Nordic Hydrology 10, 1979, 261-270

Critical Examination of the Muskingum Method

Mohammad Akram Gill
Department of Civil Engineering, Ahmadu Bello University,
Zaria, Nigeria

The Muskingum method of flood routing is critically reviewed. The kinematic wave approach is used to compare and contrast the conclusions deducible from the conventional hydrological approach. The importance of the time lag concept is emphasised.

Introduction

The Muskingum method of flood routing has been continuously receiving attention from the researchers for several reasons. Firstly, it is a simple method which can be used for flood routing without much complication as far as the procedural details are concerned. Its parameters can be calculated using the records of past historical floods. There have been attempts for the evaluation of the Muskingum parameters using the method of kinematic wave which will be examined in some details herein later. The Muskingum method has been in the lime light for its shortcomings as well which in the author's opinion are more apparent than real. For instance, it is well known that the Muskingum method produces appreciably reduced and sometimes negative values of the outflow for some small values of initial time. This has been a sore point of the Muskingum method which tended to repel many users. There had been some attempts purporting to explain this unrealistic aspect and one explanation was also offered by the author recently (1979). This aspect will again be critically examined here.

Inspite of its apparent defects, the Muskingum method is quite popular because »... the method presents the notable advantage of not requiring a knowledge of the river bed geometry as the phenomenon can be reproduced well enough on the basis of the calibration carried out using experimental data relative to the extreme sections of not significantly long reaches«, as professed by Gallati and Maione (1977). According to them further, »there is perhaps another reason for the wide spread use for the Muskingum method. The method has two parameters which can be closely linked to the speed and subsidence of the wave crest and hence the method gives often good estimates of these two important propagative characteristics«.

The method will be examined herein from the conventional hydrological viewpoint and its similarities with the hydrokinematical wave will be pointed out.

Hydrological Linear Formulation

The problem of flood routing can be formulated with reference to Fig. 1. A length of river is bounded by sections 1 and 2 and no lateral inflow is assumed within this reach. The inflow hydrograph at section 1 is known and the corresponding outflow hydrograph at section 2 is required to be determined. Using the hydrological approach, continuity equation can be written as follows,

$$\frac{dS}{dt} = I - Q \tag{1}$$

where dS/dt = rate of change of stored water between sections 1 and 2, I = rate of inflow at section 1, Q = rate of outflow from section 2, S = storage between sections 1 and 2, and t = time. If another independent relationship could be developed connecting S, I, and Q, the flood routing problem should be easily solved. Such a relationship is provided by the Muskingum equation which can be written as

$$S = K(X\,I + (1-X)Q) \tag{2}$$

where X = weighing factor which is known to vary between 0.0 and 0.5, and K = storage coefficient which is the gradient of the straight line relating S with $(X\,I + (1\text{-}X)Q)$. The parameter X is dimensionless while K has the dimension of time. Eliminating S between Eqs. (1) and (2) gives

$$I - K\,X\frac{dI}{dt} = Q + K(1-X)\frac{dQ}{dt} \tag{3}$$

If the mathematical function describing the inflow hydrograph $I(t)$ is known, Eq. (3) can be directly integrated to give the outflow $Q(t)$. If on the other hand, only the discrete values of I are known at specified time instants, the corresponding outflow values can be determined using the following equations.

$$Q_{n\,\Delta t} \equiv C_0\,I_{n\,\Delta t} + C_1\,I_{(n-1)\,\Delta t} + C_2\,Q_{(n-1)\,\Delta t} \tag{4}$$

where

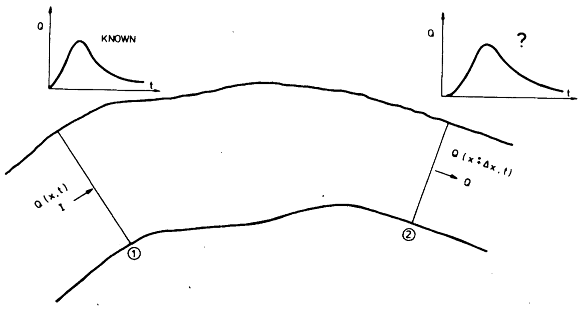

Fig. 1. Sketch of a hypothetical river reach.

