Applying Cini-Fubini approximation to the Mandelstam representation for pion-nucleon scattering amplitude, we treat explicitly the effect of pion-pion interaction to be added to the dispersion relation given by Chew, Goldberger, Low and Nambu. On account of scanty information on the contribution to the dispersion integral from high-mass states, we ought to introduce constant parameters through a subtraction. In the dispersion relations thus obtained, the integral over the branch cut extending $4\mu^2$ to $\infty$ are calculated by a method similar to that of Frazer and Fulco, but using one more subtraction in order to avoid the divergence regarding the left-hand cut. From the comparison of the dispersion relations with experimental data of pion-nucleon scattering, we investigate the property of $I=0$ (S-wave) pion-pion interaction, and with the result that the interaction is attractive and the magnitude of scattering length is $0.05-0.2 \, \mu^{-1}$. This is still one order smaller as compared with the recent experimental data. In connection with this, we briefly discuss about the necessity of another new parameter.

§ 1. Introduction

Recently, experimental evidences have appeared which might be considered to show the existence of the strong pion-pion interaction. For instance, the P-wave pion-pion scattering resonance has been found from data on single pion production in pion-nucleon collisions, and the S-wave pion-pion interaction from the experiment of $p+d\rightarrow He+\pi+\pi$. On the other hand, the existence of pion-pion scattering resonance in $J=1$ and $I=1$ state was shown to be expected from dispersion theoretic analysis of nucleon form factor by Frazer and Fulco (F-F) and by Bowcock, Cottingham and Lurie (B). The effect of the P-wave pion-pion interaction on the pion-nucleon scattering at low energies has also been investigated by B and by Frautschi and Walecka, and the effect of $I=0$ (S-wave) pion-pion interaction by Sato, Takahashi and Ueda, and by Ishida, Takahashi and Ueda (A).

In the analysis given by A, it was assumed that the imaginary part of pion-nucleon scattering amplitude included in the term which involved the pion-pion scattering amplitude explicitly could be expanded in Legendre polynomials. As shown by F-F, however, the region of convergence of this Legendre expansion is limited, so that the calculations in A are to be examined more closely. In addition, the dispersion relations in A did not satisfy crossing symmetry.

In this paper, we shall attempt to remedy these defects, applying the Cini-Fubini approximation on the Mandelstam representation for the pion-nucleon
scattering amplitudes, and using the method similar to that of F-F for the left-hand cut. To the dispersion relations given by Chew, Goldberger, Low and Nambu (CGLN), we introduce explicitly the pion-pion terms which are the integrals on \( t \), the square of the momentum transfer between two pions with reversed sign, from \( 4\mu^2 \) to \( \infty \). In this representation, the crossing symmetry is always satisfied.

At present, however, we have not much information on the contribution in the dispersion integrals from high mass states, so it will be necessary to introduce some constant parameters by a subtraction in the dispersion relations. In this paper, we shall introduce three parameters \( C_A^{(+)} \), \( C_A^{(-)} \) and \( C_B^{(-)} \) into the dispersion relations. These parameters can be inferred directly from the experimental data of pion-nucleon scattering, using the dispersion relations in the forward direction. It is important to decide how many parameters are really needed at present. In this paper, we take the standpoint that the number and properties of the parameters should be determined from comparison of the dispersion relations with experimental data.

In these dispersion relations, each pion-pion term which is the integral on \( t \) over the branch cut extending from \( 4\mu^2 \) to \( \infty \) is to be calculated. In order to perform this calculation, we must solve an integral equation including the pion-pion term by the method similar to that of F-F. The solution of this equation can be expressed as the integral on \( t \) over the left-hand cut extending from \( a=4\mu^2(1-\mu^2/4m^2) \) to \( -\infty \), its integrand being written in terms of a pion form factor and the imaginary part of pion-nucleon scattering amplitude. If the Legendre expansion for the imaginary part of pion-nucleon scattering is performed and we retain the terms only up to \( P \)-wave, then the integral included in the solution may be divergent, and we can obtain no reliable results, even though the cutoff procedure is introduced. For this reason, we again use subtraction technique for treatment of the left-hand cut. The integral is now convergent and we can safely cut off the integral at \( t=-26\mu^2 \). The subtraction constants thus introduced are also related to the pion-nucleon scattering at forward and can be estimated directly from the experimental knowledges on pion-nucleon scattering using the dispersion relations in the forward direction.

