Macroscopic Causality and Lower Limit for the Energy Derivative of the Scattering Phase Shift

—Relativistic Case—

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A relation between the macroscopic causality and lower bound of the energy derivative of the scattering phase shift is studied in the case of relativistic quantum field theory. From the requirement of the macroscopic causality the energy derivative of the real part of the phase shift must be non-negative in such an energy region that the imaginary part of phase shift does not vary rapidly with the energy, whereas in another region such inequality is not generally valid.

§ 1. Introduction

In the case of one-channel and potential scattering problems it is well known that the existence of lower bound of the energy derivative of the scattering phase shift is, fundamentally, a consequence of the macroscopic causality condition.\(^1\),\(^2\) The aim of this paper is to examine whether such a relation holds also in the case of relativistic quantum field theory. In this case the scattering phase shift must be dealt generally as a complex quantity owing to the existence of multi-channel scattering and this is a main feature different from the case of one-channel scattering.

At first one may suppose that our purpose can be achieved by making use of the analyticity properties of the scattering amplitudes. However, it is impossible to do so, because the analyticity properties of the scattering amplitudes cannot generally be derived from the requirement of macroscopic causality (see the last paragraph in § 4 of reference 3)). Therefore we shall adopt another course of study.

In order to require the macroscopic causality the scattering problem will be studied by means of the wave-packet-formalism.\(^3\) Here the “macroscopic causality” is formulated in such a way that the scattered wave cannot emerge before the incident wave arrives at the scatterer provided that the extensions of wave packets are neglected.

§ 2. Transition matrix

We consider the elastic scattering of the particles with mass \(M\) and mass \(\mu\). Denote the initial average momentum and average position of the particle
with mass $M$ by $p_1$ and $x_1$, and those of the particle with mass $\mu$ by $p_2$ and $x_2$, then the initial state of the scattering can be represented in terms of the outgoing solution of the Schrödinger equation, $|k_1, M; k_2, \mu\rangle^+$, as follows:  

$$
|p_1, x_1, M; p_2, x_2, \mu\rangle^+ = \int dk_1 dk_2 f_1(p_1, x_1, k_1) f_2(p_2, x_2, k_2) |k_1, M; k_2, \mu\rangle^+
$$

(1)

with

$$
F(p, x, k) = \exp(-ik \cdot x) f(p, k),
$$

(2)

where $F(p, x, k)$ is the wave function describing the wave packet and $f(p, k)$ is assumed to be nearly equal to zero except for $|k - p| \lesssim |\Delta p|$, $\Delta p$ being the uncertainty of the momentum of the wave packet. Here the spatial separation $|x_1 - x_2|$ has also been assumed to be sufficiently larger than the extension of the wave packet, i.e. $|x_1 - x_2| \gg |\Delta r|$, $\Delta r$ being the uncertainty of the position of the packet. Of course, $\Delta p \cdot \Delta r \approx 1$ holds by the uncertainty principle.

In a similar way the expression

$$
|p_3, x_3, M; p_4, x_4, \mu\rangle^- = \int dk_3 dk_4 f_3(p_3, x_3, k_3) f_4(p_4, x_4, k_4) |k_3, M; k_4, \mu\rangle^-
$$

(3)

is of the final state of the scattering, where $|k_3, M; k_4, \mu\rangle^-$ denotes the incoming solution of the Schrödinger equation.

Using the definitions (1) and (3) for the initial and final state of the scattering, we can now write the transition matrix for the collision:  

$$
\langle p_3, x_3, M; p_4, x_4, \mu| \exp(-i\mathcal{H}t)|p_1, x_1, M; p_2, x_2, \mu\rangle^+
$$

$$
= \left[ \int dk_1 dk_2 dk_3 dk_4 \exp\left[ -i \left( E_3(k_3) + E_4(k_4) - E_1(k_1) - E_2(k_2) \right) t \right] \right]
$$

$$
\times F_3^+(p_3, x_3, k_3) F_4^+(p_4, x_4, k_4) \langle k_3, M; k_4, \mu| k_1, M; k_2, \mu\rangle^+
$$

$$
\times F_1(p_1, x_1, k_1) F_2(p_2, x_2, k_2),
$$

(4)

where $\langle k_3, M; k_4, \mu| k_1, M; k_2, \mu\rangle^+$ is the so-called $S$-matrix and can be expanded in terms of the partial wave phase shift $\delta_i(W^2)$ in the following way:

$$
\langle k_3, M; k_4, \mu| k_1, M; k_2, \mu\rangle^+
$$

$$
= \delta^0(k_1 + k_2 - k_3 - k_4) \delta[E_1(k_1) + E_2(k_2) - E_3(k_3) - E_4(k_4)]
$$

$$
\times W/(\pi^2 K) \cdot \sum_l (2l+1) \exp[2i\delta_l(W^2)] P_l(cos \theta),
$$

