D), \]
\[ |a_A + a_B| \leq M_0(AB), \]
\[ |a_A + a_C| \leq M_0(AC), \]
\[ |a_A + a_D| \leq M_0(AD), \] (5·4)

then

\[ \sum_{i=1}^N x_i m_i^2 - \sum_J \xi_J a_J^2 - \zeta_{AB}(a_A + a_B)^2 - \zeta_{AC}(a_A + a_C)^2 - \zeta_{AD}(a_A + a_D)^2 \geq 0. \] (5·5)

The coefficients \( a_J \) are chosen in the following way.

1) \( a_A = M_0(A), \quad a_B = -M_0(B), \quad a_C = M_0(AC) - M_0(A), \quad a_D = -M_0(D). \)
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\[ a_n = M_n(B) - M_n(AC), \quad a_a + a_n = M_n(A) - M_n(B), \]

\[ a_a + a_n = M_n(AC), \quad a_a + a_n = M_n(A) + M_n(B) - M_n(AC). \]

We can easily see that they satisfy the inequalities (5·4) on account of (5·1) and (5·2).

1. \[ a_a = M_n(A), \quad a_n = -M_n(B), \quad a_a + a_n = M_n(A) + M_n(B), \]
2. \[ a_a = M_n(A), \quad a_n = -M_n(C), \quad a_a + a_n = M_n(A) + M_n(C), \]
3. \[ a_a = M_n(A), \quad a_n = -M_n(C), \quad a_a + a_n = M_n(A) + M_n(C), \]
4. \[ a_a = M_n(A), \quad a_n = -M_n(C), \quad a_a + a_n = M_n(A) + M_n(C), \]
5. \[ a_a = M_n(A), \quad a_n = -M_n(B), \quad a_a + a_n = M_n(A) + M_n(B), \]
6. \[ a_a = M_n(A), \quad a_n = -M_n(B), \quad a_a + a_n = M_n(A) + M_n(B), \]
7. \[ a_a = M_n(B), \quad a_n = -M_n(C), \quad a_a + a_n = M_n(B) + M_n(C), \]
8. \[ a_a = M_n(B), \quad a_n = -M_n(C), \quad a_a + a_n = M_n(B) + M_n(C), \]
9. \[ a_a = M_n(B), \quad a_n = -M_n(C), \quad a_a + a_n = M_n(B) + M_n(C), \]
10. \[ a_a = M_n(B), \quad a_n = -M_n(C), \quad a_a + a_n = M_n(B) + M_n(C), \]
11. \[ a_a = M_n(B), \quad a_n = -M_n(C), \quad a_a + a_n = M_n(B) + M_n(C), \]
12. \[ a_a = M_n(B), \quad a_n = -M_n(C), \quad a_a + a_n = M_n(B) + M_n(C), \]

From (5·5) with 1) we get

\[ \sum x_i m_i \geq M_n(A)^2 \xi_A + M_n(B)^2 \xi_B + [M_n(AC) - M_n(A)] \xi_C \]

\[ + [M_n(AB) - M_n(B)] \xi_D + [M_n(A) - M_n(B)] \xi_A \]

\[ + [M_n(AC) - M_n(A)] \xi_A + [M_n(AB) - M_n(AC)] \xi_B. \]

Hence one has

\[ \sum x_i m_i \geq M_n^2 \xi_A + M_n^2 \xi_B + M_n^2 \xi_C \]

\[ \sum x_i m_i \geq \text{Max}[M_n^2 \xi_A + M_n^2 \xi_B + M_n^2 \xi_{AB}, \quad M_n^2 \xi_A + M_n^2 \xi_B + M_n^2 \xi_{AC}, \]

\[ M_n^2 \xi_A + M_n^2 \xi_B + M_n^2 \xi_{AC}, \quad M_n^2 \xi_A + M_n^2 \xi_B + M_n^2 \xi_{AB}, \]

\[ M_n^2 \xi_A + M_n^2 \xi_B + M_n^2 \xi_{AC}, \quad M_n^2 \xi_A + M_n^2 \xi_B + M_n^2 \xi_{AC}, \]

\[ M_n^2 \xi_A + M_n^2 \xi_B + M_n^2 \xi_{AC}, \quad M_n^2 \xi_A + M_n^2 \xi_B + M_n^2 \xi_{AC}, \]

\[ M_n^2 \xi_A + M_n^2 \xi_B + M_n^2 \xi_{AC}. \]