In this way, we obtain the basic relations to be used for the analysis of pion-nucleon scattering as well as electromagnetic structure of nucleon. In this paper, we analyse the property of \( I=0 \) (S-wave) pion-pion interaction using these dispersion relations. Regarding the pion form factor occurring in the pion-pion term, we shall assume the scattering length approximation for the pion-pion scattering phase shift. Then, in the dispersion relations all quantities except for this unknown scattering length can be expressed in terms of phase shifts of pion-nucleon scattering, its coupling constant and the value of parameter \( C_A^{(+)} \). Therefore, comparing the dispersion relation with the experimental data, we can estimate the scattering length of pion-pion scattering. And we obtain
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the result that the \( I=0 \) (S-wave) interaction is attractive and the magnitude of scattering length \( a_8 \) is \( 0.05 \sim 0.2 \mu^{-1} \).

Before comparing the above result with experimental data, we shall arrange the approximations used in this paper in the following order: (i) Using one subtraction, we can replace phenomenologically the contributions for dispersion integrals from high mass states by an adjustable parameter \( C_A^{(+)} \). (ii) In this subtracted form, the effects of higher mass states than two pions in the pion-pion term can be neglected. (iii) We can get a reliable information about the integral over the left-hand cut by using one subtraction. (iv) We can neglect the partial pion-pion scattering amplitudes other than \( S \)- and \( P \)-waves. (v) We can approximate the \( S \)-wave pion-pion scattering phase shift by a scattering length.

These approximations are briefly examined as follows: From the fact that the dispersion relation for \( B^{(+)}(s, t) \), Eq. (Ic), is consistent with the experimental data, we can expect that (iv) is a good approximation at low energy regions. On the other hand, it is very difficult at present to correct the approximation (v), but from the form of \( F_\omega^8(t) \) the main contributions come from pion-pion scattering phase shift at low energy regions, and we may expect that (v) is also a reasonable approximation. The approximation (ii) seems to be unavoidable at present. As we use the subtracted dispersion relations, however, we expect that these effects are negligible. The approximation (iii) is questionable. Only with the assumption that the partial \( \pi-N \) scattering amplitudes other than \( S \) and \( P \)-waves are negligible on the left-hand cut, the effect of rescattering may be reasonably included in the integral over the cut by the subtraction technique. As we have no other method than the Legendre expansion for evaluation of the integral, we can not avoid the inaccuracy in the treatment of the left-hand cut.

Comparing with the experimental data of \( S \)-wave pion-pion interaction, we shall mention under the above approximations the possibility of introducing one more subtraction in our dispersion relations, Eq. (Ia'). From the most recent experiment \( p+d \rightarrow He+\pi+\pi,^9 \) \( a_8=2 \sim 3 \mu^{-1} \) has been required, which is one order larger than ours. If this large scattering length is valid, our dispersion relation for \( A^{(+)}(s, t) \) should be reformed under the above assumptions (ii) \( \sim \) (v), performing one more subtraction and thus introducing one more new parameter \( C_A^{(+)} \) in our dispersion relation. If this is the case, the effect of \( I=0 \) (S-wave) pion-pion interaction to pion-nucleon scattering at low energies is masked by \( C_A^{(+)} \), so that the dynamical effect of the interaction should be analysed at higher energies.

In § 2 dispersion relations are derived and the new parameters \( C_A^{(+)} \), \( C_A^{(-)} \) and \( C_B^{(-)} \) are introduced, and in § 3 the pion-pion term are analysed by the method similar to that of \( F-F \), but using the subtraction. In § 4, we shall report the numerical values of parameters \( C_A^{(+)} \), \( C_A^{(-)} \) and \( C_B^{(-)} \), and of the subtraction
constants \(g_A^{(+)}(0), \ g_A^{(-)}(0)\) and \(g_B^{(-)}(0)\) introduced in §3. And we analyse the property of \(I=0\) (S-wave) pion-pion interaction. Comparing the result with the recent experimental data, we briefly discuss about the necessity of new parameter \(C_A^{(+)}\). In §5 the role of the parameters is discussed.