$$C_0 = -\frac{K X - 0.5\,\Delta t}{K - K X + 0.5\,\Delta t} \tag{5}$$

$$C_1 = \frac{K X + 0.5\,\Delta t}{K - K X + 0.5\,\Delta t} \tag{6}$$

$$C_2 = \frac{K - K X - 0.5\,\Delta t}{K - K X + 0.5\,\Delta t} \tag{7}$$

$$C_0 + C_1 + C_2 = 1 \tag{8}$$

and Δt = routing interval.

Time Lag and Translatory Characteristics

In Eqs. (4) to (7), when $X = 0.5$ and $K = \Delta t$, $C_0 = C_2 = 0$, and $C_1 = 1$, with the result that

$$Q_{n\,\Delta t} \equiv I_{(n-1)\,\Delta t} \tag{9}$$

which characterizes a simple translation of a wave form in the river channel. This problem was meaningfully investigated by Kulandaiswamy (1966) and later in more details by the author (1979). It was observed that Eq. (9) is generally valid under certain restrictions imposed on the inflow hydrograph when $X = 0.5$. If no restrictions are to be imposed on the inflow hydrograph, the restriction $\Delta t = K$ should be interpreted to mean that K is a small time interval of the order of Δt. This restricts the length of the channel reach to only small magnitude through which the wave may not show any significant subsidence or distortion. It was shown in an earlier publication (Gill 1979) that if the inflow hydrograph is such that

$$\sum_{2}^{n} \frac{(-K X)^n}{n!} \frac{d^n I}{dt^n} = 0$$

solution of Eq.(3) is given by

$$Q(t + K(1-X)) = I(t - K X) \tag{10}$$

where the left hand side means Q at time $t+K$ (1-X) and similarly the left side means I at time $(t-K X)$. Obviously, Eq. (10) involves a certain time lag and describes a simple translation of the flood wave. Adding KX to the time values on both sides of Eq. (10) gives

$$Q(t + K) = I(t) \tag{11}$$

showing a time lag of K between the occurrence of I and Q values. Eq. (10) holds for all values of X. Kulandaiswamy (1966) showed that for $X = 0.5$ and

$$\sum_{3}^{n} \frac{(-K X)^n}{n!} \frac{d^n I}{dt^n} = 0$$

$$Q(t) = I(t-K) \tag{12}$$

which again is identical to Eqs. (10) and (11) showing a time lag of K. At $t=K$ in Eq. (12).

$$Q(K) = I(0) \tag{13}$$

which becomes the initial condition to be used in eliminating the constant of integration in the formal solution of Eq. (3), if the inflow hydrograph satisfies the requirements necessary for producing a translatory outflow hydrograph. The argument here will further be reinforced while examining the Muskingum method in conjunction with the kinematic wave. If however the inflow hydrograph is such that the requirements for translatory response are not satisfied, Eq. (13) will not apply; instead

$$Q(\tau) = I(0) \tag{14}$$

should be valid where τ = time lag which has its value between 0 and K. There is still no specific method available for determining τ ; method of trial and error may be handy in numerous situations. The useful conclusion arising from the analysis so far is that the use of the conventional initial condition namely

$$Q(0) = I(0) \tag{15}$$

may not always be correct as argued in the earlier publication (Gill 1979) and it is the use of Eq. (15) which indeed produces negative outflows for the time period $0<t<K$. Instead of Eq. (15), Eqs. (13) and (14) are suggested which obviate negative outflows. The time lag may however be zero in some special circumstances e.g. reservoir routing, where Eq. (15) may hold good.

