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MECHANISM OF GROUND WATER RESERVOIRS

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The article deals with unsteady ground water flow in an aquifer of limited horizontal extent. The changes in storage are taken to be dependent on a storage coefficient S, and flow rates are taken to be dependent on the coefficient of transmissibility T. Here the relation to Darcy's constant k is that $T = kD$, where D is the aquifer thickness. The equilibrium ground water surface is found and further calculations are based on the departure of the ground water level from this equilibrium value.
An expression of the impulse response function (the instantaneous unit hydrograph or IUH) is found, and its use demonstrated.
Finally, it is shown how such calculations have been used to analyse the inflow into a water reservoir in Iceland, serving as guidelines in the planning of field investigations of a major ground water reservoir in a nearby lava field.

The mechanism of a ground water reservoir is the interaction between inflow, outflow, and storage in the reservoir. The questions of storage capacity, water yield, recharge, and variation of ground water level are all part of the reservoir mechanism problem.

The problem has the advantage that it lends itself to mathematical treatment. This is an important feature, not only because of the possibility of numerical calculation, but just as much because of the qualitative information it gives about the physical behaviour of the reservoirs, and the relationship between its physical characteristics.

The Flow Equation System

We assume that the flow is practically horiozntal and can be described by

$$\vec{q} \equiv -T\,\overrightarrow{\text{grad}}\,h \tag{1}$$

the equation of continuity

$$\text{div}\,q \equiv -S\frac{\delta h}{\delta t} + R(t) \tag{2}$$

$\vec{q}$ = flow vector, $m^3/m/s$
h = ground water level, m
t = time
$R(t)$ = inflow (recharge) function, $m^3/m^2/s$
T = coefficient of transmissibility
S = storage coefficient, $m^3/m^2/m$

Inserting (1) in (2) yields

$$\frac{S}{T} \cdot \frac{\delta h}{\delta t} \equiv \text{divgrad}\,h + \frac{R(t)}{T} \tag{3}$$

Averaging (3) over a time interval t, and letting $t \rightarrow \infty$ yields:

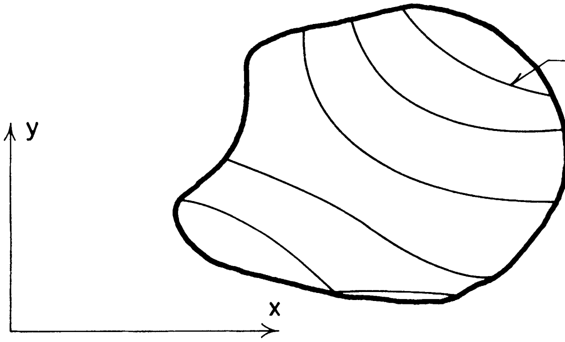
$$\text{divgrad}\,h_0 + \frac{R_0}{T} = 0 \tag{4}$$

$$h_0 \equiv \lim_{t \rightarrow \infty} \frac{1}{2t}\int_{-t}^{t} h(\tau)\,d\tau \tag{5}$$

$$R_0 \equiv \lim_{t \rightarrow \infty} \frac{1}{2t}\int_{-t}^{t} R(\tau)\,d\tau$$

To solve (4), $h_0, \frac{\delta h_0}{\delta n}$ or a linear combination of these must be given along the boundary curve of the aquifer; (n is the normal to that curve). As a rule, h_0 cannot be determined analytically, but numerical evaluation in a digital computer is very easy.

The h_0 given represents a given ground water level on the boundary. $\frac{\delta h}{\delta n}$ given represents the inflow on the boundary. ($\frac{\delta h_0}{\delta n} = 0$ is zero inflow or watertight boundary.) Any combination of these two is mathematically sufficient, but may be unrealistic in nature.



We now insert in (3)

$$h \equiv h_0 + h_1, \; R \equiv R_0 + R_1 \tag{6}$$

and get

$$\frac{S}{T} \cdot \frac{\delta h_1}{\delta t} \equiv \text{divgrad}\, h_1 + \frac{R_1}{T} \tag{7}$$

The boundary conditions for h will be the same as for h_0. But if the boundary conditions are stationary, (h_0 constant on f), then the boundary conditions for h_1 will be homogeneous.

This stationarity is not as unrealistic as it might seem to be when the runoff is to be determined. The various lakes and rivers receiving the ground water do usually have fairly stationary surface levels, varying within very narrow limits, so the error made by taking the mean values will be negligible.

