Nordic Hydrology, 9, 1978, 267-276

Spatial Correlation of Hydrologic and Physiographic Elements

Paper presented at the Nordic Hydrological Conference
(Helsinki, Finland, July-Aug, 1978)

Lars Gottschalk
Swedish Meteorological and Hydrological Institute,
Norrköping, Sweden

In an earlier paper (Gottschalk 1977) analytical expressions were given for the time correlation of hydrologic processes from the linearized equations of motion for these processes. In this paper space correlation is similarly treated. Comparison is made with empirical space correlation functions of hydrologic and physiographic elements.

Introduction

The occurrence of hydrologic and physiographic processes in one and two dimensions is well known. It is useful while analyzing hydrologic and physiographic data and is also of theoretical importance to fit empirical spatial correlation functions to a physically based model and to estimate certain parameters in the model accordingly.

For the simplest autoregressive scheme in one dimension, used in hydrology to describe time dependence, a corresponding correlation function is the exponential decay function. The continuous equation is a first order stochastic ordinary differential equation. The simplest symmetric second order autoregressive scheme can be written down as (Whittle 1954):

$$q_{rs} = a(q_{r+1,s} + q_{r-1,s} + q_{r,s+1} + q_{r,s-1}) + \varepsilon \tag{1}$$

where a is a parameter and q a variable describing the studied phenomena. We shall note that we now consider processes in the plane. Compared to the time domain, where we only can have dependence backwards to what have happened earlier, in this case we can have dependence backward and forward. Handling schemes like Eq. (1) lead to considerable theoretical difficulties. Continuous relations are easier to use to find applicable models for space correlations.

Spatial variation and particularly spatial correlation is theoretically examined in litterature by Matern (1960) Heine (1955), Whittle (1954 and 1962) and others.

Equations of Motion

A similar continous equation to Eq. (1) is (Whittle 1954):

$$\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \lambda q(x,y) = \varepsilon \tag{2}$$

where λ is a parameter and x and y space coordinates. Eq. (2) is a second order partial differential equation of elliptic type.

The equation describes steady state flow. The parameter λ describes losses proportional to $q(x, y)$. The model (2) would be applicable to a wide variety of conditions of potential flow.

We can write down a general linear equation governing unsteady flow as:

$$\alpha \frac{\partial^2 q}{\partial x^2} - \beta \frac{\partial^2 q}{\partial x \partial t} - \frac{\partial^2 q}{\partial t^2} - \gamma \frac{\partial q}{\partial x} - \delta \frac{\partial q}{\partial t} = 0 \tag{3}$$

where x is a space coordinate, t is a time coordinate and α, β, γ and δ are parameters. Eq. (3) is a linear partial differential equation of hyperbolic type describing wave phenomena. It is applicable to unsteady flow in rivers and channels. Below we shall discuss the following simplified wave equation:

$$\alpha \frac{\partial^2 q}{\partial x^2} - \frac{\partial^2 q}{\partial t^2} - \delta \frac{\partial q}{\partial t} = 0 \tag{4}$$

For the application of this equation to river flow see Kuchment (1964). Unsteady flow in channels can in many cases be approximated by a simple diffusion equation of the form:

$$\frac{\partial^2 q}{\partial x^2} = \frac{\gamma}{\alpha} \frac{\partial q}{\partial x} - \frac{\delta}{\alpha} \frac{\partial q}{\partial t} = 0 \tag{5}$$

It is then assumed that inertial forces are of negligible order. Another form of the diffusion equation more suitable for unsteady groundwater flow can be written down as:

$$\frac{\partial^2 q}{\partial x^2} - \frac{\delta}{\alpha}\frac{\partial q}{\partial t} = 0 \qquad (6)$$

Adding a loss term in accordance with Eq. (2) we get:

$$\frac{\partial^2 q}{\partial x^2} - \lambda q - \frac{\delta}{\alpha}\frac{\partial q}{\partial t} = 0 \qquad (7)$$

The equations given above are applicable to many problems in hydrology involving potential flow, diffusion processes and wave propagation.

