A Simple Model for Two-Channel Reaction in Field Theory and Its Application to the $Y^*$ Problem

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A simple two-channel model in field theory is presented in order to know characteristic features of the problems concerning $Y^*$ recently observed. Four particles, $K_-, N, \pi$ and $A$, all assumed to be spinless, are considered, and the $S$-state amplitudes are calculated in the chain approximation. The calculated amplitudes satisfy the unitarity condition and reproduce bound states, resonances, and other features of the reactions. From the examples treated it is concluded that a strong attractive force between $K_-$ and $N$ such that there is a bound state and a weak coupling between $K_-N$ and $\pi A$ channels are essential for the appearance of a sharp peak both in the $K_-N$ total cross section and the $K_-N \rightarrow \pi + A$ cross section in the unphysical region. The relation between height and width is such that the peak tends to a $\delta$-function if the width tends to an infinitesimal. This conclusion remains unchanged even if a force between $\pi$ and $A$ is varied over a wide range. According to the variety of a $\pi A$ force, the $\pi A$ cross section exhibits either a peak or a dip near the energy at which the $K_-N$ total cross section exhibits a peak. Furthermore an attractive or a repulsive $\pi A$ force contributes to make a peak broader or narrower, respectively.

§ 1. Introduction

In a recent experiment with the Berkeley $K^-$-beam, an interesting fact was reported.\(^1\)\(^2\) Namely an outgoing pion (positive or negative) in the reaction $K^- + p \rightarrow A + \pi^+ + \pi^-$ distributes along a straight line in the Dalitz plot with a fairly large probability. This suggests an existence of some kind of excited state composed of an $A$ and a pion ($Y^*$), the mass of which is nearly equal to 1380 MeV. A remarkable fact is that the half-width of that state is very narrow: about 15 MeV, as reported most recently.\(^3\) This value is much smaller than the half-width of the 3-3 resonance in pion-nucleon scattering (about 50 MeV), though a similarity to the 3-3 resonance was stressed in the earlier analysis.\(^1\)\(^3\)

If we assume an interaction of $Y^*$ weak enough to give such a narrow width, we could not explain a “large” cross section for the production of $Y^*$. In order to overcome this dilemma Takeda proposed a kind of “even-odd rule” similar to that proposed earlier to account for the copious production and long life of strange particles.\(^4\)

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\(^1\) See also a paper by Amati et al.\(^3\)

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We may introduce $Y^*$ as a new elementary particle similar to $A$ and $\Sigma$. It is, however, worthwhile to note that we can consider $Y^*$ as a bound state of the $K^-N$ system with a binding energy $\sim 50$ Mev.\textsuperscript{3,6,*} This conjecture will be supported further if $Y^*$ turns out to decay into $\pi$ and $A$ in an $S$-state, though this has not yet been unambiguously confirmed. This is because we can hardly expect a \textit{sharp} resonance in an $S$-state due to the lack of a centrifugal barrier.

If this conjecture is valid, the $Y^*$ phenomena will provide an actual example to which we can apply a field theory of unstable particles (or bound states), which have been discussed by many authors.\textsuperscript{7} It is also necessary to develop a field theoretical multi-channel problem ($K^-N$, $\pi A$, and $\pi \Sigma$ channels in the case of $Y^*$), which has been considered partly by Igi,\textsuperscript{8} and by Matthews and Salam.\textsuperscript{9,**}

Several authors have already attempted to make phenomenological analyses in terms of complex scattering lengths.\textsuperscript{5,6,11} But, since these problems are rather new to us, we consider it useful first to make more general and fundamental considerations before entering into detailed comparisons with experimental results. From this standpoint we shall investigate here a simplified two-channel model in field theory.

We shall consider four particles, $K$, $N$, $\pi$ and $A$, neglecting isospins, all assumed to be spinless, and such that $M_{\pi} + M_A < M_K + M_N$.\textsuperscript{***}

We shall calculate various amplitudes in the chain approximation method in an $S$-state, by assuming appropriate four-vertex interactions. A generalization so as to include Yukawa-type interactions is easy. An analogous treatment in a $P$-state is also possible.

This model is too simple to draw detailed quantitative conclusions on the actual $Y^*$ problem, because of the neglect of isospins, spins, parities and actual mass ratios. But we could expect to learn a number of qualitative features which are characteristic of a two-channel ($S$-state) reactions involving an unstable bound state. On the other hand, this model is, of course, not very general; rather it carries some restrictive properties. At the sacrifice of complete generality, however, we can reproduce bound states, resonances, and other features of reactions without any further approximation. Therefore we may expect to give instructive examples of more general considerations of the theory.

In § 2, the interaction Hamiltonian is given and the $S$-state amplitudes are calculated in the chain approximation. The connection between these amplitudes and the conventional quantities are established. In § 3, the general proper-

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* $K_\pm$ denotes $K^+$ or $K^0$, and $K_-$ denotes $K^-$ or $\bar{K}^0$.

** Multi-channel problems in terms of potentials have been investigated by Ross and Shaw.\textsuperscript{10}

*** Hereafter $K$ stands for $K_-$, namely $K^-$ or $\bar{K}^0$. 
ties of this model are discussed. Among them the unitarity condition plays particularly important roles in the following discussions. In § 4, several typical cases are discussed. From the examples treated we find that a strong attractive force between $K_-$ and $N$ such that there is a bound state, and a weak coupling between two channels are essential for the appearance of a sharp peak. Natures of the peaks are discussed: i.e. relations between height and width, the possible appearance of an "imaginary resonance". § 5 is devoted to the summary and concluding discussions.

§ 2. S-state amplitudes in the chain approximation

We assume the interaction Hamiltonian

$$H = g_1 KNKN + g_2 \pi\pi\pi\pi + g_3 (KN\pi\pi\pi\pi),$$

for the interactions between four particles, $K$, $N$, $\pi$ and $\Delta$, which are all assumed to be spinless for simplicity. We assume the simplified mass ratios given by

$$M_K = M_N = M_1,$$

$$M_\pi = M_\Delta = M_2 < M_1.$$

![Fig. 1. The simple chain diagram for $KN-KN$ scattering amplitude $T_1$.](image)

First we shall calculate the simple chain diagrams illustrated in Fig. 1, which result if we switch off the coupling between the $KN$ and the $\pi\Delta$ channels (the $g_3$-interaction). Denoting the scattering amplitudes of $KN-KN$ and $\pi\Delta-\pi\Delta$ as $T_1$ and $T_2$, respectively, we have

$$T_1(\xi) = \frac{g_1}{1 - g_1 R_1(\xi)} = \frac{g_1}{\Delta_1(\xi)},$$

$$T_2(\xi) = \frac{g_2}{\Delta_2(\xi)},$$

(2.2)

where

$$\xi = W^2 = -p^2, \quad (p: \text{total momentum})$$

$$R_\pi(\xi) = i(2\pi)^{-1} \int dq \frac{1}{(q^2 + M_2^2)[(p - q)^2 + M_2^2]}$$

$$= - \frac{1}{16\pi^2} \int dm^2 \frac{dm^2}{m^2 - \xi - i\epsilon} \sqrt{1 - \frac{4M_r^2}{m^2}}.$$  