In the following we shall assume h to be known and independent of t on f.

The Impulse Response Function IUH

We now calculate the IUH from the following:
Let us assume that, at time $t \equiv 0$ the aquifer, by infiltration, suddenly receives sufficient recharge water (rainfall, snowmelt) to produce a 1 m increase in ground water level. If then the recharge stops, the variation of h_1 with t will be the IUH. (Index $_1$ means from now on an IUH function).

In other words, we have

$$\frac{S}{T} \cdot \frac{\delta h_1}{\delta t} \equiv \text{divgrad}\, h_1 \tag{8}$$

With the boundary conditions

$$h_1 = 0 \text{ on } f \tag{9}$$

and the initial condition

$$h_1\,(0,x,y) \equiv 1 \tag{10}$$

(8), (9), and (10) constitute a problem analogous to that of heat conduction. The evaluation of h_1 represents no difficulty in a digital computer with sufficient core storage. If the area in question has a geometry that is simple enough, analytic expressions of h_1 may be found. As an example, to calculate the IUH for a circle with radius R we take the origin (0,0) in the centre, using axisymmetric representation of (8)

$$\frac{S}{T} \cdot \frac{\delta h_1}{\delta t} = \frac{1}{r} \cdot \frac{\delta h_1}{\delta r} + \frac{\delta^2 h_1}{\delta r^2}$$

We now insert the following dimensionless coordinates

$$t' = \frac{Tt}{SR^2}$$

$$x = r/R \tag{11}$$

We can drop the prime on t′, remembering only that from now on t means t′; our equation then becomes

$$\frac{\delta h_1}{\delta t} = \frac{1}{x} \cdot \frac{\delta h_1}{\delta x} + \frac{\delta^2 h_1}{\delta x^2} \tag{12}$$

with the boundary conditions

$$h_1(t, 1) \equiv 0 \tag{13}$$

$$h_1(0, x) \equiv 1 \tag{14}$$

Separation of the variables in (12) provides the solution

$$h_1 = e^{-\lambda^2 t}\; J_0(\lambda x) \tag{15}$$

J_0 is the Bessel function zero order first kind. (13) gives that the λ has to be a root in $J_0(\lambda) = 0$. There are infinitely many roots $\lambda_1, \lambda_2, \ldots \lambda_n$, so the complete solution to (12) becomes

$$h_1 = \sum_{n=1}^{\infty} A_n\, e^{-\lambda^2_n t}\; J_0\,(\lambda_n x) \tag{16}$$

(14) gives

$$1 = \sum_{n=1}^{\infty} A_n\, J_0\,(\lambda_n x) \tag{17}$$

We use the orthogonality of the Bessel functions to find the A_n's. We multiply (17) by $xJ_0(\lambda_m x)$ and integrate from 0 to 1. We get

$$\int_0^1 xJ_0(\lambda_m x) = \frac{J_1(\lambda_m)}{\lambda_m} \sum_{n=1}^{\infty} = A_n \int_0^1 J_0(\lambda_n x)\, J_0(\lambda_m x)\, x\, dx = A_m \frac{1}{2} J_1(\lambda_m)^2$$

which gives

$$A_m = \frac{2}{\lambda_m J_1(\lambda_m)} \tag{18}$$

and subsequently

$$h_1 \equiv 2 \sum_{n=1}^{\infty} \frac{e^{-\lambda^2_n t} J_0(\lambda_n x)}{\lambda_n J_1(\lambda_n)} \tag{19}$$

This is the IUH for the ground water level at any point within the circular area. We get the IUH for the storage from this by integrating h_1 over the area

$$\pi \cdot s_1 \equiv \int_0^1 h_1\, 2\pi\, x\, dx \equiv 2\pi \cdot 2 \sum_{n=1}^{\infty} \frac{e^{-\lambda^2_n t}}{\lambda_n J_1(\lambda_n)} \int_0^1 J_0(\lambda_n x)\, x\, dx \equiv 4\pi \sum_{n=1}^{\infty} \frac{e^{-\lambda^2_n t} J_1(\lambda_n)}{\lambda^2_n J_1(\lambda_n)}$$

We get

$$s_1 \equiv 4 \sum_{n=1}^{\infty} \frac{e^{-\lambda^2_n t}}{\lambda^2_n} \tag{20}$$

which is the runoff IUH (recession of storage).