Space correlation

We shall regard two different situations. The first one is when the boundary condition at $x = 0$ is given as $q\ (0,t) = \varepsilon\ (t)$, where $\varepsilon\ (t)$ is a random function uncorrelated in time, i.e. its covariance is given by.

$$\text{cov}_\varepsilon(\tau) = E\{\varepsilon(t)\varepsilon(t-\tau)\} = \sigma^2\delta(\tau) \qquad (8)$$

where σ is the standard deviation of the studied process. We can assume (without affecting generality) that $\sigma = 1$ and further on that the mean of the studied processes is zero.

The solution to linear partial differential equations, as examplified above, is in this case, written down as:

$$q(x,t) = \int_0^t \varepsilon(\tau)h(x,t-\tau)d\tau \qquad (9)$$

For the space covariance we derive:

$$\text{cov}(s) = E\,(q(x,t)q(x+s,t)) =$$

$$= E\,\{\int_0^t \varepsilon(\tau)h(x,t-\tau)d\tau \int_0^t \varepsilon(\tau')h(x+s,t-\tau')d\tau'\} =$$

$$= \int_0^t\int_0^t h(x,t-\tau)h(x+s,t-\tau')E\{\varepsilon(\tau)\varepsilon(\tau')\}d\tau d\tau' =$$

$$= \int_0^t h(x,t-\tau)h(x+s,t-\tau)d\tau \tag{10}$$

In the second case we shall consider the equation:

$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right) = \varepsilon(x,t) \tag{11}$$

where L is a linear operator in accordance to Eqs. (3) – (7).
$\xi(x,t)$ is a random function uncorrelated in time and space with the covariance:

$$\mathrm{cov}_{\varepsilon}(\zeta,\tau) = E\{\varepsilon(x,t)\varepsilon(x-\zeta,t-\tau)\} = \sigma^2\delta(\zeta)\delta(\tau) \tag{12}$$

We write down the solution to Eq. (11) as (we assume zero boundary conditions):

$$q(x,t) = \int_0^t\int_0^{\ell} h(t-\tau,x,\zeta)\varepsilon(\zeta,\tau)d\zeta d\tau \tag{13}$$

The covariance is derived as a generalization of Eq. (10)

$$\mathrm{cov}(s) = E(q(x,t)q(x+s,t)) =$$

$$= E\{\int_0^t\int_0^{\ell} h(t-\tau,x,\zeta)\varepsilon(\tau,\zeta)d\tau d\zeta \cdot$$

$$\int_0^t\int_0^{\ell} h(t-\tau',x,\zeta')\varepsilon(\tau',\zeta')d\tau' d\zeta'\}$$

$$= \int_0^t\int_0^t\int_0^{\ell}\int_0^{\ell} h(t-\tau,x,\zeta)h(t-\tau',x,\zeta')$$

$$E\{\varepsilon(\tau,\zeta)\varepsilon(\tau',\zeta')\}d\tau d\tau' d\zeta d\zeta'$$

$$= \int_0^t\int_0^{\ell} h(t-\tau,x,\zeta)h(t-\tau,x+s,\zeta)d\tau d\zeta \tag{14}$$

We can easily generalize to two dimensions in space and thus include Eq. (2).
We can finally write the space correlation function:

$$\rho(s) = \frac{\mathrm{cov}(s)}{\mathrm{cov}(o)} \tag{15}$$

For ρ to be well defined, it is necessary that h tends to zero a infinity and that cov(s) is finite everywhere.