(2.3)
The latter expression has been obtained using either the technique developed by Nakanishi\textsuperscript{12} or the standard method in a dispersion relation theory.\textsuperscript{13} If we divide $R_r$ into real and imaginary parts,\n\[ R_r = A_r + iB_r, \]
we have\n\[ A_r(\tilde{\xi}) = -\frac{1}{16\pi^2} \int_{\frac{4A_r^2}{3\tilde{\xi}}}^{\frac{4\tilde{\xi}}{3\tilde{\xi}}} \frac{dm^2}{m^2 - \tilde{\xi}} \sqrt{1 - \frac{4M_r^2}{m^2}}, \]
\[ B_r(\tilde{\xi}) = -\frac{1}{16\pi} \theta(\tilde{\xi} - 4M_r^2) \sqrt{1 - \frac{4M_r^2}{\tilde{\xi}}}. \]
We have cut off the integration with respect to $m^2$ at $4A_r^2$.

Next we take into account a coupling between the $KN$ and the $\pi A$ channels. As illustrated in Fig. 2, we can express the amplitudes as power series in $g_3^2$, which can be summed to yield\n
\[ J_1 = g_1 + hR_1, \]
\[ J_2 = g_2 + hR_2, \]
\[ J_3 = g_3 \]
where $J_1$, $J_2$ and $J_3$ are the amplitudes for the processes $KN-KN$, $\pi A-\pi A$ and $KN-\pi A$, respectively, and\n\[ h = g_3^2 - g_3, \]
\[ D = D_1D_2 - g_3^2 R_1R_2. \]
According to the chain approximation we have included only diagrams which
are serial combinations of "bubbles". Therefore Eqs. (2.5) do not satisfy
the requirement of crossing symmetry. The denominator $\mathcal{D}(\xi)$, common to
the three amplitudes in Eqs. (2.5), can be divided into real and imaginary
parts as follows:

$$\mathcal{D} = C - iB,$$

where

$$C = D_1D_2 - g_3A_1A_2 + hB_1B_2,$$
$$B = (g_2 + hA_1)B_2 + (g_1 + hA_2)B_1,$$

and $D_r$ is the real part of $\mathcal{D}$ given by

$$D_r = 1 - g_r A_r.$$  \hspace{1cm} (2.7)

Now we establish the connection between the $\mathcal{J}$'s given by Eqs. (2.5)
and conventional quantities. $\mathcal{J}_1$ and $\mathcal{J}_2$ are simply the scattering amplitudes. Using
the complex phase shifts $\delta_r$, we have

$$\mathcal{J}_r = \left(\frac{1}{B_r}\right) e^{i\delta_r} \sin \delta_r = -8\pi (W/k_r) e^{i\delta_r} \sin \delta_r,$$  \hspace{1cm} (2.9)

for the physical region of each reaction, where $W$ and $k_r$ are the energy and
momentum in the center of mass system, respectively, given by

$$W = \sqrt{\xi},$$
$$k_r = \sqrt{\frac{1}{4} \xi - M_r^2}.$$  \hspace{1cm} (2.10)

The imaginary parts of the $\mathcal{J}_r$ are related to the total cross section as follows:

$$\text{Im} \mathcal{J}_r = -2Wk_r \sigma_r,$$
$$\sigma_r = (\pi/k_r^2)|1 - e^{i\delta_r}|^2.$$  \hspace{1cm} (2.11)

The quantity $|\mathcal{J}_3|^2$ obviously gives the cross section for the reaction

$$K + N \rightarrow A + \pi.$$  

In addition to this reaction we are also interested in the reaction

$$K + N \rightarrow A + \pi + \pi.$$  

It is expected that the cross section of this reaction is intimately related $|\mathcal{J}_3|^2$. In order to calculate the amplitude of the above reaction, however, we must introduce the interaction Hamiltonian for the emission of a single pion. Among
many possibilities an example is given by

$$H = GNN\pi,$$  \hspace{1cm} (2.12)

in terms of which we can also construct a chain diagram. The calculation
can be simplified further by assuming a five-vertex interaction Hamiltonian
given by
Of course we can produce an effective Hamiltonian (2·13) by calculating the closed loop diagram illustrated in Fig. 3. The coupling constant $g_4$, thus interpreted, is proportional to $G_4$. Assuming, for the moment, the interaction Hamiltonian (2·13), we can calculate $\mathcal{J}_4$, the amplitude for $K+N\rightarrow A +\pi +\pi$, from the diagrams in Fig. 4. If we substitute $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ calculated before, for the shaded parts, these diagrams include all possible chain diagrams of lowest order in $g_4$.

These too can be summed to yield

$$\mathcal{J}_4 = \frac{g_4}{\mathcal{D}(\xi_1)\mathcal{D}(\xi_j)} [g_3^2 + \mathcal{D}_3(\xi_1)\mathcal{D}_1(\xi_j)],$$

(2·14)

where $\xi_1$ and $\xi_j$ respectively denote the squared momentum of the initial state and that of the “final” state excluding the pion emitted directly by the $g_4$-interaction.

In the experimental situation of reference 1), the cross section is obtained as a function of $\xi_1$, with a specified $\xi_j$. In the vicinity of the “peak”, the behavior of $|\mathcal{J}_4|^2$ as a function of $\xi_1$ can be considered to be given by

$$\sigma \sim |\mathcal{J}_4|^2 \sim \frac{g_3^2}{\mathcal{D}(\xi_1)^2} \sim \frac{g_3^2}{\mathcal{D}(\xi_j)^2} = |\mathcal{J}_3(\xi_j)|^2.$$ (2·15)

On expectation that this relation will hold also in more general cases, we shall discuss $|\mathcal{J}_3|^2$ in the following.

§ 3. General properties of the model

In this section we summarize the general properties of the amplitudes calculated in § 2.

(i) Properties of the simple chain amplitude $T_r$

We consider $T_1$, from which $T_2$ can be easily inferred. In the right-hand side of Eq. (2·4a) the integration can be performed explicitly as follows

$$-16\pi^2 A_r(\gamma) = I_4 - 2\sqrt{\frac{1}{\gamma} - 1}\tan^{-1}\sqrt{\frac{1 - (1/\lambda^2)}{(1/\gamma) - 1}}, \text{ for } \gamma > 1,$$

$$\left\{ \begin{array}{ll}
\sqrt{1 - \frac{1}{\gamma}} \log \sqrt{1 - (1/\lambda^2)} + \sqrt{1 - (1/\gamma)} & \text{for } \gamma < 1,
\end{array} \right.$$ (3·1)
where
\[ \eta = \frac{\bar{\epsilon}}{4M_1^2}, \quad \lambda_1^2 = 4A_1^2 / 4M_1^2, \]
\[ L_1 = \log \frac{\lambda_1 + \sqrt{\lambda_1^2 - 1}}{\lambda_1 - \sqrt{\lambda_1^2 - 1}}. \]