Kinematic Wave Analogy

Derivation of kinematic wave equation is given by a number of investigators e.g. Lighthill and Whitham (1955), Cunge (1969), Dooge (1973), among many others. Following derivation is reproduced from Natale and Todini (1977). Ignoring the inertial terms in the St. Venant equations,

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \qquad (16)$$

and

$$\frac{\partial y}{\partial x} \equiv S_0 - S_f \qquad (17)$$

where $A \equiv$ cross sectional area $\equiv By$, $B \equiv$ width, y = depth, S_0 = bed slope, S_f = energy slope $\equiv n^2Q^2/A^2R^{4/3}$, $n \equiv$ Manning's coefficient of friction, R = hydraulic radius, and $x \equiv$ distance in the flow direction. For a wide river channel, $R \simeq y$. Substituting for A in Eq. (16), the continuity equation becomes

$$\frac{\partial y}{\partial t} \equiv -\frac{1}{B}\frac{\partial Q}{\partial x} \qquad (18)$$

Differentiate Eq. (17) with respect to time and using Eq. (18) gives

$$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} = D\frac{\partial^2 Q}{\partial x^2} \qquad (19)$$

where

$$c \equiv \frac{5Q}{3A} \qquad (20)$$

$$D = \frac{A^2 y^{\frac{4}{3}}}{2n^2\, QB} \qquad (21)$$

In Eq. (20), $c \equiv$ celerity of the flood wave and $D \equiv$ diffusion coefficient in Eq. (21). Eq. (19) is nonlinear and difficult to solve. Solution is however obtained e.g. Dooge (1973) for a linearised version of this equation in which Q and A in Eqs. (20) and (21) are replaced by some characteristic constant values Q_0 and A_0 respectively. If in the preceding equations, Chezy's formula was used instead of Manning's for eliminating S_f, the constant 5/3 in Eq. (20) would be replaced by 3/2. In Dooge's analysis, the inertial terms were not ignored with the result that

$$c \equiv \frac{3}{2}u_0 \qquad (22)$$

$$D \equiv \frac{u_0}{2gS_0}\left(gy - \frac{u_0^2}{4}\right) \qquad (23)$$

where u_0 = characteristic constant velocity, and y_0 = characteristic constant depth. Since Dooge used Chezy's equation, the celerity equals 1.5 u_0, Eq. (22). If the $u^2_0/4$ term within the bracket in the left side of Eq. (23) is ignored, Eq. (23) reduces to Eq. (21).

Replacing the inflow and outflow by $Q(x, t)$ and $Q(x + \Delta x,t)$, Fig. 1, and expanding the outflow by Taylor's theorem, Koussis (1978) obtained the following equation by manipulating Eqs. (1) and (2) and a continuity equation.

$$\frac{\partial Q}{\partial t} + \frac{\Delta x}{K}\frac{\partial Q}{\partial x} \equiv \left[\frac{(\Delta x)^2}{K}\left(\frac{1}{2} - X\right)\right]\frac{\partial^2 Q}{\partial x^2} \qquad (24)$$

Eq. (24) resembles Eq. (19). Comparing Eq. (24) with Eq. (19), the following values of the Muskingum parameters are obtained.

$$K \doteq \frac{3\,\Delta x}{5\,Q_0/A_0} \equiv \frac{3\,\Delta x}{5\,u_0} \tag{25}$$

$$X \equiv \frac{1}{2} - \frac{3\,y_0}{10\,S_0\Delta x} \equiv \frac{1}{2} - \frac{k\,Q_0}{2\,B S_0(\Delta x)^2} \tag{26}$$

If however Eqs. (22) and (23) are used in Eq. (19) and the result compared with Eq. (24), the following results are obtained.

$$K = \frac{\Delta x}{1.5\,u_0} \tag{27}$$

$$X = \frac{1}{2} - \frac{1}{3}\left(1 - \frac{u_0^2}{4\,g y_0}\right)\frac{y_0}{S_0\Delta x} \tag{28}$$

It should be noted that in obtaining Eq. (24), the terms containing derivatives of Q higher than the second were ignored. If the form of $Q(x,t)$ is such that third and higher derivatives have significant magnitudes, Eq. (24) will only be an approximate result. The restrictions under which Eq. (24) is valid are thus comparable with those which lead to Eqs. (10), (11), and (12). If it is stipulated for instance that

$$\sum_{2}^{n} \frac{(\Delta x)^n}{n!}\frac{\partial^n Q(x,t)}{\partial t^n} \equiv 0$$

the right side of Eq. (24) will vanish giving,

$$\frac{\partial Q}{\partial t} + \frac{\Delta x}{K}\frac{\partial Q}{\partial x} \equiv 0 \tag{29}$$

which is the equation of a translatory wave. The wave form moves at a speed of $\Delta x/K$ and there is a time lag of K units of time between two reference stations, for instance 1 and 2 in Fig.1. Also Eq. (29) is valid for all values of X since it does not depend in any way on the value of X. These are precisely the same conclusions as were obtained from the examination of the Muskingum equation from a hydrological viewpoint.