Let us now derive response functions $h(x,t-\tau)$ and space correlation functions for the first situation (Eq. 10). For the simplest diffusion equation (Eq. 6) the response function is found to be

$$h(x,t-\tau) \equiv \frac{x}{2(t-\tau)^{\frac{3}{2}}\sqrt{\frac{\Pi\alpha}{\delta}}} \exp\{-\frac{x^2\delta}{4(t-\tau)\alpha}\} \tag{16}$$

and the space correlation (applying Eqs. (10) and (15)) is calculated as:

$$\rho(s) \equiv \frac{1}{(1+\frac{s}{x}+\frac{s^2}{2x^2})^2} \quad . \tag{17}$$

For the diffusion equation (Eq. 5) we find similarly:

$$h(x,t-\tau) = \frac{1}{2\delta\sqrt{\Pi}} \frac{x-\frac{\gamma}{\delta}t}{|(t-\tau)\frac{\alpha}{\delta}|^{\frac{3}{2}}} \exp\{=\frac{(x-\frac{\gamma}{\delta}t)^2}{4(t-\tau)\frac{\alpha}{\delta}}\} \tag{18}$$

and

$$\rho(s) = \frac{(x-\frac{\gamma}{\delta}t)^2 + s(x-\frac{\gamma}{\delta}t)}{(x-\frac{\gamma}{\delta}t+(x-\frac{\gamma}{\delta}t)s+\frac{s^2}{2})^2} \tag{19}$$

Dividing a river into linear subsystems by the method of Kalinin and Miliukov (1958) the response function is expressed as:

$$h(n,t) \equiv \frac{1}{\Gamma(n)}(\frac{t}{k})^{n-1} e^{-t/k} \tag{20}$$

where the river is devided into n subsystems with equal time constants k. We can now find the space correlation as the correlation between an outflow from the m-th $(m<n)$ reservoir and the n-th reservoir. Using Eq. (10) we find:

$$\mathrm{cov}(n-m) = \frac{\Gamma(n+m-1)}{\Gamma(n)\Gamma(m)}\frac{k}{2}(\frac{1}{2})^{n+m-2} \tag{21}$$

and

$$\rho(n-m) = \frac{\Gamma(n+m-1)}{\Gamma(2n-1)} \frac{\Gamma(n)}{\Gamma(m)} \left(\frac{1}{2}\right)^{m-n} \tag{22}$$

For the second situation we apply Eqs. (14) and (15). Eq. (6) gives for this situation the following response function:

$$h(x,\zeta,t-\tau) = \frac{1}{\sqrt{\Pi 4(t-\tau)\frac{\alpha}{\delta}}} \quad \exp\{-\frac{(x-\zeta)^2}{4(t-\tau)\frac{\alpha}{\delta}}\} \tag{23}$$

Insertion in Eq. (14) will give us a non-convergent integral expression. The correlation structure of Eq. (16) has been thoroughly investigated by Whittle (1962). Generalization for two dimensions also gives non-convergent expressions while in case of three dimensions the expression for covariance is of the principal form cov(*s*) = const/*s*. This expression is not finite for $s=0$ and can thus not be used to derive the correlation function. Analyzing the effect of autocorrelation (in time) in $\varepsilon(x,t)$. Whittle derives for a similar three dimensional case the following principal formula:

$$\mathrm{cov}(s) = \mathrm{const}(1-\exp(-as))\frac{1}{as} \tag{24}$$

which is finite for $s = 0$. Adding a loss term proportional to $q(x,t)$ as in Eq. (7) gives convergent expressions in all three dimensions.

The reponse function of Eq. (5) for the second situation is:

$$h(x,\zeta,t-\tau) = \frac{1}{\sqrt{4\Pi(t-\tau)\frac{\alpha}{\delta}}} \{\exp| -\frac{(x-\frac{\gamma}{\delta}t-\zeta)^2}{4(t-\tau)\frac{\alpha}{\delta}}| - \exp| -\frac{(x-\frac{\gamma}{\delta}t-\zeta)^2}{4(t-\tau)\frac{\alpha}{\delta}}|\} \tag{25}$$

Also in this case we get non-convergent integral expressions.