(5)

where

$$
W^2 = [E_3(k_3) + E_4(k_4)]^2 - (k_3 + k_4)^2,
$$

$$
K^2 = [W^2 - (M^2 + \mu^2)][W^2 - (M^2 - \mu^2)]/4W^4,
$$

$$
cos \theta = 1 - 2\delta^0/K^2.
and
\[ 4\mathcal{F} = (k_3 - k_1)^2 - [E_3(k_3) - E_1(k_1)]^2. \]

§ 3. Condition of the scattering

Let us examine the conditions for \( p_1, x_1, p_2, x_2 \) and \( p_4, x_4, p_4, x_4 \) for getting non-vanishing transition matrix (4). We set \( \delta_i(W^2) = \delta_{iR}(W^2) + i\delta_{iI}(W^2) \) and introduce the Fourier transform of \( \exp[-2\delta_{iI}(W^2)] \) by
\[
\exp[-2\delta_{iI}(W^2)] = \int_{-\infty}^{\infty} d\alpha \exp(2i\alpha W^2) E_i(\alpha) \tag{6}
\]
with
\[
E_i'(\alpha) = E_i(-\alpha), \tag{6}'
\]
in order to take into account the effect of the variation of \( \exp[-2\delta_{iI}(W^2)] \). It should be noted that the Fourier transform \( E_i(\alpha) \) is not unique, since \( \exp[-2\delta_{iI}(W^2)] \) is defined only in the physical region \( W^2 \geq (\mu + \mu)^2 \).

In order that the transition matrix (4) does not vanish, the variation of the phase of the integrand (4) with respect to \( k_i \)'s must be equal to zero apart from the errors of the extensions of the packets, when \( k_i \)'s vary around \( p_i \)'s:
\[
\delta k_1 \cdot x_3 + \delta k_1 \cdot x_1 - \delta k_1 \cdot x_5 - \delta k_1 \cdot x_5 - \delta k_1 \cdot V_1(p_3) t - \delta k_1 \cdot V_1(p_1) \tag{7}
\]
where \( V(p) = p/E(p) \), \( \delta_{iI}(W^2) = \left[ \frac{d\delta_{iR}(W^2)/dW^2}{W^2 - \mu^2} \right] \) and \( W^2 = [E_5(p_5) + E_4(p_4)] \) in the centre-of-mass system \( (p_3 + p_4 = 0) \). Here we have neglected the effects of the variations of \( \cos \theta \) and \( W/K \) in (5). As is easily seen, this approximation becomes better at high energy \( W^2 \gg (\mu + \mu)^2 \). \( \delta k_i \)'s in (7) are restricted by virtue of the conservation law of the total energy momentum as follows:
\[
\delta k_1 + \delta k_3 - \delta k_4 = 0 \tag{8}
\]
and
\[
\delta k_1 \cdot V_1(p_3) + \delta k_3 \cdot V_1(p_1) - \delta k_1 \cdot V_3(p_3) - \delta k_1 \cdot V_3(p_1) = 0. \tag{9}
\]
Using the undeterminate multiplier \( T \), we get from (7), (8) and (9)
\[
x_3 - x_1 \approx -[V_2(p_2) - V_1(p_3)] T, \tag{10}
\]
\[
x_1 - x_3 \approx [V_4(p_4) - V_3(p_3)] [t - T - \beta W_2] \tag{11}
\]
with
\[
\beta = 4[\delta_{iI}'(W^2) + \alpha],
\]
where \( \approx \) means that the relations (10) and (11) hold if the extension of the wave packet is neglected which is of the order \( \Delta p \) or \( \Delta r \) and if the effects of \( \cos \theta \) and \( W/K \) in (5) is omitted.
The relations (10) and (11) represent the following fact that the incident particles which are initially localized at the positions $x_1$ and $x_2$ collide with each other at the time $T$, then the scattered particles emerge after the time $\beta W_p$ and finally arrive at the positions $x_3$ and $x_4$. (See Fig. 1.) The time interval $(0, T)$ stands for the initial state of the scattering and the time interval $(T + \beta W_p, t)$ stands for the final state of the scattering. The four-vector $PQ$ is time-like.

Fig. 1. Schematic representation of the relations (10) and (11).

