$$I_3(r,z,\nu_\alpha;s) = \frac{2\nu_\alpha z E(m)}{\pi\sqrt{(r+s)^2+\nu_\alpha^2 z^2}[(r-s)^2+\nu_\alpha^2 z^2]}$$

$$I_4(r,z,\nu_\alpha;s) = \frac{1}{\pi r\sqrt{(r+s)^2+\nu_\alpha^2 z^2}}\left[K(m)+\frac{(r^2-s^2-\nu_\alpha^2 z^2)E(m)}{(r-s)^2+\nu_\alpha^2 z^2}\right]$$

$$I_5(r,z,\nu_\alpha;s) = \begin{cases} \dfrac{2}{\pi r^2\sqrt{(r+s)^2+\nu_\alpha^2 z^2}}\left[\dfrac{(r-s)\nu_\alpha^2 z^2}{r+s}\Pi(m,p)-s^2K(m)\right]+\dfrac{2\sqrt{(r+s)^2+\nu_\alpha^2 z^2}}{\pi r^2}E(m)-\dfrac{2\nu_\alpha z}{r^2} & r>s \\ \dfrac{2}{\pi}\left[\dfrac{\sqrt{4r^2+\nu_\alpha^2 z^2}}{r^2}E(m)-\dfrac{1}{\sqrt{4r^2+\nu_\alpha^2 z^2}}K(m)\right]-\dfrac{\nu_\alpha z}{r^2} & r=s \\ \dfrac{2}{\pi r^2\sqrt{(r+s)^2+\nu_\alpha^2 z^2}}\left[\dfrac{(r-s)\nu_\alpha^2 z^2}{r+s}\Pi(m,p)-s^2K(m)\right]+\dfrac{2\sqrt{(r+s)^2+\nu_\alpha^2 z^2}}{\pi r^2}E(m) & r<s \end{cases}$$

$$I_6(r,z,\nu_\alpha;s) = \begin{cases} -\dfrac{2\nu_\alpha z}{\pi\sqrt{(r+s)^2+\nu_\alpha^2 z^2}}\left\{\dfrac{1}{r^2}\left[K(m)+\dfrac{r-s}{r+s}\Pi(m,p)\right]+\dfrac{E(m)}{[(r-s)^2+\nu_\alpha^2 z^2]}\right\}+\dfrac{2}{r^2} & r>s \\ -\dfrac{2\nu_\alpha z}{\pi\sqrt{4r^2+\nu_\alpha^2 z^2}}\left[\dfrac{1}{r^2}K(m)+\dfrac{1}{\nu_\alpha^2 z^2}\mathrm{E(m)}\right]+\dfrac{1}{r^2} & r=s \\ -\dfrac{2\nu_\alpha z}{\pi\sqrt{(r+s)^2+\nu_\alpha^2 z^2}}\left\{\dfrac{1}{r^2}\left[K(m)+\dfrac{r-s}{r+s}\Pi(m,p)\right]+\dfrac{E(m)}{[(r-s)^2+\nu_\alpha^2 z^2]}\right\} & r<s \end{cases}$$

and

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-m^2\sin^2\theta}}\,d\theta$$

$$E(m) = \int_0^{\frac{\pi}{2}} \sqrt{1-m^2\sin^2\theta}\,d\theta$$

$$\Pi(m,p) = \int_0^{\frac{\pi}{2}} \frac{1}{(1-p^2\sin^2\theta)\sqrt{1-m^2\sin^2\theta}}\,d\theta$$

$$m^2 = \frac{4rs}{(r+s)^2+\nu_\alpha^2 z^2},\quad p^2 = \frac{4rs}{(r+s)^2}$$

DISCUSSION

K. Komvopoulos[1]

This paper is another welcome addition to the field of contact mechanics of Professor Keer's group. Although the analysis yields unattractive integral equations which require cumbersome numerical iterations for solution, it sheds light to interesting contact mechanics aspects, especially the effect of transverse material isotropy to the stress fields arising in normal and sliding contacts. However, there are a few points that need some further clarification and discussion.

The results of the maximum shear stress for zinc, shown in Fig. 2(*a*), deviate from the Hertzian solution. Although such distortion of the subsurface stress field is not encountered with β-quartz (Fig. 2(*b*)), the maximum shear stress is closer to the surface than that predicted by the Hertz theory. Does this suggest that transverse isotropy may shift the location of the maximum shear stress upward or downward without aggravating the spatial distribution of the Hertzian stress field? What combination of elastic constants leads to this behavior? A discussion about the effect of transverse isotropy on the resulting maximum shear stress would be elucidating. Is it possible to determine an "optimum" transverse isotropy from the yield limit standpoint? How does the theory apply to laminated and fibrous/particulate composite half-spaces?

With regard to the interpretation of the pressure profiles shown in Fig. 4, it should be noted that the deviation from

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the Hertzian solution depends not only on the layer thickness but also on the indentation depth [A1]. Thus, the argument according to which only medium coating thicknesses produce pressure distributions significantly different from the Hertzian pressure profile cannot be generalized.

Although the analysis is for transversely isotropic multi-layered half-spaces, the bulk of the results is for single-layered media which have been analyzed in the recent past in quite depth. In particular, the results presented for indentation and sliding of the single-layered half-space, i.e., Figs. 6 through 9, and the interpretation of the associated possible coating failure mechanisms are essentially the same with those already reported in the literature for plane-strain normal contact [A2]. The similarity between normal and sliding contact solutions stems from the relatively low friction coefficient ($f = 0.1$) adopted by the authors. Is this associated with convergence problems which often arise with high friction coefficient values? For comparison purposes, a higher friction coefficient, e.g. $f = 0.5$, should have been selected.