The response function of the wave equation (Eq. 4) can be written (Kuchment 1964):

$$h(x,\zeta,t-\tau) = \alpha^2 \beta \sum_{n=1}^{\infty} (1 - \exp\{-\frac{(t-\tau)}{\delta}\}) \sin(\frac{\Pi n}{\ell} x) \quad \cos(\frac{\Pi n}{\ell}\zeta) \tag{26}$$

It is assumed that only a small number of terms in the summation should be accounted for. With this assumption we derive the following principal form for the correlation function:

$$\rho(s) = \frac{1}{N} \sum_{n=1}^{N} \cos\left(\frac{\Pi n}{\ell} s\right) \tag{27}$$

Finally we derive the response function for the potential equation (Eq. 2).

$$h(x,y) = \frac{1}{2\Pi} K_0(\lambda s) \tag{28}$$

and the space correlation function:

$$\rho(s) = \lambda s \, K_1(\lambda s) \tag{29}$$

where $s = \sqrt{x^2 + y^2}$ and the K:s are modified Bessel functions of the second kind. We shall discuss below the application of the derived space correlation functions to different hydrologic situations.

Application of Theoretical Correlation Functions

We shall note at once that discussing the correlation structure of hydrologic phenomena a time interval should first be defined. Due to the non-stationarity of the processes involved in hydrologic phenomena and their different time scales the character of a process studied can change in dependence on the time interval we consider. The space correlation function along a river reach for time intervals which have the same magnitude as the time scale of the river runoff process can at some distance change from negative to positive values and vice versa. For this situation the analytical correlation function Eq. (27) derived from the wave equation may be suitable (Fig. 1).

For large time intervals, say years, we do not observe negative space correlation along a river reach. This is examplified in Fig. 2 where space correlation coefficients along eight Swedish river systems are plotted against the distance between observation points.

For this situation analytical expressions Eqs. (17), (19) and (24) derived from the diffusion equations and Eq. (22) derived from the Kalinin and Miliukov model will give a good fit (Fig. 2).

The suggested models assume random fluctuation in space and time around a mean inflow to a river reach. This is a good approximation for many situations. For the Swedish mountain rivers at the beginning of snowmelt this condition is, however, not fulfilled. For the starting month of snowmelt negative correlation is derived between the mountain and lowland observation points. For other months the correlation is positive very much like the annual correlation.

The analytical correlation function derived from the different equations (Eq. (17) and (24)) from theoretical considerations can also be applicable to one and two dimensional groundwater flow.

Hydrologic elements are highly dependent on physiography. Physiographic

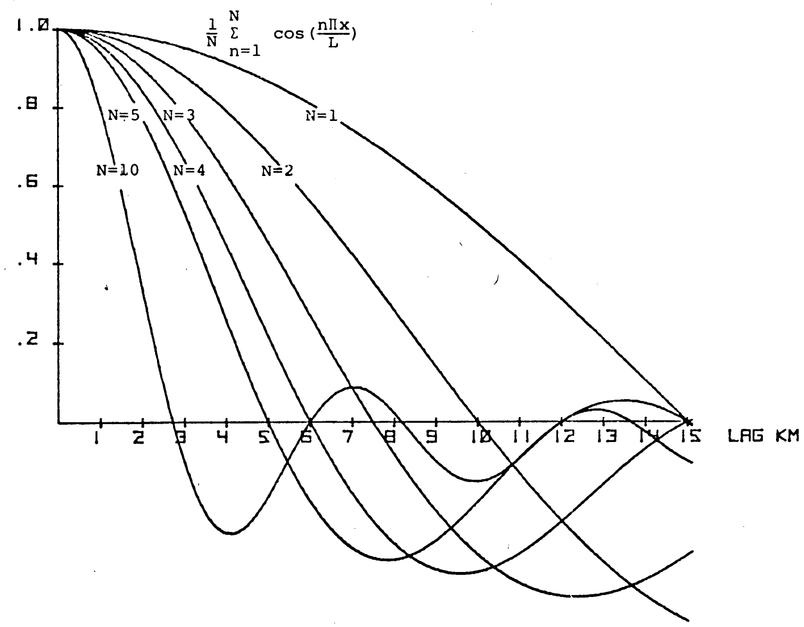

Fig. 1. Theoretical correlation function (Eq. 27).