Now relax the restriction on the form of $Q(x,t)$ namely that

$$\sum_{3}^{n} \frac{(\Delta x)^n}{n!}\frac{\partial^n Q(n,t)}{\partial t^n} \equiv 0$$

which can be compared with the restriction imposed by Kulandaiswamy (1966) on the form of the inflow hydrograph. Assume further that $X = 0.5$, for which Eq. (24) again reduces to Eq. (29) and the preceding remarks in respect of translatory characteristics again hold good. It is thus seen that there are no contradictions between the conclusions arrived at from both approaches namely the so called hydrological approach and the hydrokinematic wave approach.

Although the kinematic approach appears to be elegant and algebraic expressions are obtained using which the evaluation of the Muskingum parameters is feasible, certain difficulties still remain. For instance, in the expressions for

K and X, reference values of Q_0, u_0, and y_0 are needed. The bothersome question is as to what values of these parameters should be used? The variation of Q and the associated variables in any flood routing situation is considerable. In the numerical computations of flood routing, some authors preferred to use the variable temporal values of Q and A for the evaluation of K and X. The parameters K and X are thus rendered functions of Q and other variables instead of being constant as assumed in the kinematic approach. The diffusion equation, Eq. (24), is valid essentially for constant values of K and X; if these parameters are to be treated as variables, a more complex equation would result instead of Eq. (24). Owing to these difficulties, sometimes unrealistic values (e.g. negative) of X are obtained.

The usual hydrological method of determining K and X is free from such faults. Historical flood data can be used for the evaluation of K and X. A method of least squares has been proposed by the author (1977) for computing K and X which eliminates the element of subjectivity involved in the traditional method of trial and error.

Should X = 0.5 Always Produce a Translatory Wave?

The question whether $X = 0.5$ should always produce a translatory wave has already been partially explored in an earlier publication (Gill 1979). The question is further discussed here using the argument of the kinematic wave. It is obvious as seen already that the parabolic diffusion equation, Eq. (24), is reduced to a simple translatory wave equation, Eq. (29), for $X = 0.5$. This leads to the conclusion that for $X = 0.5$, the wave is a translatory one but this conclusion is not always correct since it is subject to the restrictions in regards to third and higher derivatives of $Q(x,t)$. In order to emphasizs this point, let a typical inflow hydrograph be routed for $X = 0.5$ so as to check the results. Let the inflow hydrograph be defined by the following equation,

$$I = \alpha \frac{t}{K_0} \exp\left(-\frac{t}{K_0}\right) \tag{30}$$

where α = a constant having the dimensions of discharge, and K_0 = a coefficient of storage not necessarily equal to K of the Muskingum equation. Using Eq. (30) in Eq. (3) gives the following result for $X = 0.5$,

$$Q = \alpha \frac{3t-K}{2K_0-K} \exp\left(-\frac{t}{K_0}\right) - 3\alpha \frac{K_0 K}{(2K_0-K)^2} \exp\left(-\frac{t}{K_0}\right) + \beta \exp\left(-\frac{2t}{K}\right) \tag{31}$$

where β = constant of integration and $2K_0 > K$. For $K = K_0$, Eq. (31) gives

$$Q = \alpha \frac{3t-4K}{K} \exp\left(-\frac{t}{K}\right) + \beta \exp\left(-\frac{2t}{K}\right) \tag{32}$$

Comparing Eq. (32) with Eq. (30) it is obvious, that the routed flood wave is not translatory even when $\beta = 0$ in Eq. (32). It is thus clear that $X = 0.5$ does not always produce a translatory wave. A translatory wave for $X = 0.5$ is produced only under the circumstances which yield Eq. (24.). If the cumulative effect of third and higher derivatives is not negligible, a translatory wave will not be produced.