The function \(-16\pi^2 A_1(\eta)\) given by Eq. (3.1) diverges logarithmically at \(\eta = \lambda_1^2\). As easily noted, however, the behavior of the function near the cutoff energy changes depending strongly on the method of cutoff. Therefore we should be interested mainly in the energy region far below the cutoff, since we know at present no definite method of cutoff. In order to see the general behavior of \(-16\pi^2 A_1(\eta)\) in the energy region far below the cutoff, we may put \(\lambda_1 \to \infty\) in the right-hand side of Eq. (3.1), except for the constant term \(L_1\). In this approximation we can prove that

\[ \frac{d}{d\eta} [-16\pi^2 A_1(\eta)] \geq 0, \quad \text{for } \eta \leq 1. \quad (3.2) \]

In the neighbourhood of the threshold we find

\[ -16\pi^2 A_1(\eta) \approx -16\pi^2 A_1(1) - \begin{cases} \frac{\pi \kappa_1}{M_1}, & \text{for } \eta < 1, \\ \frac{\eta^2}{2M_1^2}, & \text{for } \eta > 1, \end{cases} \quad (3.3) \]

where \(k_1\) is the relative momentum in the center of mass system given by Eq. (2.10) and \(\kappa_1\) is its analytically continued quantity given by

![Fig. 5. A plot of \(-16\pi^2 A_1(\eta)\) with \(\lambda_1 = 20\). There is also plotted \(D_1(\eta)\) with \(g_1 = -24.13\), which will be used in Case 1, Case 5 and Case 6 (§ 4).](https://example.com/plot.png)

* The inequality for \(\eta < 1\) can be proved directly from Eq. (2.4a), since the denominator \(m^2 - \bar{\epsilon}\) never vanishes below the threshold.
Eqs. (3.3) show that $-16\pi^2A_1(\gamma)$ exhibits a "cusp" at the threshold. In Fig. 5 we have plotted $-16\pi^2A_1(\gamma)$ with $\lambda_1=20$, in which we can see all the general features implied in Eqs. (3.2) and (3.3).

On the other hand, $D_1(\gamma)$ exhibits several distinguished behaviors according to the different values of $g_1$.

For $g_1>0$, corresponding to a repulsive force, $D_1(\gamma)$ never vanishes and is larger than unity for $\gamma<1$. If we confine ourselves to the energy region far below the cutoff such that

$$-16\pi^2A_1(\gamma)>0,$$

we have

$$D_1(\gamma)>1,$$

for $\gamma>1$ also. A schematic plot is given in Fig. 6(a).

For $-16\pi^2/L_4<g_1<0$, we have

$$0<D_1(\gamma)<1,$$

in the region mentioned above. This case corresponds to such a weak attractive force that there is neither bound state nor resonance (Fig. 6(b)).

For $g_1=-16\pi^2/L_4$, the bottom of the "cusp" touches the abscissa which represents the zero of $D_1(\gamma)$ (Fig. 6(c)). This case corresponds to a zero energy bound state (or resonance).

For $g_1<-16\pi^2/L_4$, $D_1(\gamma)$ vanishes at two points, one below the threshold ($\gamma_1$) and one above the threshold ($\gamma_1'$) (Fig.6(d)). The points $\gamma_1$ and $\gamma_1'$ can be considered to correspond to a bound state and a resonance, respectively.

If we arbitrarily make $g_1$ negatively large we would have a negative $\gamma_1$, which is a kind of "ghost". In order to exclude such a ghost, which may be related with the lack of crossing symmetry in our model, we must assume a lower limit on $g_1$ as

$$g_1 \geq -\frac{16\pi^2}{L_4-2\sqrt{1-(1/h^2)}}. $$

Summarizing we have the following theorems:
**Theorem I**

The inequality

\[ \frac{1}{L_1 - 2\sqrt{1 - \left(\frac{1}{\lambda_1}\right)^2}} \leq \frac{g_1}{16\pi^2} \leq \frac{1}{L_1}, \]

must be satisfied if a bound state is to exist.

**Theorem II**

There exists either one bound state or no. If there exists one (no) bound state, we have necessarily one (no) resonance also. This is a kind of Levinson's theorem.\(^{14}\) If the binding energy tends to zero, then the resonance energy also tends to zero (a zero energy resonance).

We can further prove the following theorem from the monotonic decrease (increase) of \(-16\pi^2 A_1(\eta)\) at \(\eta > 1\) \((\eta < 1)\) (Eq. (3·2)).

**Theorem III**

\(\eta_1\)' has a maximum which is obtained when \(\eta_1\) is a minimum. This maximum, which depends on the cutoff, becomes largest for \(\lambda = \infty\).

We shall add a remark on the resonance here obtained. As a consequence of the inequality in Eq. (3·2), \(D_1(\eta)\) changes its sign from negative to positive as \(\eta\) increases through \(\eta_1\). This, combined with Eq. (2·9), leads to the conclusion that the (real) phase shift \(\delta_i\) goes through \(\pi/2\) decreasing. In examining more in detail we find that the cross section exhibits no sharp peak near the resonance energy. These facts are in agreement with what we know about an S-wave potential scattering.\(^{15}\)

(ii) **Unitarity condition**

Using the expression given by Eqs. (2·5) we can derive the following formulae:

\[
\begin{align*}
\text{Im } \mathcal{J}_1 &= B_1 |\mathcal{J}_1|^2 + B_2 |\mathcal{J}_2|^2, \\
\text{Im } \mathcal{J}_2 &= B_2 |\mathcal{J}_2|^2 + B_1 |\mathcal{J}_3|^2, \\
\text{Im } \mathcal{J}_3 &= B_1 \mathcal{J}_3 \mathcal{J}_1^* + B_2 \mathcal{J}_2 \mathcal{J}_3^*. 
\end{align*}
\]

These are combined to the equation

\[ \text{Im } \langle f \mid \mathcal{J} \mid i \rangle = \sum_{n=1,2} \langle f \mid \mathcal{J}^* \mid n \rangle B_n \langle n \mid \mathcal{J} \mid i \rangle, \]

which turns out to imply the unitarity condition for \(\mathcal{J}\), if we note that

\[ -B_n = k_n/8\pi W, \]

from Eqs. (2·4b) and (2·10). From Eqs. (3·5-1) and (3·5-2) we find

\[
\begin{align*}
\text{Im } \mathcal{J}_1 &\leq 0, \\
\text{Im } \mathcal{J}_2 &\leq 0,
\end{align*}
\]
in both the physical and unphysical regions.