Restriction for the support of each term in the integral representation (2·23) is easily derived from (5·8). For instance, that of the first term is obtained
from (5·8) by replacing \( \sum x_i m_i^2 \) by \( \kappa \), \( \zeta_{AD} \) by 0 and \( \zeta_h \) by \( z_h \) for \( h \notin \{AD\} \). The restriction for support (5·8) yields, for example,

\[
\sum x_i m_i^2 \geq \frac{1}{2} \sum_j M_j^2 \zeta_j + \frac{1}{3} \left( M_{AB}^2 \zeta_{AB} + M_{AC}^2 \zeta_{AC} + M_{AD}^2 \zeta_{AD} \right),
\]

but one cannot prove \( \sum x_i m_i^2 \geq \sum M_j^2 \zeta_j \) in general (see Appendix B).

The integration parameters in (2·22) are given by

\[
\alpha = \sum x_i m_i^2 - M_A^2 \zeta_A - M_B^2 \zeta_B - M_C^2 \zeta_C - M_D^2 \zeta_D - (M_A^2 + M_B^2 + M_C^2 + M_D^2) \zeta_{AB},
\]

\[
x = \frac{\zeta_{AB} - \zeta_{AC} - \zeta_{AD}}{\zeta_{AB} + \zeta_{AC} - 2 \zeta_{AD}}
\]

for \( \zeta_{AB} > \zeta_{AD} \) and \( \zeta_{AC} > \zeta_{AD} \), and so on. In order to assure \( \alpha \geq 0, \beta \geq 0 \) and \( \gamma \geq 0 \) from (5·8), it is necessary and sufficient that

\[\exists b_i (i = 1, 2, \ldots, 12) \quad b_i \geq 0, \quad \sum_{i=1}^{12} b_i = 1,\]

\[
M_A^2 + M_B^2 + M_C^2 + M_D^2 \leq (b_3 + b_5 + b_7 + b_9) M_{AB}^2
\]

\[
+ (b_1 + b_6 + b_8 + b_{12}) M_{AC}^2 + (b_2 + b_4 + b_9 + b_{12}) M_{AD}^2
\]

(5·11a)

and

\[
(b_1 + b_2 + b_3 + b_4 + b_5) M_A^2 \geq M_A^2,
\]

\[
(b_1 + b_2 + b_7 + b_8 + b_{12}) M_B^2 \geq M_B^2,
\]

\[
(b_2 + b_4 + b_7 + b_8 + b_{12}) M_C^2 \geq M_C^2,
\]

\[
(b_2 + b_6 + b_9 + b_{11} + b_{12}) M_D^2 \geq M_D^2.
\]

(5·11b)

As is easily seen, (5·11a) is satisfied only when

\[
M = M_A = M_B = M_C = M_D = \frac{1}{2} \text{Max}(M_{AB}, M_{AC}, M_{AD})
\]

(5·12)

because of inequalities \( M_A + M_B \geq M_{AB} \), etc. If at least two among \( M_{AB}, M_{AC} \) and \( M_{AD} \) are equal to \( 2M \), (5·11b) is equivalent to

\[
\text{Max} M_j^2 \leq \frac{1}{2} \sum_j M_j^2 \leq M^2.
\]

(5·13)

If \( M_{AB} > \text{Max}(M_{AC}, M_{AD}) \), (5·11b) becomes

\[
M_A^2 + M_B^2 \leq M^2, \quad M_C^2 + M_D^2 \leq M^2
\]

(5·14)

instead of (5·13).

§ 6. Discussions

We have investigated support properties for the integral representations
(2.22) and (2.23). Our main results are (3.11), (3.15) and (5.8). Of course, these restrictions for supports are not necessarily the best ones.