§2. Kinematics and dispersion relations

Let the 4-momenta of the pions be \(k_1\) and \(k_2\), and those of the nucleon \(p_1\) and \(p_2\). Define the variables

\[
s = -(p_1 + k_1)^2, \quad \bar{s} = -(p_1 - k_2)^2, \quad t = -(k_1 - k_2)^2.
\]

They are connected by the relation

\[
s + \bar{s} + t = 2m^2 + 2\mu^2,
\]

where \(m\) and \(\mu\) are the masses of nucleon and pion respectively. In the process \(\pi(k_1) + N(p_1) \rightarrow \pi(k_2) + N(p_2)\), these variables are expressed in the center of mass system as follows:

\[
s = W^2 = (E + \omega)^2,
\]

\[
\bar{s} = (E - \omega)^2 - 2k^2(1 + \cos \theta),
\]

\[
t = -2k^2(1 - \cos \theta),
\]

where \(E\) and \(\omega\) are the energy of nucleon and pion respectively, \(W\) is the total energy of the system, \(k\) is the magnitude of 3-momenta of pion, and \(\theta\) is the scattering angle \((\cos \theta = k_1 \cdot k_2 / k^2)\). In the process \(\pi(k_1) + \pi(-k_2) \rightarrow \bar{N}(-p_1) + N(p_2)\), on the other hand, they are written in the center of mass system

\[
s = -p^2 - q^2 + 2pq \cos \varphi,
\]

\[
\bar{s} = -p^2 - q^2 - 2pq \cos \varphi,
\]

\[
t = 4(q^2 + \mu^2) = 4(p^2 + m^2),
\]

where \(p\) and \(q\) are the magnitudes of the 3-momenta of nucleon and pion respectively, and \(\varphi\) is the angle between incident pion and final nucleon.

The amplitudes \(A^{(\pm)}(s, t)\) and \(B^{(\pm)}(s, t)\) for the process I defined by Chew, Goldberger, Low and Nambu are decomposed into partial wave amplitudes as follows:

\[
A^{(\pm)} = \left[ (W + m) / (E + m) \right] f_1^{(\pm)} - \left[ (W - m) / (E - m) \right] f_2^{(\pm)},
\]

\[
B^{(\pm)} = \left[ 1 / (E + m) \right] f_1^{(\pm)} + \left[ 1 / (E - m) \right] f_2^{(\pm)},
\]

where \(f_1\) and \(f_2\) are, suppressing superscripts \((\pm)\) referred to charge state,

\[
f_1 = \sum_{i=\pm}^m f_{i+}(s) P_{i+}(x) - \sum_{i=\mp}^m f_{i-}(s) P_{i-}(x),
\]

\[
f_2 = \sum_{i=\pm}^m f_{i+}(s) P_{i-}(x) - \sum_{i=\mp}^m f_{i-}(s) P_{i+}(x),
\]

\[
f_3 = \sum_{i=\pm}^m f_{i+}(s) P_{i-}(x) + \sum_{i=\mp}^m f_{i-}(s) P_{i+}(x),
\]

\[
f_4 = \sum_{i=\pm}^m f_{i+}(s) P_{i-}(x) - \sum_{i=\mp}^m f_{i-}(s) P_{i+}(x),
\]

\[
f_5 = \sum_{i=\pm}^m f_{i+}(s) P_{i+}(x) + \sum_{i=\mp}^m f_{i-}(s) P_{i-}(x),
\]

\[
f_6 = \sum_{i=\pm}^m f_{i+}(s) P_{i+}(x) - \sum_{i=\mp}^m f_{i-}(s) P_{i-}(x),
\]

\[
f_7 = \sum_{i=\pm}^m f_{i+}(s) P_{i-}(x) + \sum_{i=\mp}^m f_{i-}(s) P_{i+}(x),
\]

\[
f_8 = \sum_{i=\pm}^m f_{i+}(s) P_{i+}(x) + \sum_{i=\mp}^m f_{i-}(s) P_{i-}(x),
\]

\[
f_9 = \sum_{i=\pm}^m f_{i+}(s) P_{i-}(x) + \sum_{i=\mp}^m f_{i-}(s) P_{i+}(x),
\]

\[
f_{10} = \sum_{i=\pm}^m f_{i+}(s) P_{i+}(x) + \sum_{i=\mp}^m f_{i-}(s) P_{i-}(x),
\]
In the above, \( f_{i \pm} \) is the scattering amplitude in the state of parity \((-1)^i\) and total angular momentum \(j = l \pm 1/2\), and is written in terms of phase shift
\[
f_{i \pm} = e_{i \pm} \sin \delta_{i \pm}/k. \tag{2.7}
\]