Eq. (32) can also be used for explaining the initial condition. For instance, if Eq. (15) is used as initial condition as is routinely done, Eq. (32) gives

$$Q = \frac{\alpha}{K}(3t-4K)\exp\left(-\frac{t}{K}\right) + 4\alpha\exp\left(-\frac{2t}{K}\right) \tag{33}$$

Assume $\alpha = 1{,}200$ m^3/sec., and $K = 6$ hours as typical values. The outflow hydrograph using Eq. (32) is plotted in Fig. 2 and is seen to be marred by initial negative values. If on the other hand, Eq. (14) is used as initial condition, the problem of determining the value of the time lag is to be faced. From the hydrograph of Eq. (33), it can be seen that there is a turning point somewhere between $t = 0$ and $t = K$. If this turning point is raised such that $Q = 0$ instead of being negative at this point, the corresponding value of time should then be the correct value of the time lag. In order to determine the time lag, differentiate Eq. (32) with respect to time and set $(dQ/dt)_{t=\tau} = 0$; this fixes the position of the turning point on the time axis. This gives

$$\frac{3\alpha\tau}{2K} - \frac{7\alpha}{2} \equiv -\beta\, e^{-\tau/k} \tag{34}$$

At $t = \tau$, $Q = 0$, therefore from Eq. (32)

$$\frac{\alpha}{K}(3\tau-4K) = -\beta\, e^{-\tau/k} \tag{35}$$

Solving Eqs. (34) and (35) gives $\tau = K/3$. Using now the initial condition $t = K/3$, $Q = 0$, find the constant term β in Eq. (32). The complete solution is given as

$$Q \equiv \frac{\alpha}{K}(3t-4K)\exp\left(-\frac{t}{K}\right) + 3\alpha\exp\left(-\frac{2t}{K}+\frac{1}{3}\right) \tag{36}$$

Using $\alpha = 1{,}200$ m^3/sec., and $K = 6$ hours, Eq. (36) is also plotted in Fig. 2 for comparison with Eq. (33). Eq. (36) eliminates the initial negative outflows although the actual hydrograph for higher values of t is not substantially changed. Some relatively small changes occur only in the rising part of the hydrograph; the falling hydrograph remains effectively unchanged. The peak is slightly increased in Eq. (36).

It must be emphasised that as in Eq. (30), I is undefined or zero for all values of t less than zero even though the function defining I itself is continuous for $-\infty<t<\infty$, similarly in Eq. (36) Q is undefined or zero for all values of t less than the time lag

The translatory wave defined by Eq. (29) has a time lag of K units of time. There may however be some other translatory waves with a time lag different from K; at least one of such waves was identified and discussed by the author in an earlier publication (Gill 1979). If the inflow hydrograph is defined by

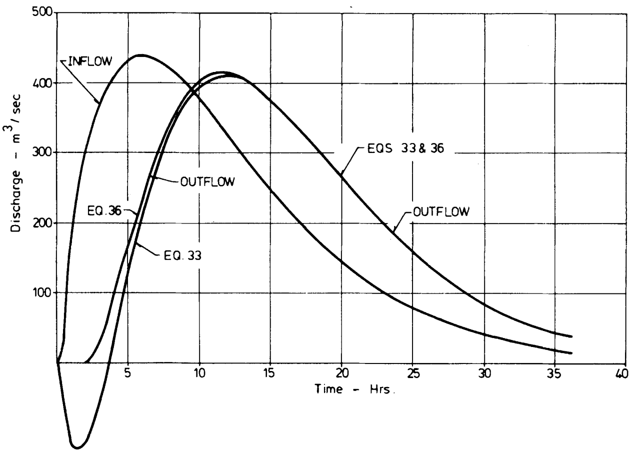

Fig. 2. Routed outflow hydrographs for explaining the use of correct initial condition.