If the Mandelstam representation

\[
\int_{m_{AB}}^\infty ds' \int_{m_{AC}}^\infty dt' \frac{\sigma_{12}(s', t')}{(s'-s-i\varepsilon)(t'-t-i\varepsilon)} + \int_{m_{AB}}^\infty ds' \int_{m_{AD}}^\infty dt' \frac{\sigma_{23}(t', u')}{(t'-t-i\varepsilon)(u'-u-i\varepsilon)}
\]

holds, the integral representation (2.22) immediately follows from it by using the Feynman identity and integrating each term by parts. Then the restrictions for supports become

\[
\alpha \geq M_{AB}^2 x + M_{AC}^2 (1-x),
\beta \geq M_{AB}^2 y + M_{AD}^2 (1-y),
\gamma \geq M_{AB}^2 z + M_{AC}^2 (1-z).
\]

So the restrictions for supports, (3.11), for the nucleon-nucleon scattering are good, while the restriction, (3.15c), of \( \rho_{123} \) for the pion-nucleon scattering should be improved to \((m_N + m_\pi)^2\), otherwise the Mandelstam representation must break down.

When \( t \) is fixed to \( t' = -4\delta \leq 0 \), the integral representation (2.22) is reduced to a single dispersion relation if

\[
\alpha \geq 0,
\beta \geq 0,
\gamma \geq M^3 \cdot \text{Min}(z, 1-z)
\]

with

\[
M^3 \geq M_{a1}^2 + M_{a2}^2 + M_{a3}^2 + M_{a4}^2 + 4\delta.
\]

The spectral variables are defined as follows:

First term: \( s' = \frac{\alpha + 4\delta}{x} (1-x) \),

Second term: \( u' = \frac{\beta + 4\delta}{1-y} \),

Third term:

\[
s' = \frac{\gamma - z (M_{a1}^2 + M_{a2}^2 + M_{a3}^2 + M_{a4}^2)}{1-2z} \quad \text{for} \quad z < \frac{1}{2},
\]

\[
u' = \frac{\gamma - (1-z) (M_{a1}^2 + M_{a2}^2 + M_{a3}^2 + M_{a4}^2)}{1+2z} \quad \text{for} \quad z > \frac{1}{2}.
\]
If \((6\cdot 2)\) is satisfied, we naturally have the normal dispersion relation for
\[-M_{ab}^2 \leq 4d^2 \leq M_{ab}^2 + M_{ab} - (M_a^2 + M_b^2 + M_c^2 + M_d^2), \tag{6\cdot 6}\]
where \(d^2\) may become negative.

A similar consideration also applies to the case in which \(s\) or \(u\) is fixed. In the general case, \(\alpha \geq 0, \beta \geq 0\) and \(\gamma \geq 0\) are not always assured. Such an example will be presented in Appendix B. We have obtained \((5\cdot 12)\) with \((5\cdot 13)\) or \((5\cdot 14)\) as the condition for \(\alpha \geq 0, \beta \geq 0\) and \(\gamma \geq 0\). As this condition is too stringent, our results in the general case will be far from practical uses, but it will be of theoretical interest especially when it is compared with results obtained from the axiomatic theory. Namely, the integral representation \((2\cdot 10)\) with \((5\cdot 8)\) reveals notable characteristics of perturbation theoretical results irrelevant to particle-number conservation laws.

**Appendix A**

**Proof of \((2\cdot 2)\)**

1') We have proved in §6 of N that
\[V = \sum_{i,j} x_i m_i^2 - \frac{1}{U} \sum_{i,j} W^{(ij)} k_i k_j \tag{A\cdot 1}\]
where \(W^{(ij)}\) is the \(U\)-function in the graph in which external lines \(A_i\) and \(A_j\) are joined. Hence if \(x_1, x_2, \ldots, x_{n+1} \in W^{(ij)}\)
\[\forall P(A_i, A_j) \exists \nu_k \in F \nu_k \in P(A_i, A_j), \tag{A\cdot 2}\]
where \(F = \{\nu_1, \nu_2, \ldots, \nu_{n+1}\}\). Therefore when all the lines of \(F\) are opened, \(A_i\) and \(A_j\) are separated (i.e. \(F \in \mathbb{Z}\)). Hence
\[\exists S \in S \ F \supset S. \tag{A\cdot 3}\]

