Muon Pair Creation by a High Energy Electron

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The cross section for the muon pair creation by an electron is investigated by the Feynman-Dyson method. Its numerical values are calculated at the incident energy about 1 Bev. At the angles of emitted muons \(\theta_+ < \theta_- < 45^\circ\), the main contribution to the cross section comes from the Feynman diagrams in which the virtual photon has a time-like four-momentum.

§ 1. Introduction

As an interesting example of the quantum electrodynamics, the pair creation of Dirac particles by a charged particle was investigated by several authors. The exact treatment was so complicated that various approximations were adopted, e.g. the Weizsäcker-Williams method or similar ones. Their results seemed to coincide with the experiments of cosmic rays, but not with the direct measurements of the so-called trident process. Murata et al. treated this problem at the energies higher than 10 Bev, by the orthodox method based on the Feynman-Dyson's recipe.

At the present time, muons are usually considered to have only the electromagnetic and the weak interactions. Many experiments and theoretical works have been made to find out any unusual behaviour of muons different from electrons. These works mainly concern the anomalous magnetic moment of the muon, and intend to find the structure of the vertex part of the muon interacting with the external photon, which has a space-like four-vector of energy-momentum. Recently, Bjorken and Drell proposed an experiment of muon pair creation as a test of quantum electrodynamics. In the experiment proposed by them, the main contribution to the cross section comes from the Feynman diagrams 2a and 2b in Fig. 2, where the virtual photon has a space-like four-momentum. They ignored the diagrams 1a and 1b in Fig. 1, in which the virtual photon has a time-like four-vector of energy-momentum; this case has not been treated elsewhere.

In this paper, we propose an experiment in which the main contribution arises from the diagrams 1a and 1b. The condition for this case is satisfied.
when the angles and the energies of the emitted muons are not very large, i.e. 
\[ \theta_+ \approx \theta_- \approx 45^\circ \] and \[ E_+ = E_- \approx 490 \text{ Mev} \text{, at the incident energy } E_0 = 1 \text{ Bev}. \]

The general formulas for the cross section are derived in § 2 by the Feynman-Dyson method, and their numerical values are calculated in § 3 as functions of the energies and angles of emitted muons. The adaptability of these results is discussed in § 4, where we see that the characteristic features of the muon distribution are retained in the wider energy region of the incident electron: \( E_0 < 28 \text{ Bev} \).

### § 2. General formulas for the cross section

Throughout this paper, we use the natural unit \( \hbar = c = 1 \), and the following notation:

- \( P_0(p_0, iE_0) \), \( P(p, iE) \) : The initial and the final energy-momentum four-vectors of the electron with the rest mass \( M \).
- \( P_+ (p_+, iE_+) \) : The energy-momentum four-vectors of the created muons with the rest mass \( \mu \).
- \( K(k, iK_0) \) : The energy-momentum four-vector of the virtual photon.
- \( q = p_0 - p - p_+ - p_- \) : The momentum transferred to the target nucleus.

The target nucleus with the electric charge \( Z_e \) is considered as a fixed source of the static Coulomb field. From the diagrams 1a and 1b, we obtain the differential cross section \( d\sigma_1 \):

\[
d\sigma_1 = \frac{(Z_e^4)}{\pi^4} \frac{E_0}{|p_0|} \frac{p_+ E_+ p_- E_-}{K^4 q^4} \frac{1}{|E_+ E_-|} \left[ A + B + C \right] dE_+ dE_- d\Omega_+ d\Omega_- d\Omega,
\]

\[
A = \frac{4E_0^2 - q^2}{2(pq) + q^2} \left[ 2 - \frac{2(vq)}{E} - \frac{(v, p_0 - q)}{E_0} + \frac{(p, p)(p_-, p_0 - q) + (p_-, p)(p_+, p_0 - q)}{E E_0 E_+ E_-} \right] + \frac{\mu^2}{E E_0 E_+ E_-} \left\{ \frac{\mu^2}{E E_0 E_+ E_-} \right\},
\]

\[
B = \frac{4E_0^2 - q^2}{2(pq) + q^2} \left[ 2 - \frac{2(vp_0)}{E_0} - \frac{2(v, p + q)}{E} + \frac{(p, p_0)(p_-, p_0 + q) + (p_-, p_0)(p_+, p_0 + q)}{E E_0 E_+ E_-} \right] + \frac{\mu^2}{E E_0 E_+ E_-} \left\{ \frac{\mu^2}{E E_0 E_+ E_-} \right\},
\]

\[
C = \frac{1}{2(pq) + q^2} \left[ \frac{2(vq)}{E} - \frac{(p, q)(p_+ p_0) + (p_+ p_0)(p_+, p_+ + q)}{E E_0 E_+ E_-} + \frac{\mu^2}{E E_0 E_+ E_-} \right].
\]
where  $v = \frac{1}{2} \left( \frac{p_+}{E_+} + \frac{p_-}{E_-} \right)$.