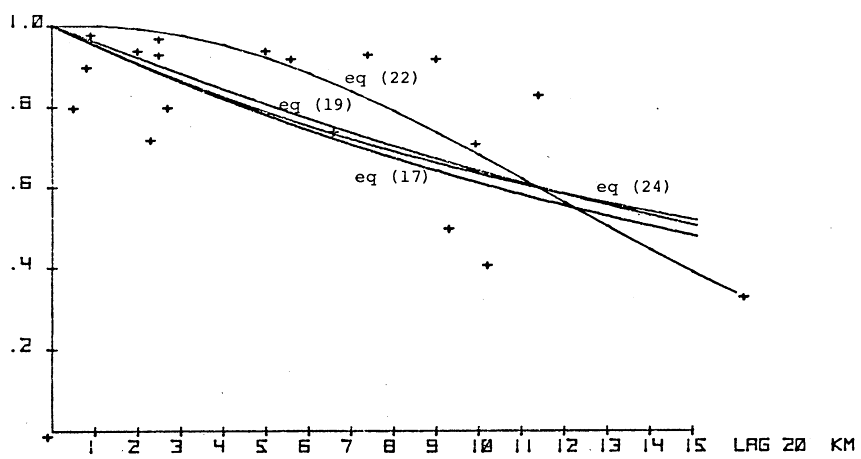

Fig. 2. Theoretical correlation functions (Eqs. (17), (19), (22) and (24)) and observed space correlation along Swedish rivers.

data are available from maps and today also in digital form for grid-nets. Fundamental statistical analysis of such digital physiographic data has been done by I. Krasovskaia (1977) for hydrological purposes. Hydrologic data are always derived as irregular point values and usually are not dense enough for spatial interpolation and integration hydrologic fields. Dense digital regular data sets of physiography together with empirical relations between hydrology and physiography will highly improve such interpolation and integration.

In a large scale landscape are approximated by trend surfaces of different orders. The correlation functions of such surfaces are easily derived. The texture characteristics of the landscape can advantageously be described by the analytical correlation function Eq (29) calculated from the potential equation.

The correlation function gives a thorough description of spatial variations in case of approximately normally distributed data. Precipitation data are highly non-normal and this is why it could be doubtful to use the correlation function in this case to describe spatial dependence. An alternative is to devide precipitation data into classes and use conditional probalities to describe the dependence in space. This is valid for short time intervals. Precipitation values for months and longer time intervals behave nearly normally and in this case the correlation function is again valid. Expression (29) based on the potential equation would give an applicable correlation function.

The correlation function of snow cover depths is also well described by the analytical function Eq. (29) based on the potential equation.

References

Gottschalk, L. (1977) Correlation structure and time scale of simple hydrologic systems. *Nordic Hydrology, Vol. 8 No. 3,* 129-140.

Heine, V. (1955) Models for two-dimensional stationary processes in the plane. *Biometrika 42.*

Kalinin, G. P., and Miliukow, P. I. (1958) Priblizhennyi raschet neustanovivshegosya dvizheniya vodnykh mass. Trudy TSIP vyp 66 Gidrometeoizdat.

Krasovskaia, I. (1978) Digital pshysiographic map data and its analysis for hydrological purposes. 1st Report on NFR project »Spatial variation of hydrology and physiography«. SMHI.

Kuchment, L. S. (1964) Obobśhennaya formula dlya rascheta gidrografa stoka. Trudy TSIP vyp 133 Gidrometeoizdat.

Matern, B. (1960) Spatial variation. Medd Statens Skogsforskningsinst. 49 (5), University of Stockholm.

Whittle, P. (1954) On stationary processes in the plane. *Biometrika 41.*

Whittle, P. (1962) Topographic correlations, power-law covariance functions, and diffusion. *Biometrika 49.*

Received: 1 November, 1978

Adress:

Swedish Meteorological and Hydrological Institute,
Fack,
601 01 Norrköping
Sweden.