The present study shows that the maximum tensile stress at the surface decreases with increasing thickness for $h/a < 0.6$. However, it has been shown that the maximum tensile stress may be encountered at the surface when the layer is thin ($h/a < 0.45$) or at the layer interface for intermediate and thick layers ($h/a > 0.5$) and, moreover, it exhibits a discontinuity at the interface [A2]. This suggests that surface and interfacial microcracking are dominant failure mechanisms of thin and intermediate or thick layers, respectively. The results of Fig. 8 are for the top region of the substrate and, therefore, are not representative of the actual stress at the bottom of the layer, where the relatively higher tensile stresses are produced [A2]. Nonetheless, the conclusion that a thin layer results in a higher tensile stress at the surface is in agreement with the predictions of [A2].

Furthermore, delamination of the layer (debonding) due to high shear stresses at the layer-substrate interface is controlled by the layer thickness [A2]. The enhancement of this failure mechanism was found to occur for layer thicknesses of $h/a = 0.35$. This value is remarkably close to that reported in the present analysis. Also, the observations that the interfacial shear stress is more localized near the contact edge and that it reaches higher values for medium coating thicknesses are in agreement with the results of [A2].

Additional References

A1 Komvopoulos, K., 1989, "Elastic-Plastic Finite Element Analysis of Indented Layered Media," ASME JOURNAL OF TRIBOLOGY, Vol. 111, pp. 430–439.

A2 Komvopoulos, K., 1988, "Finite Element Analysis of a Layered Elastic Solid in Normal Contact With a Rigid Surface," ASME JOURNAL OF TRIBOLOGY, Vol. 110, pp. 477–485.

Authors' Closure

Our paper presented a general approach for the three-dimensional contact analysis of a spherical indenter in sliding contact on a transversely isotropic multi-layered half-space. In order to demonstrate the method, three examples were chosen: analysis of a transversely isotropic half-space for two representative materials, zinc and β-quartz, the effective moduli of a two-layered system having different combinations of layer thicknesses and investigations of critical stress components which may cause different coating failure for various thicknesses of Al_2O_3 ceramic coating deposited on a 52100 steel substrate.

Our response to the discusser's points is as follows:

1) Transversely isotropic materials have material properties which are isotropic in the transverse plane and with different properties in its normal direction (see Eqs. (1)–(6)). Therefore, the location of maximum shear stress may shift upward or downward due to the transversely isotropy and it may also deviate from the contact axis for some classes of hexagonal crystals (see Dahan and Zaraka, 1977). However, the contact profile still has a Hertzian stress distribution (see solutions presented by Lin et al. (1991) and Chen (1969)). Since the mechanical properties, such as yield limit and fracture toughness, in the transverse plane are different from those in the normal direction for transversely isotropic materials, more general theories to describe the anisotropic feature of yielding and fracture criteria are required.

2) It is well known that the distributions of contact pressure may deviate from the Hertzian solution for layers due to the layer thickness and the elastic properties of the layer relative to those of substrate. However, as indicated in the paper, the results shown in Fig. 4 are only used to compare the contact pressure profile between different thicknesses of Al_2O_3 ceramic coating. We did not focus on material properties for the coating other than those given for this figure.

3) As indicated in the paper, three representative examples have already been given to illustrate the method, where Fig. 2 is the analysis of transversely isotropic half-space and Fig. 3 is for the analysis of a two-layered half-space. The results in Figs. 6–9 show the critical stress components for Al_2O_3 ceramic coating with different thicknesses: thin, intermediate and thick coatings. For the coating thickness as thin as several microns, transversely isotropic properties are chosen for the coating layer and isotropic properties for the substrate. Furthermore, only the coefficient of friction $f = 0.1$ (lubricated sliding contact) is considered in the paper. Although results for the case of $f = 0.5$ (dry contact) are not shown, they can be easily obtained from the present method and there are no convergence problems associated with differing coefficients of friction. It is noted that higher stress fields will develop and move to the surface at the trailing edge of the contact area as the coefficient of friction increases.

4) The authors agree with the discusser that the interfacial micromaking in the coating is a dominant mode for intermediate and thick layers. However, the results in Fig. 8 are used to show how the substrate can be protected from cracking failure by the evidence that compressive stresses are developed at the top region of substrate.

Finally, the authors have two further comments on the discussion:

1) It is inappropriate to describe the derived integral equations in the paper as unattractive. As a matter of fact, the integral equation approach, which is analogous to the boundary element method, has an important role in contact mechanics. Numerous contact problems have been formulated into integral equations and solved either analytically or numerically in the literature (see Gladwell, 1980).

2) The discusser's assertion that cumbersome numerical iterations are required for solutions is not correct. On the contrary, much of the computational effort for the analysis of the multi-layered structure has been greatly reduced to simple matrix operations by using the propagator matrix and associated mathematical treatment. At this point the developed method can very effectively perform three-dimensional contact analysis. The present approach also provides a more direct way to determine the contact radius and contact pressure and allows the calculation of the displacement and stress fields for layer thickness as thin as $a/h = 20$.

Additional Reference

Dahan, Marc and Zarka, Joseph, 1977, "Elastic Contact Between a Sphere and a Semi Infinite Tranversely Isotropic Body," *Int. J. Solids Structures*, Vol. 13, pp. 229–238.