It may be shown that the discharge IUH, which we shall call q_1, is

$$q_1 = - \frac{ds_1}{dt} = 4 \sum_{n=1}^{\infty} e^{-\lambda^2_n t} \tag{21}$$

which then is the discharge IUH (run-off flow rate).

Use of the Response Function

When the response function (IUH) is known, the output of the system in question, being the run-off from or the ground water level variation in a catchment area, is given by the convolution integral

$$y(t) = \int_0^{\infty} g(\tau)\, X(t - \tau)\, d\tau \tag{22}$$

Here $y(t)$ symbolises the output, $g(\tau)$ the IUH function, and $X(t)$ the input. If the problem were to find the ground water level variation at a point in the area considered in the preceding chapters, we would insert the h_1 for g, R(t) for X, and get the desired water level variation. A convolution integral like (22) is an easy and very quick process to handle in a digital computer.

In practice, however, it is very rare that such a direct use can be made of the IUH. The information available is seldom accurate enough, and the model itself cannot handle the inhomogeneity of the aquifer which is always present, at least to some extent. Nevertheless, statistical features of the input will be reflected in the output according to (22).

A closer examination of (22) reveals that when (21) is inserted for $g(\tau)$, then (22) satisfies the equation:

$$y_n + \frac{1}{\lambda^2_n} \cdot \frac{dy_n}{dt} = \frac{4}{\lambda^2_n} X(t) \qquad (23)$$

$$y = \sum_{n=1}^{\infty} y_n \qquad (24)$$

(23) and (24) make is clear that such a ground water reservoir is equivalent to a system of infinitely many parallel linear reservoirs, as shown in Figure 1. Each of the reservoirs has an input equal to $\frac{4}{\lambda^2_n}$ times the total input into the whole system, and each linear reservoir has a time constant equal to $\frac{1}{\lambda^2_n}$. Thus the linear reservoirs receive smaller and smaller parts of the total input when n increases and their time constants get smaller and smaller.

(23) can be written as

$$y_n + \frac{\alpha_n}{4} \cdot \frac{dy_n}{dt} = \alpha_n \cdot X(t) \qquad (25)$$

or in words: our system behaves like infinitely many parallel linear reservoirs, where the input into the n'th reservoir is α_n times the total input, and the time constant of the n'th reservoirs is $\alpha_n/4$.

$$\alpha_n = \frac{4}{\lambda^2_n} \quad \text{(see Table 1)} \qquad (26)$$

Table 1.

n	α_n	$\Sigma\alpha_n$
1	0.6912	0.6912
2	0.1312	0.8224
3	0.0534	0.8758
4	0.0293	0.9051

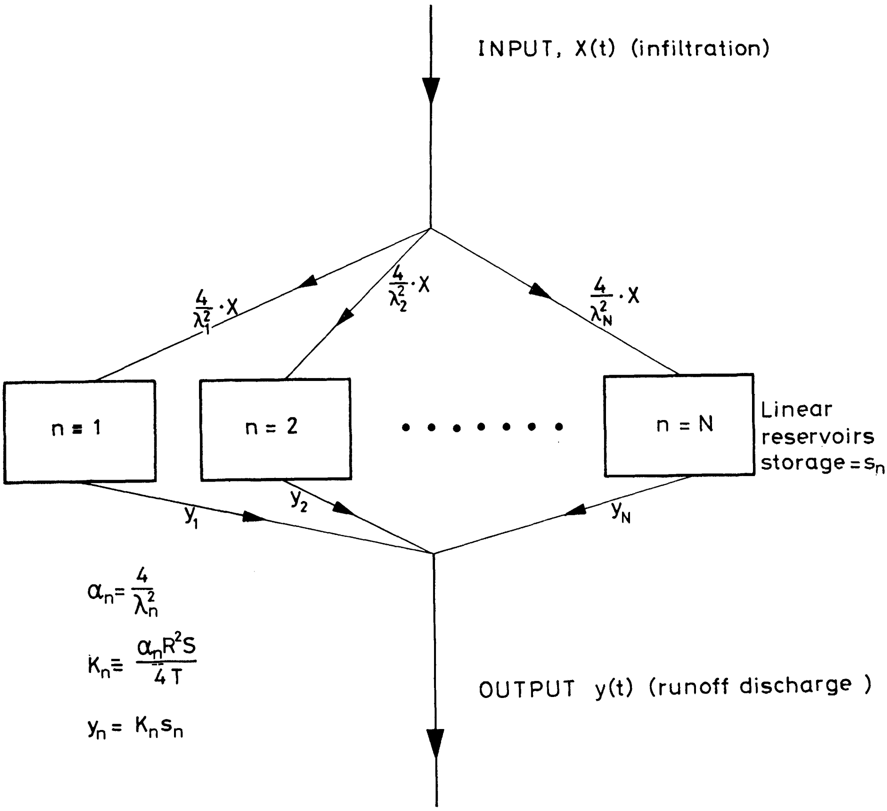


Fig. 1.
Reservoir model.