(iii) **Scattering lengths**

We can prove that \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) as given by Eqs. (2·5) behave as

\[
\mathcal{J}_r(\xi) \approx \mathcal{J}_r(4M_r^2) \left\{ \begin{array}{ll}
1 + \frac{1}{16\pi} \frac{k_r}{M_r} \mathcal{J}_r(4M_r^2) & , \text{for } \xi \geq 4M_r^2, \\
1 - \frac{1}{16\pi} \frac{k_r}{M_r} \mathcal{J}_r(4M_r^2) & , \text{for } \xi \leq 4M_r^2.
\end{array} \right.
\]

This equation leads to the introduction of a scattering length defined by

\[ a_r = \frac{1}{16\pi M_r} \mathcal{J}_r(4M_r^2), \quad (3.8) \]

in terms of which we can write

\[
\mathcal{J}_r(\xi) \approx 16\pi M_r a_r \left\{ \begin{array}{ll}
\frac{1}{1 + ik_r a_r}, & \text{for } \xi \geq 4M_r^2, \\
\frac{1}{1 - k_r a_r}, & \text{for } \xi \leq 4M_r^2.
\end{array} \right.
\]

(3.9)*

When the two channels are coupled \( \mathcal{J}_r \) is in general complex, therefore \( a_r \) is also complex and may be decomposed as

\[ a_r = b_r + ic_r. \]

The imaginary part \( c_r \) is negative from (3·6). The behavior of \( \mathcal{J}_r \) near the threshold, given by (3·9), is schematically illustrated in Fig. 7.

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*This definition of \( a \) corresponds to

\[ k \cot \delta = -\frac{1}{a} + \ldots. \]
(iv) **Renormalization**

In the simple chain case the renormalization procedure is easy. Namely we can write

\[ \mathcal{D}_r(\tilde{\xi}) = 1 - g_r R_r(\tilde{\xi}) = D_r(4M_r^2) - g_r [R_r(\tilde{\xi}) - A_r(4M_r^2)], \]

then we have

\[ T_r(\tilde{\xi}) = \frac{\tilde{\alpha}_r}{1 - \tilde{\alpha}_r R_r(\tilde{\xi})}, \]

where

\[ \tilde{\alpha}_r = \frac{g_r}{D_r(4M_r^2)} = \frac{g_r}{1 + g_r L_r/16\pi^2}, \]

and the finite part of \( R_r \) is given by

\[ R_r(\tilde{\xi}) = R_r(\tilde{\xi}) - A_r(4M_r^2) = A_r(\tilde{\xi}) - A_r(4M_r^2) + iB_r(\tilde{\xi}). \]

From Eq. (3·8) we find

\[ \tilde{\alpha}_r = 16\pi M_r a_r. \]

In the compound chain case the situation is not as simple. We can prove that the divergence can be absorbed into two scattering lengths, \( b_1 \) and \( b_2 \), and that the behavior of the functions is insensitive to the cutoff. In the following, however, we shall not try to apply the renormalization procedure consistently. It is worthwhile to observe that a cutoff gives some kind of inverse length as seen from Eq. (3·11).

§ 4. **Special cases**

In this section we shall consider several special cases which serve to investigate the general features of the \( Y^* \) problem. Numerical values, i.e. mass ratios, cutoff momenta, binding energies, etc., are chosen for the sake of easiness of computation only. The parameters used are summarized in Table I.

It is convenient to divide the energy region into three parts:

- **Region I**: \( \xi \leq 4M_2^2 \), or \( \eta \leq \eta_0 \),
- **Region II**: \( 4M_2^2 \leq \xi \leq 4M_1^2 \), or \( \eta_0 \leq \eta \leq 1 \),
- **Region III**: \( 4M_1^2 \leq \xi \), or \( 1 \leq \eta \).

\( (\eta_0 = 4M_3^2/4M_2^2) \)

We shall mainly consider Region II. The behavior of the KN-KN scattering amplitude near and above the threshold (Region III) can be inferred from the scattering lengths by using Eq. (3·7). In Region II,

\[ B_1 = 0, \]
so that $\mathcal{A}$ and $\mathcal{B}$ in Eq. (3·7) can be simplified to

$$\mathcal{A} = D_1 D_2 - g_5^2 A_1 A_2,$$  \hspace{1cm} (4·1a)

and the unitarity condition (3·5) becomes

$$\text{Im } \mathcal{J}_1 = B_2 |\mathcal{J}_3|^2,$$

$$\text{Im } \mathcal{J}_2 = B_2 |\mathcal{J}_3|^2,$$

$$\text{Im } \mathcal{J}_3 = B_2 \mathcal{J}_3 \mathcal{J}_3^*.$$

Case 1. $g_5$ is negative and large such that there is a bound state of the KN system in Region II; $g_5=0$, and $g_5^2$ is small.

In this case Eqs. (4·1) can be simplified further to

$$\mathcal{A} = D_1 - g_5^2 A_1 A_2,$$  \hspace{1cm} (4·2a)

$$\mathcal{B} = g_5^2 A_1 B_2.$$  \hspace{1cm} (4·2b)
$g_3^2 \ll |g_1|$, we have the approximate forms for $\mathcal{A}$ and $\mathcal{B}$ near $\eta_1$:

$$\mathcal{A} \approx -\alpha (\eta - \eta_1^*)$$

$$\mathcal{B} \approx \frac{g_3^2}{g_1} B_2(\eta_1) \sim -\frac{1}{16\pi} \frac{g_3^2}{g_1},$$

where

$$\alpha \approx \theta \frac{-\tilde{b}_1}{1 - \eta_1} > 0, \quad (0 < \theta < 1)$$

is the slope of $\mathcal{A}$ near $\eta_1$ and is insensitive to $g_3^2$. From Eqs. (4.3) we have a one-level formula for $\mathcal{J}_3$:

$$\mathcal{J}_3 \approx -\frac{g_3}{\alpha} \frac{1}{\eta - \eta_1^* + i\gamma/2},$$

where the half-width $\gamma/2$ is given by

$$\frac{\gamma}{2} \approx -\frac{1}{16\pi \alpha} \frac{g_3^2}{g_1} > 0.$$

Fig. 8. Example 1. $-\text{Im } \mathcal{J}_1$, $-\text{Im } \mathcal{J}_2$ and $|\mathcal{J}_3|^2$ exhibit a sharp peak. The curve $-1/B_2$ represents the kinematical upper bound for $-\text{Im } \mathcal{J}_2$. The behavior of $\text{Re } \mathcal{J}_4$ shows that $\tilde{a}_2$ goes through $\pi/2$ increasing as $\eta$ increases through $\eta_1^*$. 
Both the width $\Gamma$ and the difference $\eta_1 - \eta_1^*$ are of order $g_3^2$. We have further

$$|\mathcal{J}_3|^2 = \frac{1}{B_2} \text{Im} \mathcal{J}_1 \approx \frac{g_3^2}{\alpha^2} \frac{1}{(\eta - \eta_1^*)^2 + \Gamma^2/4},$$

$$\mathcal{J}_2 = g_3 A_1 \mathcal{J}_3 \approx - \frac{g_3^2}{g_3^4} \frac{1}{\eta - \eta_1^* + i\Gamma/2}$$

An example is shown in Fig. 8. The following points should be noted:

(i) The second term of $\mathcal{O}$ given by Eq. (4.2a) makes the zero point $\eta_1$ of $D_1$ shift to the left ($\eta_1^*$). This means that the $g_3$-interaction is some kind of attractive force. The difference $\eta_1 - \eta_1^*$ is, of course, of order $g_3^2$.