The terms $A$ and $B$ correspond to the diagrams 1a and 1b respectively, and $C$ is the interference between them. In the same way, the diagrams 2a and 2b give the differential cross section $d\sigma_2$:

\[ d\sigma_2 = \frac{(Ze)^2}{\pi^4} \frac{E_0}{|p_0|} \frac{E_+ E_- E_0}{K^4 q^4 D_+^4 D_-^4} dE_+ dE_- d\Omega_+ d\Omega_- d\Omega, \]

\[ N = \frac{\mu^2}{E_+ E_-} \left[ \frac{4(E_+ D_+ - E_- D_+)^2 - (D_+ - D_-)^2 q^2}{E_0 E_0} \right] \left[ 1 - \frac{(pp_+)}{E_0 E_0} \right] \]

\[ + 4D_+ D_- \left[ \left( \frac{pq}{E_0 E_0} \right) - q^2 \right] + 4(E_+ D_+ - E_- D_+)^2 \left[ 1 - \frac{(pp_+)}{E_0 E_0} \right] \left[ 1 - \frac{(pp_-)}{E_0 E_0} \right] \]

\[ + \left[ 1 - \frac{(pp_+)}{E_0 E_0} \right] \left[ 1 - \frac{(pp_-)}{E_0 E_0} \right] + 2D_+ (E_+ D_+ - E_- D_+) \left\{ \left( \frac{pq}{E_0 E_0} \right) \left[ 1 - \frac{(pp_+)}{E_0 E_0} \right] \right\} \]

\[ + \left( \frac{pq}{E_0} \right) \left[ 1 - \frac{(pp_+)}{E_0 E_0} \right] - \left( \frac{pq}{E_0} \right) \left[ 2 - \frac{(pp_+)}{E_0 E_0} - \frac{(pp_-)}{E_0 E_0} \right] \]

\[ - 2D_+ (E_+ D_+ - E_- D_+) \left\{ \left( \frac{pq}{E_0} \right) \left[ 1 - \frac{(pp_+)}{E_0 E_0} \right] + \left( \frac{pq}{E_0} \right) \left[ 1 - \frac{(pp_-)}{E_0 E_0} \right] \right\} - \left( \frac{pq}{E_0} \right) \left[ 2 - \frac{(pp_+)}{E_0 E_0} - \frac{(pp_-)}{E_0 E_0} \right] \]
\[ + D^2 \sum q \left[ \left( 1 + \frac{(p \cdot p_\perp)}{E_\perp E_\perp} \right) \left( 1 - \frac{(p \cdot p_\perp)}{E_\perp E_\perp} \right) + \left( 1 + \frac{(p_\perp \cdot p_\perp)}{E_\perp E_\perp} \right) \left( 1 - \frac{(p_\perp \cdot p_\perp)}{E_\perp E_\perp} \right) \right] \]
\[ - 2 \left( \frac{p \cdot q}{E_\perp} \right) \left[ \left( \frac{pq}{E} \left( 1 - \frac{(p \cdot p_\perp)}{E_\perp E_\perp} \right) \right) + \left( \frac{pq}{E} \left( 1 - \frac{(p_\perp \cdot p_\perp)}{E_\perp E_\perp} \right) \right) \right] \]
\[ + D^2 \sum q \left[ \left( 1 + \frac{(p_\perp \cdot p_\perp)}{E_\perp E_\perp} \right) \left( 1 - \frac{(p_\perp \cdot p_\perp)}{E_\perp E_\perp} \right) + \left( 1 + \frac{(p_\perp \cdot p_\perp)}{E_\perp E_\perp} \right) \left( 1 - \frac{(p_\perp \cdot p_\perp)}{E_\perp E_\perp} \right) \right] \]
\[ - 2 \left( \frac{p \cdot q}{E_\perp} \right) \left[ \left( \frac{pq}{E} \left( 1 - \frac{(p \cdot p_\perp)}{E_\perp E_\perp} \right) \right) + \left( \frac{pq}{E} \left( 1 - \frac{(p_\perp \cdot p_\perp)}{E_\perp E_\perp} \right) \right) \right] \]
\[ - 2 \left( \frac{p \cdot q}{E_\perp} \right) \left[ \left( \frac{pq}{E} \left( 1 - \frac{(p \cdot p_\perp)}{E_\perp E_\perp} \right) \right) + \left( \frac{pq}{E} \left( 1 - \frac{(p_\perp \cdot p_\perp)}{E_\perp E_\perp} \right) \right) \right] \]
\[ \text{where } D = 2(p \cdot q) + q^2. \]