It will be seen that about 70 % of the total output is from res. No. 1, and res. No.'s 1 through 4 give 90 % of the total output. As res. No. 1 has the biggest time constant, the effect of it will be dominating.

We can now define a physical time constant for each reservoir

$$K_n = \frac{\alpha_n R^2 S}{4T} \tag{27}$$

Now we are able to find the discharge from each reservoir in the case of periodic input. We suppose that a periodic infiltration is given by a Fourier series, and we examine one of the frequencies

$$R(t) = A \cos 2\pi ft \tag{28}$$

The discharge will be

$$q(t) \equiv 4 \sum_{n \equiv 1}^{\infty} \int_0^{\infty} e^{-\lambda^2_n \tau} \ A \cos 2\pi f (t - \tau) \, d\tau$$

$$= 4A \sum_{n \equiv 1}^{\infty} \frac{\lambda^2_n \cos 2\pi ft + 2\pi f \sin 2\pi ft}{(2\pi f)^2 + \lambda_n^4}$$

With

$$\tan \beta_n \equiv \frac{2\pi f}{\lambda^2_n} = \frac{\alpha_n \pi f}{2} \tag{29}$$

we get

$$q_n(t) \equiv 4A \frac{\cos (2\pi ft - \beta_n)}{\sqrt{(2\pi f)^2 + \lambda_n^4}} \tag{30}$$

Introducing α_n we get

$$q_n(t) \equiv A \, \alpha_n \cos \beta_n \cos (2\pi ft - \beta_n) \tag{31}$$

Now, the input into each reservoir is:

$$R_n(t) = \alpha_n A \cos 2\pi ft$$

We see that the output of each of the reservoirs is a cosine function with the same frequency as the input, but the phase is delayed by β_n; we have

$$0 < \beta_n < \frac{\pi}{2} \tag{32}$$

The amplitude is diminished by the factor

$$\frac{Max \, q_n(t)}{Max \, R_n(t)} \equiv \cos \beta_n \tag{33}$$

We now get the following picture of the reservoir effect:

If $\alpha_1 f \ll 1$ (small time constant of the reservoir), all the β's are small, because α_n decreases as n increases. Then $\cos\beta_n$ is of order 1, and the q_n's will be of same order of magnitude as the R_n's, and the phase difference between q_n and R_n, the angle β_n, will be small. This is the case of a small reservoir that has only a small effect on the input.

If, on the other hand, $\alpha_1 f \gg 1$ (large time constant of the reservoir), β_1 will be of the order $\frac{\pi}{2}$, and accordingly q_1 will be very small. As 69 % of the water goes through the first reservoir in our model, the variations in discharge will be small compared to those of the input, and will be observed 1/4 of a cycle later than those of the input.

CONCLUSION

A ground water reservoir with time-independent boundary conditions is characterized by an average ground water table, and variations can be treated as departures from this average value. The average ground water table can be found if the storage coefficient S, the coefficient of transmissibility T, and the average infiltrations are known, together with the boundary values.

The reservoir treated in Chapter 5 responds to variations in the infiltration as a series of infinitely many parallel linear reservoirs (Figure 2). The major proportion of the water goes through the first reservoirs, and these have the largest time constant, so the flow is mainly characterized by these reservoirs.

Due to the linearity of the reservoirs the output characteristics are functions of the ratio of the time constant of the reservoir and the input period under consideration. The character of the reservoir can be accurately judged from

$$K_1 = \frac{\alpha_1 R^2 S}{4T} = \frac{0.69\, R^2 S}{4T} \tag{34}$$

which is the time constant of the largest (first) linear reservoir. If K_1 is larger than the period of the infiltration. We have the case of a large reservoir, variations in the runoff discharge are small and $^1/_4$ of a cycle behind the infiltration.

The above conclusions retain their validity when the infiltration is no longer simply periodic, but consists merely of a whole specter of periods. The case of an input thus composed is easily treated when the linear reservoir series model is used.