(ii) The coupling between two channels (the $g_3$-interaction) produces a width of order $g_3^2$. The relation between the width and the height of the peak is of special interest. In the limit where $g_3^2$ tends to an infinitesimal, we have

$$\text{Im} \mathcal{J}_1 = B_1 |\mathcal{J}_3|^2 \sim -\delta(\eta - \eta_1).$$

We can naturally expect that $\text{Im} \mathcal{J}_1$ tends to a $\delta$-function, since a stable $Y^*$ appears if the coupling is switched off leaving $\eta_1$ unchanged. In this case $\mathcal{J}_1$ contains a term like

$$\frac{1}{\bar{\xi} - M_0^2 + i\epsilon},$$

where $M_0^2$ is the square of the $Y^*$ mass.

On the other hand, it seems strange that $|\mathcal{J}_3|^2$ also contains a $\delta$-function, since the reaction $K+N \rightarrow \pi + \Lambda$ will not occur if the coupling is switched off. But we must include a stable $Y^*$ into the complete set of states for $g_3 = 0$.

The unitarity condition (3.5-1) should be modified to

$$\text{Im} \mathcal{J}_1 = -\pi g_3^2 \delta(\bar{\xi} - M_0^2) + B_1 |\mathcal{J}_1|^2,$$

where $g_3$ is the "coupling constant" between $K$, $N$ and $Y^*$. The attenuation of the $KN$ beam ($-\text{Im} \mathcal{J}_1$) below the threshold now results from the reaction

$K+N \rightarrow Y^*$.

In our model a $Y^*$ is detected either by observing a peak in the energy spectrum of $\pi$ which has been produced together with $Y^*$, or by observing $\pi$ and $\Lambda$ which have decayed from $Y^*$. The latter observation is replaced by the observation of a stable $Y^*$ if the $g_3$-interaction is switched off. We can consider, therefore, the right-hand side of Eq. (3.5-1) or Eq. (4.9) as representing the squared amplitude for detecting a $Y^*$ in the final state, irrespective of

---

* See the paper by T. Goto for the proof that the complete set of states can be formed from the scattering states alone, not including the unstable particle state.

** Of course other particles, e.g., pions, must be emitted in order that this reaction actually occurs.
whether \( g_3^2 \) is zero or not. Then we can easily understand that the integral of the squared amplitude does not depend on \( g_3^2 \), because we should detect "one" \( Y^* \) in any way, if "one" \( Y^* \) is produced at all.

We can make another interpretation. The height of the peak of \(|\mathcal{J}_2|^2\) is \( \sim g_3^2/l^2 \sim 1/l^1 \), which, multiplied by \( l^1 \), has given a value independent of \( g_3^2 \). The width \( l^1 \) represents a probability of \( Y^*-\)decay per unit time, while the height \( (\sim 1/l^1) \) represents a duration of a \( Y^* \). The total probability, integrated with respect to time, is a product of the above two quantities, and then independent of \( l^1 \).

(iii) \(-\text{Im} \mathcal{J}_2\) exhibits also a peak, but its height is bounded by a factor which is purely kinematical and independent of \( g'\)'s and the cutoff. In fact we have

\[
-\text{Im} \mathcal{J}_2(\eta_1^*) = -\frac{1}{B_4(\eta_1^*)} = 8\pi \frac{W}{k_2} |_{\eta_1^*},
\]

which turns out to be equal to the kinematical upper limit in an \( S\)-state scattering. In examining \( \text{Re} \mathcal{J}_2 \) we find that the phase shift of \( \pi A-\pi A \) scattering goes through \( \pi/2 \) increasing as \( \eta \) increases through \( \eta_1^* \).

(iv) We find

\[ b_1 > 0, \quad b_1^2 > c_1^2. \]

Therefore, \(-\text{Im} \mathcal{J}_1\) exhibits a "cusp" at the threshold of the \( KN \) channel (see Fig. 7).

Case 2. \( g_1 \) is positive, \( g_3=0 \), and \( g_3^2 \) is small.

In the single-channel scattering we have the following three cases from Eqs. (3·11) and (3·13):

\[ b_1 > 0 \quad \text{for} \quad g_1 > 0, \]

\[ b_1 < 0 \quad \text{for} \quad -16\pi^2/L_A < g_1 < 0, \]

\[ b_1 > 0 \quad \text{for} \quad g_1 < -16\pi^2/L_A. \]

For each case \( D_1(\tilde{\sigma}) \) has already been plotted in Fig. 6. A positive \( b_1 \) implies that the absolute value of \( D_1(\tilde{\sigma}) \) decreases when \( \tilde{\sigma} \) decreases from \( 4M_1^2 \). Therefore, we cannot necessarily conclude that there is a bound state only because \( b_1 \) is positive.

The situation will be similar if a weak channel coupling is taken into account. In Fig. 9, we shall show an example in

![Fig. 9. Example 2. \(-\text{Im} \mathcal{J}_1\times10^6\) exhibits no sharp peak in spite of the cusp-type behavior at the threshold.](https://academic.oup.com/ptp/article-abstract/26/3/391/1943518)
which $g_1 > 0$. The inequality $b_1^2 > a_1^2$ is still satisfied and $-\text{Im} \mathcal{J}_1$ exhibits a cusp at the threshold. But $\mathcal{A}$ never vanishes in Regions I and II, and consequently neither $-\text{Im} \mathcal{J}_1$ nor $|\mathcal{J}_3|^2$ exhibits any sharp peak. On extrapolating these quantities by the rigid use of the scattering length approximation (see §5, (v)), we obtain a peak in the space-like region of $\xi$.

**Case 3.** $g_1 = g_2 = 0, g_3^2 \neq 0$

As noted in Case 1, (i), the $g_3$-interaction is an attractive one. Thus we may expect the possibility that the $g_3$-interaction causes both the formation of a bound state and its decay into $\pi$ and $\Lambda$. We find, however, that the width is essentially broad in this case. Examples are shown in Fig. 10. Furthermore, $\mathcal{A}(\eta)$ is found not to behave as simply as in Case 1. In the present example, $\mathcal{A}(\eta)$ behaves somewhat like a parabola, and two peaks appear.

Now we shall take into account the effect of the $g_2$-interaction. If $g_2$ is very small its effect is simple. A small positive $g_2$ shifts $\eta_1$ to the right and a small negative $g_2$ to the left. If $g_2$ is of the comparable order of magnitude as $g_1$, however, the behaviors of the amplitudes are so complicated that general treatments are rather difficult. In the following we shall, at first, consider the case in which only $g_2$ and $g_3$ are effective, neglecting $g_1$, and subsequently con-

![Fig. 10. Example 3. $\mathcal{A}(\eta)$ behaves like a parabola and much smaller than $\mathcal{B}(\eta)$. $|\mathcal{J}_3|^2$ exhibits no sharp peak.](https://academic.oup.com/ptp/article-abstract/26/3/391/1943518/10)
Consider some interesting cases in which $g_1$ and $g_2$ are of the comparable order of magnitude.