These formulas are exact ones without any approximations. Besides (1) and (2), the cross section includes another term corresponding to the interference between (1) and (2). Nevertheless, as we are interested in the case \( d\sigma_1 \gg d\sigma_2 \), the interference term is not calculated here.

### § 3. Energy and angular distribution of muons

For the purpose of the experimental design, the emitted directions of muons are fixed and the angles of the final electrons are integrated. We adopt the following notation for the angles. The \( z \)-axis is parallel to \( p_\perp + p_\perp \) and the azimuthal angles are measured from the plane of the muon pair as shown in Fig. 3.

For the simplicity we take \( \theta_0 = 0, \theta_+ = \theta_- \) and \( E_+ = E_- = E \), then the integration over the variables \( \theta \) and \( \varphi \) is performed by the elementary method. We obtain from (1) and (2),

\[
\left( \frac{d\sigma_1}{dE dE d\Omega d\Omega} \right)_{\theta_+ = \theta_- = \epsilon} = 2 \left( \frac{Ze^3}{\pi^3} \right)^2 \left( \frac{e^2}{m} \right)^2 \frac{m^2 E \rho_{E^2 k^2}^3}{(4E^2 - k^2)^2} \left[ A + B + C \right],
\]
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\[ A = \left\{ - \frac{k^2}{E \varepsilon} + \frac{\mu^2}{\varepsilon^3} \left( 1 - \frac{p^2}{E E_0} \right) - \frac{p^\mu p^\mu \sin^2 \theta}{E E \varepsilon^2} \right\} \left( 4 E E_0 I_{1u} - I_{1s} \right) \]

\[ + \frac{p^2}{E E_0} \left( \frac{k^2}{2 \varepsilon^2} - E_0 - E \frac{\mu^2}{\varepsilon} \right) \left( 4 E E_0 I_{1u} - I_{1s} \right) \]

\[ + \frac{p^2}{E E \varepsilon^2} (1 + \cos^2 \theta) \left( 4 E E_0 I_{1u} + I_{1s} - I_{1u} \right) \]

\[ + \left\{ \frac{1}{2 - \frac{E}{E_0}} \left( \frac{2 - k}{\varepsilon} \right) \left( 1 - \frac{k}{E_0} \right) \left( 1 - \frac{k}{E_0} \right) \right\} I_{1s} \]

\[ B = \left\{ \frac{1}{2 - \frac{E}{E_0}} \left( \frac{2 - k}{\varepsilon} \right) \left( 1 - \frac{k}{E_0} \right) \left( 1 - \frac{k}{E_0} \right) \right\} \left( \frac{4 E E_0 I_{1u} - I_{1s}}{(p_0 - k)^2 - p^2} \right) \]

\[ + \left\{ \frac{2}{(p_0 - k)^2 + p^2} \right\} \left( \frac{4 E E_0 I_{1u}}{(p_0 - k)^2 - p^2} \right) \]

\[ C = \left\{ \frac{2}{(p_0 - k)^2 + p^2} \right\} \left( \frac{4 E E_0 I_{1u}}{(p_0 - k)^2 - p^2} \right) \]

\[ + \left\{ \frac{2}{(p_0 - k)^2 + p^2} \right\} \left( \frac{4 E E_0 I_{1u}}{(p_0 - k)^2 - p^2} \right) \]

\[ \left\{ \frac{2}{(p_0 - k)^2 + p^2} \right\} \left( \frac{4 E E_0 I_{1u}}{(p_0 - k)^2 - p^2} \right) \]

\[ \left\{ \frac{2}{(p_0 - k)^2 + p^2} \right\} \left( \frac{4 E E_0 I_{1u}}{(p_0 - k)^2 - p^2} \right) \]