2') **Lemma.** In order that \(x_1, x_2, \ldots, x_{n+1}/U\) is non-vanishing when \(x_i = 0 \ (\forall i \in G-S)\) where \(S\) is an intermediate state, it is necessary and sufficient that
\[F = \{\nu_1, \nu_2, \ldots, \nu_{n+1}\} \supset S. \tag{A\cdot 4}\]

*Proof.* Since \(S\) intersects \(\nu(S) - 1\) independent circuits, the number of independent circuits included in \(G-S\) is \(n - \nu(S) - 1 = n + 1 - \nu(S)\). So \(U\) has \(n + 1 - \nu(S)\)-th order zero point at \(x_i = 0 \ (\forall i \in G-S)\). On the other hand, the order of zero point of the numerator is
\[\nu(F \cap (G-S)) = n + 1 - \nu(S) \quad \text{if} \ F \supset S, \tag{A\cdot 5}\]
on account of \(n + 1 = \nu(F) = \nu(F \cap S) + \nu(F \cap (G-S))\). Thus the lemma is obtained.

(Q.E.D.)
3') From Theorem 5-1 of N we have

\[ U = U^{(H')} U^{(H_c)} U^{(R)} + O(\varepsilon_S^{s(S)}) \]  \hspace{1cm} (A·6)

where \( \varepsilon_S = \text{Max} \ x_i \) and where \( R \) is the reduced graph corresponding to an intermediate state \( S \). So for \( \varepsilon_S = 0 \) we get

\[ W_{S}/U = \prod_{i \in S} x_i/U^{(R)} = (\sum_{i \in S} 1/x_i)^{-1}. \]  \hspace{1cm} (A·7)

On the other hand, for \( \varepsilon_S = 0 \)

\[ \forall S' \ni S \ W_{S'}/U = 0. \]  \hspace{1cm} (A·8)

**Proof.** Let \( H'_1 \) and \( H'_2 \) be the two subgraphs obtained by the Schnitt \( S' \).

Since \( S' - (S' \cap S) \triangleq \phi \), we denote one of its elements by \( I \). Then we have

\[ \exists A \ \forall P(AI) \text{ in } H'_1 \text{ or } H'_2 \]

\[ P(AI) \cap S \triangleq \phi \]  \hspace{1cm} (A·9)

because of the definition of \( S \). For instance, we assume \( A \) is an external line of \( H'_1 \). Then we see that \( H'_1 \cap S \) is a generalized intermediate state of \( H'_1 \).

Now, from the lemma if \( W_{S'}/U \neq 0 \) for \( \varepsilon_S = 0 \) then one has

\[ \exists x_{v_1} x_{v_2} \ldots x_{v_m} \subseteq U^{(H'_1)} U^{(H_c)} \ (m = n + 1 - \nu(S')) \]

\[ F' = S' + \{v_1, v_2, \ldots, v_m\} \ni S. \]  \hspace{1cm} (A·10)

Therefore \( H'_1 \cap F' \) includes an intermediate state (i.e. a subset of \( H'_1 \cap S \)) of \( H'_1 \). This is inconsistent with the definition of \( U^{(H')} \) (because each term of the \( U \)-function corresponds to a set of independent integration momenta (see § 3 of N)).

(Q.E.D.)

4') We write

\[ V' = \sum_{i=1}^{S} m_i x_i^2 + \sum_{S} W_{S}/U \]  \hspace{1cm} (A·11)

From 3') we get

\[ \forall S \subseteq S \ V' = V^{(R)} \quad \text{for } x_i = 0 \quad (\forall i \in G - S), \]  \hspace{1cm} (A·12)

where \( V^{(R)} \) is the \( V \)-function for the reduced graph \( R \):

\[ V^{(R)} = \sum_{S} m_i x_i^2 + \sum_{S} W_{S}/U^{(R)} \]  \hspace{1cm} (A·13)

So, (A·12) is rewritten as

\[ \forall S \subseteq S \ V - V' = 0 \quad \text{for } x_i = 0 \quad (\forall i \in G - S) \]  \hspace{1cm} (A·14)

on account of Theorem 5-1 of N. If \( V - V' \) were not identically zero, it would be a linear combination of such terms as \( x_{v_1} x_{v_2} \ldots x_{v_m}/U \). Since no cancellation cannot happen by putting \( x_i = 0 \), one must have
\[ \forall S \subseteq \mathcal{S} \quad x_1 x_2 \cdots x_{n+1}/U = 0 \quad \text{for} \quad x_i = 0 \quad (\forall i \in G - S). \quad (A.15) \]

Hence from the lemma
\[ \forall S \subseteq \mathcal{S} \quad F = \{ \nu_1, \nu_2, \cdots, \nu_{n+1} \} \triangle S. \quad (A.16) \]

This is inconsistent with (A.3). Therefore \( V - V' = 0 \).