PLANNING OF RESERVOIR INVESTIGATIONS IN ICELAND

In connection with the development of the River Thjorsa at Burfell in Iceland, a water storage is in the course of construction in Lake Thorisvatn (Figure 2). Withdrawals from the Thorisvatn storage are to take place during low discharge periods in wintertime, but the bulk of inflow comes with the springtime snow melt.

The inflow into the future store is estimated at 50 m^3/sec on average, and an estimated 40 % comes from the nearby Veidivatnahraun, a porous lava field which is known to be a major ground water reservoir.

The most efficient use of the water stored can be implemented only if good predictions of the inflow into the Thorisvatn reservoir in the draught period exist at the beginning of the period. As all precipitation during wintertime is

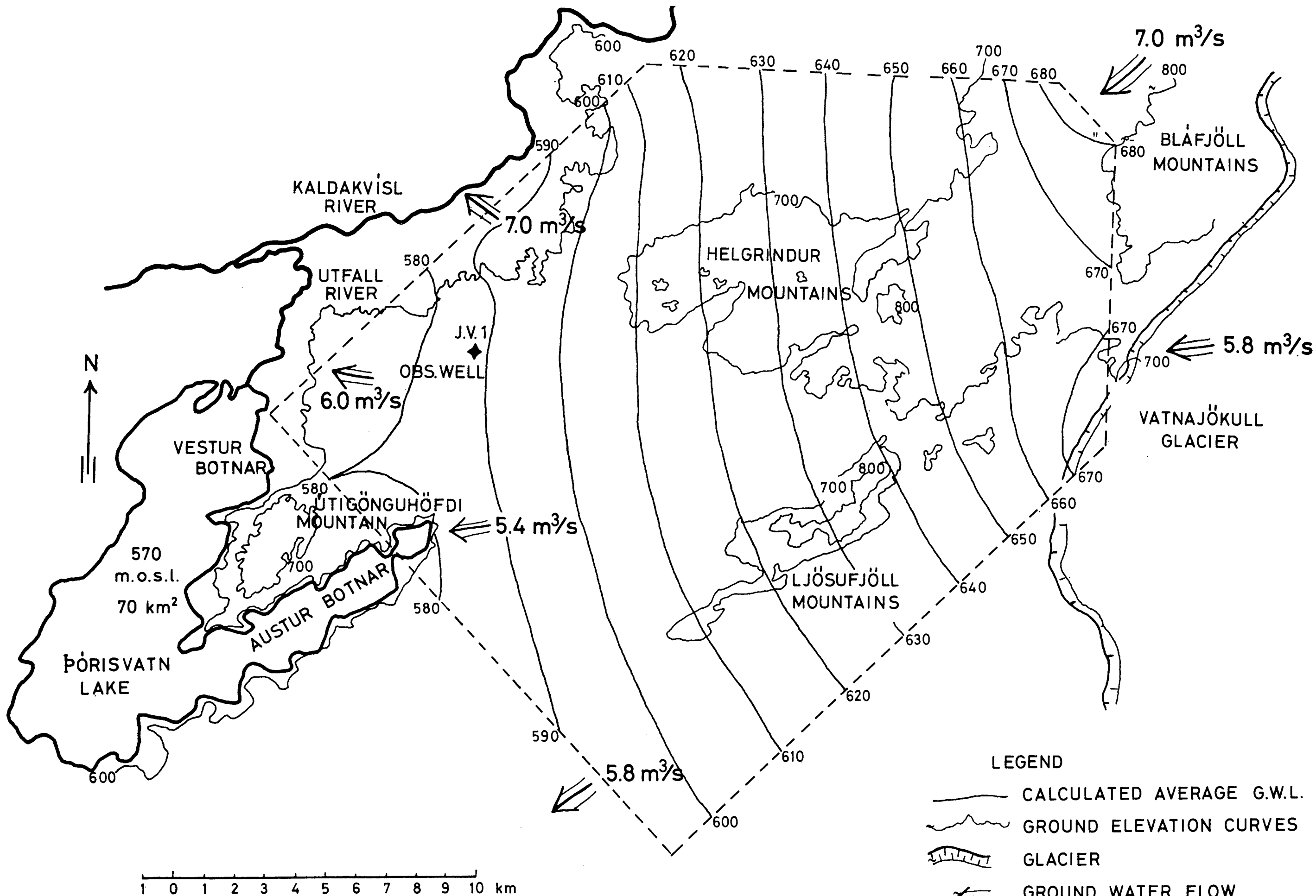


Fig. 2.
The Thorisvatn area.

snow, the only inflow is from the ground water, the major proportion of which comes from Veidivatnahraun. A hydrologic investigation of the area is therefore being conducted by the National Energy Authority of Iceland.