**Case 4.** $g_1 = 0$, $g_2 > 0$, $g_2^2 \neq 0$

We are interested in this case because there is some evidence that the $K$-meson interaction (here the $g_1$-interaction) is weaker than the pion interaction (here the $g_2$-interaction). We shall consider only the case of $g_2 > 0$, since a negative $g_2$ turns out not to produce any resonance-like behavior.

We have

$$\mathcal{A} = D_2 - g_2^3 A_1 A_2,$$

$$\mathcal{B} = (g_2 + g_2^2 A_1) B_2.$$  \hspace{1cm} (4·11a)

(4·11b)

It should be noted that $\mathcal{B}$ is no longer of definite sign. In fact it is possible for $\mathcal{B}(\eta)$ to change sign in Region II, if $g_2$ is positive and large enough. This is illustrated schematically in Fig. 11, where $\eta_4$ is the zero point of $\mathcal{B}(\eta)$ given by

$$A_1(\eta_4) = -g_2/g_2^2.$$  \hspace{1cm} (4·12)

If the slope of $\mathcal{B}(\eta)$ near $\eta_4$ is large enough, $\mathcal{B}(\eta)$ can be approximated by

$$\mathcal{B}(\eta) \approx \beta (\eta - \eta_4),$$  \hspace{1cm} (4·13b)

where

$$\beta \sim -\frac{1}{16\pi} g_2^2 \frac{dA_1}{d\eta} \bigg|_{\eta=\eta_4} \sim -\frac{1}{64\pi^2} \frac{g_2^3}{4\pi} L_3 > 0.$$  \hspace{1cm} (4·14)

Also we have

$$\mathcal{A}(\eta) \sim \mathcal{A}(\eta) = 1.$$  \hspace{1cm} (4·13a)

The latter equality is obtained by substituting Eq. (4·12) in Eq. (4·11a). From Eqs. (4·13a) and (4·13b) we have

$$G \sim i \frac{g_1}{\eta - \eta_4 + i\Gamma/2},$$  \hspace{1cm} (4·15)

where $\Gamma$ is given by

$$\Gamma/2 = 1/\beta > 0.$$  \hspace{1cm} (4·16)

The right-hand side of Eq. (4·15) is of a form of the one-level formula, with the real and imaginary parts interchanged in the denominator. This can also be considered to be a kind of resonance, which we shall call an *imaginary resonance*.

We have
and we can expect that $|\mathcal{J}_2|^2$, and consequently $-\text{Im} \mathcal{J}_1$, exhibit a peak.

On the other hand we have

\[
\mathcal{J}_2 \approx i \frac{1}{B_2(\eta_4)} \frac{\eta - \eta_4}{\eta - \eta_4 + i \Gamma/2},
\]

\[
\text{Im} \mathcal{J}_2 \approx \frac{1}{B_2(\eta_4)} \frac{(\eta - \eta_4)^3}{(\eta - \eta_4)^2 + \Gamma^2/4}.
\]

It is noted that $-\text{Im} \mathcal{J}_2$ exhibits a "dip" at $\eta = \eta_4$ instead of a peak. At this energy we find

\[
\delta_\eta = 0.
\]

Now we shall examine more specifically whether we can really expect a sharp resonance in the present case. From Eq. (4.16) and Eq. (4.14) we must assume a large $g_\eta^2$ in order to have a small $\Gamma$. This leads necessarily to a large $g_\eta$ from Eq. (4.12). These large constants cause $\mathcal{A}(\eta)$ also to have a zero point (sometimes more than one) near $\eta_4$ so that a simple one-level formula as in Eq. (4.15) is no longer valid. Moreover we find

---

**Fig. 12.** Example 4. An imaginary resonance occurs at $\eta_4$ where $\mathcal{I}^2(\eta_4) = 0$. $-\text{Im} \mathcal{J}_2$ exhibits a dip at $\eta_4$. But the behaviors of various quantities are too complicated to apply simple one-level formulae. Consequently there occurs no sharp peak.
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unless very extreme values are chosen for the parameters. These facts make the problem so complicated that we can hardly expect a sharp peak as simply as in Case 1. An example is shown in Fig. 12.

Case 5. Case 1 is modified by assuming a large negative \( g_2 \) such that there is a bound state in the \( \pi A \) system.

Eq. (4·1b) can be put into the form

\[
\mathcal{B} = (g_2 D_1 + g_3^2 A_1) B_2 = g_2 \left( D_1 + \frac{g_3^2}{g_2} A_1 \right) B_2, \tag{4·19}
\]

which shows that an imaginary resonance can occur irrespective of the sign of \( g_2 \), if \( D_1 \) changes sign in the region considered.

The energy \( \eta_1^* \) of a bound state of \( \pi A \) system, which we may identify as a \( \Sigma \), is determined by

\[
D_1 - g_3^2 (A_1 A_2 / D_1) = 0, \tag{4·20}
\]

if \( D_1 \) does not vanish in Region I. In Case 1 the second term on the right-hand side of Eq. (4·20) is negative because \( D_1 \) is positive in Region I. Therefore, the energy of \( \Sigma \), denoted by \( \eta_1^* \), shifts to the left of \( \eta_2 \), the zero point of \( D_2(\eta) \).

We shall consider three examples in which \( \eta_1^* \), another zero point of \( D_2(\eta) \) which corresponds to a resonance in \( \pi A \) scattering, lies at \( \eta_2 \), 1, and to the right of 1.

In Example 5.1, in which \( \eta_2 = \eta_1^* = \eta_0 \),

\( g_2 \) is given by

\[
g_2/16\pi^2 = -1/L_2. \tag{4·21}
\]

Even with this condition (zero binding energy), there is a bound \( \Sigma \) because \( \eta_2^* < \eta_2 \). In this case the product \( D_1 D_2 \) is small in Region II and

\[
|D_1 D_2| \ll g_3^2 |A_1|
\]

can be satisfied unless \( g_3^2 \) is extremely small. If the above inequality is satisfied we can approximate \( \mathcal{A}(\eta) \) as

\[
\mathcal{A} \approx -g_3^2 A_1 A_2, \tag{4·22a}
\]

which varies slowly except near both edges of Region II. On the other hand we can approximate \( \mathcal{B} \), given by (4·19), as

\[
\mathcal{B} \approx a'(\eta - \eta_1^{**}), \tag{4·22b}
\]

where

\[
\eta_1^{**} - \eta_1 = +0(g_3^2/|g_2|),
\]
and
\[ \alpha' \approx g_2 B_2 \alpha - (g_2/16\pi)\alpha > 0. \]  
(4.23)

(\( \alpha \) is given by Eq. (4.4).)