**Appendix B**

**Counterexample against \( \alpha \geq 0, \beta \geq 0 \) and \( \gamma \geq 0 \)**

One might suppose that the weight functions in (2.22) would vanish in general unless \( \alpha \geq 0, \beta \geq 0, \gamma \geq 0 \) if stability conditions are satisfied. This is indeed the case in the non-trivial lowest order. For Fig. 1 we obtain
\[ \alpha \geq x(m_1 - m_2)^2 + (1 - x)(m_2 - m_4)^2 \geq 0, \quad (B.1) \]
where
\[ \alpha = \sum_{i=1}^{4} x_i m_i^2 - x_1 x_2 M_a^2 - x_2 x_3 M_b^2 - x_1 x_4 M_c^2 - x_3 x_4 M_d^2 \]
\[ x = \frac{x_1 x_5}{x_1 x_5 + x_2 x_4}. \quad (B.2) \]

But we will show that this expectation is not correct in general.

Consider Fig. 2, in which the expressions for \( U \) and \( W_s \) are as follows:
\[ U = x_1 x_2 x_3 + x_1 x_2 x_5 + x_1 x_3 x_6 + x_1 x_2 x_4 \]
\[ + x_1 x_3 x_6 + x_1 x_4 x_5 + x_1 x_4 x_6 + x_1 x_2 x_4 \]
\[ + x_1 x_3 x_4 + x_2 x_3 x_5 + x_2 x_4 x_5 + x_2 x_4 x_6 \]
\[ + x_1 x_5 x_6 + x_1 x_2 x_6 + x_3 x_4 x_6 + x_5 x_5 x_6, \]
\[ W_A = x_1 x_2 x_4 (x_3 + x_5 + x_6), \]
\[ W_B = x_1 x_3 x_5 (x_2 + x_4 + x_6), \]
\[ W_C = x_2 x_3 x_6 (x_1 + x_4 + x_5), \]
\[ W_D = x_4 x_3 x_6 (x_1 + x_2 + x_3). \quad (B.3) \]
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\[ W_{12} = x_2 x_3 x_4 x_5, \]
\[ W_{10} = x_1 x_3 x_4 x_6, \]
\[ W_{10} = x_1 x_2 x_5 x_6. \]

Hence, when
\[ \forall i \quad m_i = m, \]
\[ M_a = M_b = M_c = M_d = M, \]
\[ \forall i \quad x_i = 1/6 + \varepsilon y_i \quad (\varepsilon : \text{infinitesimal}) \]
with
\[ \sum_{i=1}^{n} y_i = 0, \]
we have

\[ V = m^2 - \frac{M^2}{6} - \frac{\varepsilon}{16} [((y_3 + y_3 + y_4 + y_5)s + (y_1 + y_3 + y_4 + y_5)t + (y_1 + y_2 + y_3 + y_5)u]. \]

On the other hand, stability conditions are satisfied if \( M < 3m \). Therefore if
\[ 6m^2 < M^2 < 9m^2, \]
the lower limits of integrations over \( \alpha, \beta \) and \( \tau \) in (2·22) are indeed minus infinity in spite of the presence of stability conditions. This means that no dispersion relation holds for this example.

It might be worthwhile to notice that this example is realizable as a reduced graph if there exists the pseudoscalar meson (say \( \pi^0 \)) whose mass is larger than 340 Mev.

References

5) Y. Nambu, Nuovo Cimento 6 (1957), 1064.

Note added in proof: From our integral representation (2·22) we can derive the partial wave dispersion relation which has so far been derived under the assumption of the Mandelstam representation. Detailed account will be discussed elsewhere.