where

\[ I_{1u} = \int_0^\pi (\cos \theta)^i \sin \theta \, d\theta \]

\[ \left\{ \frac{2}{(p_0 - k)^2 + p^2} \right\} \left( \frac{4 E E_0 I_{1u}}{(p_0 - k)^2 - p^2} \right) \]
And
\[
\left( \frac{d\sigma_2}{dE_2 dE d\Omega d\Omega} \right)_{\theta_+^{\pm} = \phi_+^{\pm} = \theta,-} = \frac{2 (Ze)^2 e^2}{m^3} \left( \frac{e^2}{m} \right)^3 \left( \frac{E_0^2}{E_0 - p_0^2} \right) \sin^2 \theta_+^{\pm}.
\]

(4)

In the derivation of (4), we used an approximation \( E \gg m \). This is satisfied with an error less than 1%, when \( E_+ = E_- < 490 \text{ Mev} \) at \( E_0 = 1 \text{ Bev} \). The numerical values of (3) and (4) are calculated at \( E_0 = 1 \text{ Bev} \) and the results are listed in Table I.

Table I. Numerical values of \( \langle d\sigma \rangle Z^2 dE_+ dE_0 d\Omega d\Omega \rangle_{\phi_+^{\pm} = \phi_+^{\pm} = \theta,-} \) and \( \langle d\sigma \rangle Z^2 dE_+ dE_0 d\Omega d\Omega \rangle_{\phi_+^{\pm} = \phi_+^{\pm} = \theta,-} \) in \( \text{cm}^2 \text{ Bev}^{-2} \text{ strad}^{-2} \).

<table>
<thead>
<tr>
<th>( E_0 = E_- ) (Bev)</th>
<th>10°</th>
<th>20°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>1.61 \times 10^{-34}</td>
<td>3.59 \times 10^{-35}</td>
<td>7.52 \times 10^{-36}</td>
<td>1.59 \times 10^{-36}</td>
<td>4.50 \times 10^{-37}</td>
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<td>(\sigma_1)</td>
<td></td>
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</tr>
<tr>
<td>0.30</td>
<td>5.98 \times 10^{-37}</td>
<td>1.00 \times 10^{-36}</td>
<td>7.28 \times 10^{-37}</td>
<td>2.81 \times 10^{-37}</td>
<td>9.78 \times 10^{-37}</td>
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<td>(\sigma_2)</td>
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<td>0.40</td>
<td>8.55 \times 10^{-34}</td>
<td>1.40 \times 10^{-34}</td>
<td>2.10 \times 10^{-35}</td>
<td>1.29 \times 10^{-35}</td>
<td>2.29 \times 10^{-37}</td>
</tr>
<tr>
<td>(\sigma_3)</td>
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</tr>
<tr>
<td>0.45</td>
<td>7.58 \times 10^{-36}</td>
<td>5.23 \times 10^{-36}</td>
<td>1.83 \times 10^{-36}</td>
<td>3.59 \times 10^{-37}</td>
<td>8.81 \times 10^{-38}</td>
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<td>(\sigma_4)</td>
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<td>2.42 \times 10^{-34}</td>
<td>2.98 \times 10^{-35}</td>
<td>9.38 \times 10^{-37}</td>
<td>8.10 \times 10^{-38}</td>
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<td>(\sigma_5)</td>
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<td></td>
<td>2.84 \times 10^{-35}</td>
<td>9.06 \times 10^{-36}</td>
<td>2.02 \times 10^{-36}</td>
<td>2.92 \times 10^{-37}</td>
<td>6.24 \times 10^{-38}</td>
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<td>(\sigma_6)</td>
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<td>7.32 \times 10^{-37}</td>
<td>4.85 \times 10^{-38}</td>
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<td>(\sigma_7)</td>
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<td>1.01 \times 10^{-35}</td>
<td>1.94 \times 10^{-36}</td>
<td>2.54 \times 10^{-37}</td>
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<tr>
<td>(\sigma_8)</td>
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</table>

At the angles \( \theta_+ = \theta_- < 45° \) and at the energies \( E_+ = E_- < 490 \text{ Mev} \), the main contribution to the cross section comes from the diagrams 1a and 1b, and the contribution from the diagrams 2a and 2b can be neglected. At \( \theta_+ = \theta_- = 60° \), they become comparable to each other.