From
\[ A_1(\eta_1) = 1/g_1, \]
and Eq. (4.22a) we have
\[ A_1(\eta_1^*) \approx A_1(\eta_1) = -(g_2^2/g_1) A_2(\eta_1). \]  
(4.24)

Thus we have
\[ \mathcal{S} \approx -i\alpha'(\eta - \eta_1^*) - i\Gamma/2, \]  
(4.25)

which exhibits an imaginary resonance with a half-width
\[ \Gamma/2 = (g_2^2/\alpha' g_1) A_2(\eta_1). \]  
(4.26)

\[ |\mathcal{S}_3|^2 \] takes the form
\[ |\mathcal{S}_3|^2 \approx \frac{g_2^2}{\alpha^2} \frac{1}{(\eta - \eta_1^*)^2 + \Gamma^2/4}. \]  
(4.27)

---

Fig. 13. Example 5-1. Both \( |\mathcal{S}_3|^2 \) and \(-\text{Im}\mathcal{S}_1\) exhibit a sharp peak, while \(-\text{Im}\mathcal{S}_2\) exhibits a dip. \( |\mathcal{S}_3|^2 \) with \( g_2 = 0 \) (\( |\mathcal{S}_3(0)| \)) is also plotted for the sake of comparison. Peaks at \( \eta = \eta_0 \) come from the zero energy resonance in \( \pi A - \pi A \) scattering with \( g_2 = 0 \).
The relation between the width and the height of the peak is again similar to that of a δ-function if \( g_z^2 \) tends to an infinitesimal.

An example is shown in Fig. 13. In the Figure, we have also plotted \( |\mathcal{S}_z^{(0)}|^2 \), with \( g_z = 0 \). The half-width \( \Gamma/2 \) of \( |\mathcal{S}_z^{(0)}|^2 \) calculated from Eq. (4·6) is \( 2.56 \times 10^{-4} \). Comparing this with \( \Gamma/2 \) of \( |\mathcal{S}_z|^2 \), about \( \sim 0.015 \), we find that the \( g_z \)-interaction here considered makes the peak appreciably broader. This feature is common to the three examples in Case 5. \(-\text{Im } \mathcal{S}_z \) exhibits a dip at \( \eta^{**} \).

We see that \( |\mathcal{S}_z|^2 \) exhibits another peak at \( \eta = \eta^0 \). This comes from the fact that \( A(\eta) \) becomes a minimum due to the vanishing of the first term of the right-hand side of Eq. (4·1a). This peak, however, will hardly be observed in the actual processes, e.g. \( K^- + p \rightarrow A + \pi + \pi \), since this peak appears at a threshold and a vanishing phase volume of the final products should be multiplied.

Example 5-2, in which
\[
\eta_2' = 1,
\]
is the next typical example. This example is, however, found to be very similar to Example 5-1 except that there is no peak at \( \eta = \eta^0 \). An example is shown in Fig. 14. At \( \eta \sim 1 \), \(-\text{Im } \mathcal{S}_z \) is large (close to the kinematical upper bound)

Fig. 14. Example 5-2. Curves are similar to those in Fig. 13.
In Example 5-3, parameters were chosen in order that

\[ \eta_1' \gg 1. \]

In our model, this condition is, at the same time, apt to give a very light \( \Sigma \). This is connected to the restriction stated in Theorem III in §3.

There appears again an imaginary resonance similar to those in Examples 5-1 and 5-2. It is noted that a conventional resonance appears also. Namely Eq. (4.1a) can be written as

\[ \omega_0 = D_3 \left( D_1 - g_3 A_1 A_2 \right), \quad (4.28) \]

since \( D_3 \) does not vanish (and is negative) in Region II. Eq. (4.28) shows that \( \eta_1^* \) lies to the right of \( \eta_1 \), the difference being proportional to \( g_3^2 \). In the curve in Fig. 15, the two peaks at \( \eta_1^* \) and \( \eta_1^{**} \), respectively, overlap and form a peak much broader than that with \( g_3 = 0 \).

- \( \text{Im} \mathcal{J}_2 \) exhibits both a dip at \( \eta_1^{**} \) and a peak at \( \eta_1^* \).

![Fig. 15. Example 5-3. There occur both a conventional and an imaginary resonances. Two peaks each in \(|\mathcal{J}_2|^2\) and \(-\text{Im} \mathcal{J}_1\) overlap with each other to form a single peak.](image)
Case 6. Case 1 is modified by a large positive $g_b$.

We shall choose $g_2$ large enough to give a zero point of $B$ in Region II. Eq. (4.19) shows that $\gamma_{1}^{**}$ lies to the left of $\gamma_1$, the difference being proportional to $g_2^2/g_2$. Also a conventional resonance occurs at $\gamma_{1}^{*}$, which is shifted to the left of $\gamma_{1}^{*}$ by a distance proportional to $g_2^2$. At these energies we find

$$\mathcal{A}(\gamma_{1}^{*}) = 0,$$

$$B(\gamma_{1}^{*}) = \frac{g_2^2}{g_1+hA_2}B_2 = \frac{g_2^2}{g_1A_2B_2}B_2,$$

and

$$\mathcal{A}(\gamma_{1}^{**}) = \frac{g_2^2}{h},$$

$$B(\gamma_{1}^{**}) = 0.$$

We note that $D_2$ is positive and "large" (i.e. not so small as in the $g_2<0$ case) for $g_2>0$. This has the two effects of increasing the slopes of $\mathcal{A}(\gamma)$ at $\gamma=\gamma_{1}^{*}$ as seen from Eq. (4.28), and of reducing $B(\gamma_{1}^{*})$ as seen from Eqs.

![Graph](https://example.com/graph.png)

**Fig. 16.** Example 6. There occur both a conventional and an imaginary resonances. The peak at $\gamma=\gamma_{1}^{*}$ each in $|s_{2}|^2$ and $-\text{Im} s_{2}$ is much more prominent than that at $\gamma=\gamma_{1}^{**}$ and there results practically a single peak. The scale of abscissa has been enlarged because peaks are very narrow.
The combination of these two effects produces a very sharp peak in $|\mathcal{S}|^{-2}$ at $\gamma = \gamma_1^*$. An example, shown in Fig. 16, really exhibits such a sharp peak. There occurs a sharp peak also at $\gamma_1^{**}$ (an imaginary resonance). In the present example the peak at $\gamma_1^*$ (a conventional resonance) is much more prominent than the peak at $\gamma_1^{**}$, and there results practically a single peak. The half-width of the resultant peak is estimated as $\sim 0.75 \times 10^{-4}$, about one-third of that in $|\mathcal{J}_3^{(0)}|^2$.

Also $-\text{Im } \mathcal{J}_2$ exhibits a very sharp peak, with a dip closely adjacent.

§ 5. Summary and concluding discussions

(i) The cases considered in § 4 can be classified into two groups.

A. Case 1, Case 5, Case 6

In these cases we have assumed a strong attractive force between $K$ and $N$ such that there is a bound state in Region II, and a weak coupling between two channels. We have shown that one or two sharp peaks can occur in $|\mathcal{J}_3|^2$, $-\text{Im } \mathcal{J}_1$, and $-\text{Im } \mathcal{J}_2$, for $\pi A$ interactions varying over a fairly wide range. A peak occurs either as a result of a conventional or an imaginary resonance, or both.