One possible method is to drill a series of ground water wells and register the variation of the ground water level. By continually comparing ground water levels with withdrawals and amount of water stored, enough experience to warrant efficient prediction could be gained in time. But then the problem is where to place the observation wells in order to gain the maximum information.

A theoretical investigation of the reservoir mechanism of Veidivatnahraun was performed on the lines set out in the preceding chapters. The results of the calculations of the average ground water level are shown in Figure 3. The calculations were carried out on an IBM 1620 II computer owned by the University of Iceland.

In the calculations the infiltration was put equal to the precipitation and then several runs were made on the computer with various T and the result compared with all the knowledge available of the actual situation in the area, including observations of ground water level variation in the observation well I.V.1 in 1969. These observations showed a mean water level of 587.3 m above sea level and a variation of 2.6 m.

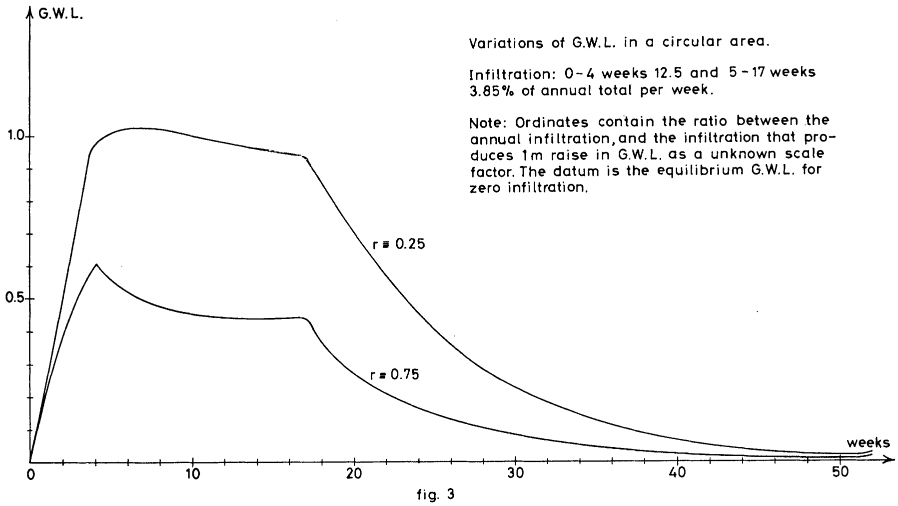


Fig. 3.
Calculated G. W. L.

The result given in Figure 2 is for $T = 0.2\ m^2/sec$. It shows an inflow of 12.8 m^2/sec into the area through the side facing the Vatnajökull glacier. The presence of this inflow has not been physically established, but is strongly indicated by the fact that many small melt-water rivers in this area disappear into the ground. The outflow is 24.2 m^2/sec, 7.0 to the river Kaldakvisl, 6.0 to the river Utfall, 5.4 to Thorisvatn in Austurbotnar, and 5.8 m^3 to the area south of Thorisvatn; this water cannot be caught in the future storage system.

In order to investigate the character of the water level variation curves, water elevation curves were calculated according to (19), the results being shown in Figure 3. A $\frac{R^2T}{S}$ value of 50 weeks was used as a likely value. Two curves are shown in Figure 3, one for a point R/4 from the centre, the other for a point 3R/4 from the centre (R/4 from the edge). The curves are calculated for the following infiltration.

15.5 – 15.6 First $^1/_2$ of total precipitation at equal rate (snow melt).

15.6 – 15.10 Second $^1/_2$ of total precipitation at equal rate (summer rain).

The water level curves differ principally in the following way.

Closer to the centre the curve has a well defined maximum height, but the times when the infiltration changes are not very apparent.

Closer to the edge, the time when the snow melt stops is clearly seen, but the maximum water level height is maintained for only a very short time, the time when infiltration stops, and recession also begins being seen but not so clearly.

These results have been interpreted in the way that observation wells close to the centre of the area would give a good indication of the amount of water stored and observation wells close to the edge can be used to indicate the time for significant changes in the infiltration. Altogether, these observations would form a good basis for judging the inflow to the Thorisvatn storage system in wintertime.

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