$$I \equiv I_0 + A\,(1-\cos a\,t) \qquad (37)$$

where I_0, A, and a are constants, it can be shown that the routed hydrograph for X = 0.5 is given by

$$Q \equiv I_0 + A\,(1-\cos(a\,t-a\tau)) \qquad (38)$$

where $\tau = (2/a)$ arctan $(aK/2)$. The time lag in this case is less than K and approaches the value of K only for small values of $aK/2$. The cumulative affect of higher derivatives of I in this case is not negligible. However, all the odd derivatives can be expressed in term of the first derivative and all the even derivatives in terms of the second derivative. The coefficients of the first and second derivatives are thus altered in such away that the time lag $\tau = (2/a)$ arctan $(aK/2)$ is obtained instead of $\tau = K$.

Conclusions

The Muskingum method of flood routing is critically reviewed in the light of some recent work published by the author and the kinematic wave approach which has been used previously and recently also by some investigators for the evaluation of the Muskingum parameters. The linear hydrological method of using the Muskingum equation leads to conclusions which are comparable with those deducible from the linear kinematic wave theory. The linear kinematic approach is however somewhat delimited in its scope because for X = 0.5, it always predicts a translatory wave which may not actually be the case. The hydrological method is free from this fault.

The concept of time lag is emphasised particularly for initial values of time. Conventionally, it is assumed that the inflow and outflow hydrographs start changing from initial steady flow condition simultaneously. It is emphasised that this in fact is not always the case. The outflow hydrograph will only start changing after the lapse of a certain time from the moment when the inflow hydrograph started changing. The use of the time lag concept eliminates the initial appreciably reduced or sometimes even negative values of the routed outflow hydrograph.

Acknowledgment

The author is indebted to Professor S. Oleszkiewicz, Head, Department of Civil Engineering, Ahmadu Bello University, Zaria, for his continuous encouragement in research matters. The diagrams used in this paper were prepared by Mrs. J. Brezezina.

References

Chow, V.T., (1959) *Open Channel Hydraulics,* McGraw – Hill Book Co. Inc., New York.

Cunge, J.A., (1969) On the subject of a flood propagation computation method (Muskingum method), *Journal of Hydraulic Research, International Association for Hydraulics Research, Vol. 7,* No. 2, pp. 205-230.

Dooge, J.C. I., (1973) Linear Theory of Hydrologic Systems, Agricultural Research Service Technical Bulletin, No. 1468.

Gallati, M., and Maione, U., (1977) Perspective on mathematical models of flood routing, Proc. of the workshop held at the IBM Scientific Centre, Pisa, Italy, John Wiley & Sons, London, pp. 169-179.

Gill, M.A., (1977) Routing of floods in river channels, *Nordic Hydrology, Vol. 8,* pp. 163-170.

Gill, M.A., (1978) Flood routing by the Muskingum method, *Journal of Hydrology, Vol. 36,* No. 3/4.

Gill, M.A,, (1979) Translatory characteristics of the Muskingum method of flood routing, *Journal of Hydrology, Vol. 40,* pp. 17-29.

Koussis, A., (1978) Theoretical estimation of flood routing parameters, *Journal of the Hydraulics Division, ASCE, Vol. 104,* No. HY1, pp. 109-115.

Kulandaiswamy, V.C., (1966) A note on Muskingum method of flood routing, *Journal of Hydrology, Vol. 4,* pp. 273-276.

Lighthill, M.J., and Whitham, G.B., (1955) On kinematic waves: I-Flood movement in long rivers, *Proc. Royal Society (London), A, Vol. 229.*

Linsley, R.K., Kohler, M.A., and Paulhus, J.L.H., (1949) *Applied Hydrology,* McGraw – Hill Book Co. Inc., New York.

Natale, L., and Todini, E., (1977) A constrained parameter estimation technique for linear models in Hydrology, Proc. of the workshop held at the IBM Scientific Centre, Pisa, Italy, John Wiley & Sons, London, pp. 110-147.

First received: April 26, 1979
Revised version received: July 6, 1979

Address: Department of Civil Engineering,
Ahmadu Bello University,
Zaria,
Nigeria.