§ 4. Concluding remarks

As we took the special angles \( \theta_0 = 0 \) and \( \theta_+ = \theta_- \) in § 3, the deviation from this special case must be considered. The main contribution to the cross section arises from...
the term $B$ in (3) and sharply depends on the denominator $q^2[2(pq) + q^2]$. This denominator is expanded for a small value of $\theta_0$ and the ratio of the cross section for the two cases is obtained:

$$\frac{\sigma(\theta_0)}{\sigma(\theta_0=0)} \approx 1 - \left[ \frac{p(p+k)}{q^2} + \frac{2p\cdot k}{2(pq) + q^2} \right] \theta_0^2 = 1 - c \theta_0^2$$  \hspace{1cm} (5)

where the coefficient $c$ depends on $\theta_0$ and $E_\perp$. The approximate values of $\theta_0$ are given in Table II, under the condition that the deviation $c \theta_0^2$ is 1% and 5% of the maximum intensity. When the muon direction $\theta_\perp$ fluctuate, the situation is almost the same as $\theta_0$.

In the preceding section we calculated the cross section at the fixed incident energy 1 Bev, but the characteristic features obtained there are not changed in the wider energy region. Since the cross section (3) depends sharply on the minimum value of the transferred momentum, the most predominant contribution comes from the term $B$ in (3) as long as the following inequality is satisfied,

$$E_0 > 2\mu \Rightarrow E_{0m} > 2\mu$$  \hspace{1cm} (6)

where $E_0/2 \approx \xi$ is the energy of the created muon and $E_{0m}/2\mu \approx E$ is the final energy of the electron, while the transferred momentum takes its minimum value $q_{min} \approx 2\mu^2/E_0$. The inequality (6) corresponds to the energy region $0.2$ Bev $< E_0 < 28$ Bev. This lower limit of the incident energy, however, is not adequate for the following reason.

If the incident energy is small as $E_0 \sim 2\mu$, the transferred momentum becomes large as $q \sim \mu$, which has the Compton wavelength of $1.7 \times 10^{-13}$ cm. Then the target nucleus is not regarded as a point source, and the effect of the charge distribution in the nucleus must be considered. The same situation arises also at higher energies when the emitted angles are large. The target nucleus which can be regarded as a point source is determined by the inequality $(q_{min})^{-1} \gtrsim \rho A^{1/3}$ (nuclear radius). The minimum value of the transferred momentum and the maximum mass number of the target nucleus at this condition are given in Table III.

Table II. The angles $\theta_0$ which give the deviation of 1% and 5% of the maximum intensity.

<table>
<thead>
<tr>
<th>$\theta_0$</th>
<th>$\theta_0^2=1%$</th>
<th>$\theta_0^2=5%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_\perp = \theta_\perp$</td>
<td>$0^\circ$</td>
<td>$15^\circ$</td>
</tr>
<tr>
<td>$\theta_0$</td>
<td>$0.1^\circ$</td>
<td>$0.4^\circ$</td>
</tr>
<tr>
<td>$\theta_\perp$</td>
<td>$0.2^\circ$</td>
<td>$0.9^\circ$</td>
</tr>
</tbody>
</table>

Table III. The minimum value of the transferred momentum $q_{min}$ and the maximum mass number $A$.

<table>
<thead>
<tr>
<th>$\theta_0$</th>
<th>$0^\circ$</th>
<th>$10^\circ$</th>
<th>$20^\circ$</th>
<th>$30^\circ$</th>
<th>$45^\circ$</th>
<th>$60^\circ$</th>
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</thead>
<tbody>
<tr>
<td>$q_{min}$</td>
<td>$0.21\mu$</td>
<td>$0.39\mu$</td>
<td>$0.70\mu$</td>
<td>$1.0\mu$</td>
<td>$1.5\mu$</td>
<td>$1.7\mu$</td>
</tr>
<tr>
<td>$A = (r_0 q_{min})^{-3}$</td>
<td>$180$</td>
<td>$13.0$</td>
<td>$4.2$</td>
<td>$1.5$</td>
<td>$0.6$</td>
<td>$0.3$</td>
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</tbody>
</table>
In the measurements at large angles, light nuclei must be used as targets, which recoil at the scattering and decrease the cross section further. Therefore, the experiment should be designed at the conditions that the incident energy is as high as possible and the emitted angles of muons are as small as possible. These conditions will not only increase the cross section but also decrease the ambiguities due to the complicated mechanisms mentioned above.

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