§ 4. Condition of macroscopic causality and concluding remarks

The scattered waves cannot emerge before the incident waves collide with each other, so that $\beta W_p$ must be non-negative,\* i.e.

$$\delta_{ir}^{2}(W_p^2) + \alpha \geq 0,$$

provided that the extensions of the wave packets and the effects of $\cos \theta$ and $W/K$ in (5) are neglected. It is well valid at high energy $W_p^2 \gg (M + \mu)^2$.

The condition (12) implies that the transition matrix (4) cannot contain the contribution from $\alpha$ smaller than $-\delta_{ir}^{2}(W_p^2)$. This requirement is satisfied if we are able to make $E_i(\alpha)$ zero for $\alpha < -\delta_{ir}^{2}(W_p^2)$ by making use of non-uniqueness of $E_i(\alpha)$. This possibility, however, is not always assured. The above requirement is also satisfied if the contribution from $\alpha < -\delta_{ir}^{2}(W_p^2)$ vanishes after integration over $\alpha$ from $-\infty$ to $+\infty$, even though $E_i(\alpha)$ does not identically vanish for $\alpha < -\delta_{ir}^{2}(W_p^2)$.

(1) If the first possibility for $E_i(\alpha)$ is true, we have the following relation from (6), (6)' and (12):

$$\exp[-2\delta_{ir}^{2}(W_p^2)] \approx \int_{-\delta_{ir}^{2}(W_p^2)} \, da \, \exp(2i\alpha W_p^2) E_i(\alpha)$$

\* Suppose $\beta W_p$ is not smaller than $-t_0$ apart from the errors of the extensions of wave packets, where the extension of the interaction is given by $t_0[V(p_2) - V(p_1)]$. Then in another Lorentz frame the extension of the interaction becomes of the order $(t_0(1 - \beta^2))\Delta V$, $\Delta V$ being the relative velocity in the new Lorentz frame. If $\beta$ takes a sufficiently large value, this extension can be made measurable by a macroscopic observation whatever small $t_0$ may be. Therefore, in order that $(t_0(1 - \beta^2))\Delta V$ be always of microscopic order, $t_0$ must be zero. (See the last paragraph in § 6 in reference 3.)
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with

\[ E_i(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dW^2 \exp(-2i\alpha W^2) \exp[-2\delta_{ii}(W^2)], \]  

(14)

where \( \varphi = \delta_{ii}(W_p^2) \) must always be non-negative, as is easily seen. Inserting (14) into (13), we get a new type of relation between the real part and imaginary part of l-wave phase shift:

\[ \exp[-2\delta_{ii}(W_p^2)] \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} dW^2 \frac{\sin[2(W^2-W_p^2)\delta'_{II}(W_p^2)]}{W^2-W_p^2} \exp[-2\delta_{ii}(W^2)], \]  

(15)

which holds well at high energy \( W_p^2 \gg (M+\mu)^2 \). As is easily seen, the unphysical region \(- (M+\mu)^2 < W^2 < (M+\mu)^2 \) hardly contributes to the integration in (15), if we are concerned with the high energy \( W_p^2 \gg (M+\mu)^2 \). The region \( W^2 \leq -(M+\mu)^2 \) relates to the region \( W^2 \geq (M+\mu)^2 \) by virtue of the crossing symmetries. Note that \( \delta'_{II}(W_p^2) \) must always be non-negative.

(2) The second possibility for \( E_i(\alpha) \), however, may be a general case in which we cannot obtain any new condition about the complex phase shift from the requirement of the macroscopic causality.

(3) In a particular case in which the imaginary part of phase shift is constant within some energy region, that is, we need not take into account the effect of the variation of \( \exp[-2\delta_{ii}(W^2)] \), \( (\alpha=0) \), we have the following condition in place of (12):

\[ \delta'_{II}(W_p^2) \geq 0 \]

(16)

within such energy region. This inequality is well valid at high energy \( W_p^2 \gg (M+\mu)^2 \).

In the case of one-channel and potential scatterings Wigner showed the following inequality: \(^1\)

\[ \delta'_i(k) \geq -a, \]

(17)

where \( a \) is the radius of the potential. In the theory of potential scatterings, the causality may be violated by the range \( 2a \) of the potential at most. However, the violation by more than the range \( 2a \) is unacceptable. The inequality (17) is an immediate consequence of this fact. \(^2\)

In the covariant field theory, however, such an extension of the interaction, whatever small it may be, always leads to the measurable violation of the macroscopic causality owing to the Lorentz covariance (see the footnote). The inequality (16) is an immediate consequence of this requirement in some energy region.
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References