B. Case 2, Case 3, Case 4

In these cases we have failed to present examples in which there occurs a sharp peak. This situation seems to be valid more generally.

We can, thus, conclude that the existence of a bound state in the $K N$ system and a weak coupling between two channels are essential for a sharp peak (or peaks) in the various amplitudes. We can understand this very naturally:

Let us consider the process

$$K^- + p \rightarrow A + \pi^+ + \pi^-.$$

We can divide this process into two steps, say,

$$K^- + p \rightarrow Y^{*+} + \pi^- \rightarrow A + \pi^+.$$

The initial $K^- p$ system falls into a bound state $Y^{*+}$ after the emission of a $\pi^-$. The $Y^*$ thus formed cannot decay into a $K^-$ and an $N$ from the law of energy conservation. It continues to exist until $K^-$ and $N$ bound in the $Y^*$ change into $\pi$ and $A$. If the probability of this change is small, the lifetime of $Y^*$ is long, and we observe a narrow peak in the cross section.$^*$ The integrated value of the peak, which gives the total yield of the process, is

$^*$ This possibility was already suggested by G. Takeda.
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determined essentially by the probability of the first step and does not depend on that of the second step, i.e. whether \( Y^\ast \) is long lived or not. Copious production and long life, which seem to contradict each other at first sight, are, thus, ascribed to two distinct interactions giving rise to the formation and to the decay of a bound state, the former being strong and the latter weak.

(ii) In Case 5 and Case 6, we have found that an attractive \( \pi A \pi A \) force makes the peak broader and a repulsive one sharper. We could understand this effect, at least qualitatively, since an attractive (a repulsive) interaction in the final \( \pi A \) state will make the transition into that state more probable (unprobable). If the \( \pi A \pi A \) force turns out to be a repulsive one with an appropriate strength, a channel coupling not very weak would be allowed in order to reproduce a narrow peak.

(iii) We should be particularly interested in the conclusion that a large negative \( g_1 \) and a small \( g_3^2 \) are required in our model; \( g_1^2/g_3^2 \) amounts to \( 10^2 \sim 10^3 \), though these values depend possibly on the special choice of the numerical parameters and on the nature of a \( \pi A \pi A \) force. It seems strange that the \( K_N \) interaction is very strong, because there is some evidence that the \( K_N \) interaction is somewhat weaker than the \( \pi N \) interaction.\(^{17} \) We should note, however, that the two interactions, \( K_N \) and \( K_N \) interactions, are not necessarily identical with each other only from general principles, e.g. charge conjugation invariance, or charge independence.

We can, of course, calculate the amplitudes for \( K_N \) scattering from the graphs which are crossing symmetric to that in Fig. 1. But there is some doubt in applying our model to such graphs straightforwardly, since this model has been so invented as to be particularly suitable for the \( S \)-wave \( K_N \) interaction.\(^* \)

The interaction Hamiltonian (2.1) need not be considered to be a truly elementary one; rather it may be interpreted more reasonably as an effective Hamiltonian derived from more fundamental ones, e.g. an appropriate set of Yukawa interactions.

(iv) It may happen that the more correct inclusion of the \( \Sigma \)-particle would change the results appreciably. In particular, it may be troublesome if we are forced to assume a large \( g_5 \), the coupling constant between \( K_N \) channel and \( \pi \Sigma \) channel. In this connection it seems interesting that we have obtained the examples in which a \( \Sigma \) results as a bound state of the \( \pi A \) system (Case 5). If we can reproduce similar cases with spins, parities, iso-spins, and mass ratios being taken into account correctly, then we would have another merit.

Namely, \( \Sigma \), resulting as an \( S \)-state bound state of the \( \pi A \) system, has a parity opposite to that of \( A \). This leads to a conclusion that \( Y^\ast \) decays into \( \Sigma \) and \( \pi \) in a \( P \)-state, which will be consistent with the preliminary experi-

\(^* \) Crossing symmetric graphs cause scatterings in higher wave states as well as an \( S \)-state.
mental result that the ratio of \( Y^* \to \Sigma + \pi \) to \( Y^* \to \Lambda + \pi \) is very small \((\sim 0.08)\). \(^{1,2}\)

In any case it will be necessary to develop a three-channel model including the \( \pi \Sigma \) system for further investigations.

(v) We have listed the scattering lengths \( b_1 \) and \( c_1 \) in Table I. A direct comparison of these values with the realistic ones, e.g. Dalitz solution,\(^6\) may be meaningless, because of the crudeness of the model and the arbitrariness of parameters chosen, though they are found to give rather reasonable values and will provide some reference in more realistic considerations.\(^9\) Here, we shall only summarize the results very briefly without discussing any more in detail.

We see that \( b_1 \) in Case 1, Case 5 and Case 6 are not much different from \( b_1^{(0)} \), with \( g_2 = 0 \), while \( c_1 \) varies to some extent. In particular, \( c_1 \) in Example 6 is very small. This is evidently due to the repulsive \( \pi \Lambda \) interaction (a positive \( g_2 \) makes \( D_2 \) large and gives a large \( \mathcal{C} \)) and will be valid generally.

If the “scattering length approximation” is valid, \( \text{Im} \mathcal{J}_1 \) below the threshold can be written as

\[
-\text{Im} \mathcal{J}_1 \simeq 16\pi M_1 \frac{-c_1}{1-2\kappa_1 b_1 + \kappa_1^2 |a_1|^2},
\]

from the second equation in Eqs. (3·9). We can calculate \( \kappa_i \), at which the right-hand side of Eq. (5·1) becomes maximum, as

\[
\kappa_i = \frac{b_i}{b_i^2 + c_i^2},
\]

and correspondingly we have

\[
\bar{\gamma}_1 = 1 - \frac{b_1^2}{M_1^2 (b_1^2 + c_1^2)^2},
\]

which are also listed in Table I. We find that they give positions of the peaks not inconsistent with the correct ones, at least in Case 1, Case 5 and Case 6, but are far from giving information about detailed features of the resonance, e.g. whether it is a conventional resonance or an imaginary one, whether the peaks, if there are two, overlap or not, and so on.

(vi) There is reported also an existence of \( K^* \), which can be considered to be an excited state in the \( \bar{\kappa}\pi \) system.\(^2\) The width of the observed peak is again so small \((\sim 10 \text{ Mev})\) that we cannot reproduce it in a \( P \)-state chain approximation applied to the \( K\pi \) system.\(^8\) Moreover there are no appropriate two-particle system in which a bound state can appear and be identified as \( K^* \), in analogy with the case of \( Y^* \) (the mass of \( K^* \) is \( 884 \text{ Mev} \)). Therefore, the introduction of \( K^* \) as a new elementary particle may be required.

*Note that \( \tilde{a} \) is a dimensionless quantity. In order to obtain a true scattering length \( a \) in unit of, say, the pion Compton wavelength, we must divide \( \tilde{a} \) by a factor \( 8\pi (2M_1/M_\pi) \), which is about 258, if we replace \( 2M_1 \) by \( M_\Lambda + M_K \) tentatively.
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