For the process III, the partial wave decomposition of \( A^{(\pm)}(s, t) \) and \( B^{(\pm)}(s, t) \) are given by, as done by P-F and B,
\[
A^{(\pm)}(t, \cos \varphi) = (8\pi/p^2) \sum (J+1/2) (pq)^J \left\{ mf^{(\pm)J}_{-}(t) \cos \varphi P_j'(\cos \varphi) / \sqrt{J(J+1)} \right. \\
- f^{(\pm)J}_{+}(t) P_j(\cos \varphi) \right\}, \tag{2.8a}
\]
\[
B^{(\pm)}(t, \cos \varphi) = 8\pi \sum (J+1/2) (pq)^{-J} f^{(\pm)J}_{-}(t) P_j'(\cos \varphi) / \sqrt{J(J+1)}, \tag{2.8b}
\]
where \( J \) is the total angular momentum of the state, and subscripts \( \pm \) are referred to definite helicities of nucleon anti-nucleon pair; \(+\) for the same helicity and \(-\) for the opposite helicity. The inverse of Eqs. (2.8a) and (2.8b) are
\[
f^{(\pm)J}_{\pm}(t) = \frac{1}{8\pi} \left\{ - \frac{p^2}{(pq)^J} A^{(\pm)J}_{\pm}(t) + \frac{m}{(2J+1)(pq)^{-J}} \left\{ (J+1) B^{(\pm)J}_{\pm}(t) + JB^{(\pm)J}_{\pm}(t) \right\} \right\}, \tag{2.9a}
\]
\[
f^{(\pm)J}_{-}(t) = \frac{1}{8\pi} \frac{\sqrt{J(J+1)}}{(2J+1)(pq)^{-J}} \left\{ B^{(\pm)J}_{\pm}(t) - B^{(\pm)J}_{\pm}(t) \right\}, \tag{2.9b}
\]
where
\[
\left[ A^{(\pm)J}_{\pm}(t) ; B^{(\pm)J}_{\pm}(t) \right] = \int_{-1}^{1} dx P_{J}(x) \left[ A^{(\pm)J}(s(x), t) ; B^{(\pm)J}(s(x), t) \right]. \tag{2.10}
\]

The \((\pm)\) amplitudes are written in terms of the total isotopic spin as follows:

For process I
\[
A^{(+)} = (1/3) \left[ A^{(1/2)} + 2 A^{(3/2)} \right], \quad A^{(-)} = (1/3) \left[ A^{(3/2)} - A^{(1/2)} \right], \tag{2.11}
\]
and for process III
\[
A^{(+)} = (1/\sqrt{6}) A^{(0)}, \quad A^{(-)} = (1/2) A^{(3)}. \tag{2.12}
\]

Finally, the amplitudes \( A^{(\pm)}(s, t) \) and \( B^{(\pm)}(s, t) \) satisfy the crossing relations
\[
A^{(\pm)}(s, \bar{s}, t) = \pm A^{(\pm)}(s, t), \tag{2.13a}
\]
\[
B^{(\pm)}(s, \bar{s}, t) = \mp B^{(\pm)}(s, t). \tag{2.13b}
\]

Applying the Mandelstam representations for \( A^{(\pm)}(s, t) \) and \( B^{(\pm)}(s, t) \), we can infer the analytic properties of these amplitudes. With \( t \) fixed, the Mandelstam representation for the pion-nucleon scattering amplitude \( A^{(+)}(s, t) \), for example, leads to the dispersion relation given by CGLN,
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\[ A^{(+)}(s, t) = \frac{1}{\pi} \int_{(m+\rho)^2}^{\infty} ds' \text{Im} A^{(+)}(s', t) \left\{ \frac{1}{s' - s} + \frac{1}{s' - \bar{s}} \right\}. \quad (2.14) \]

The absorptive part \( \text{Im} A^{(+)}(s', t) \), when expressed as the sum of products of two amplitudes through unitarity condition, is divided into two parts:

\[ \text{Im} A^{(+)}(s', t) = \text{Im} A^{(+)}_{\text{el}}(s', t) + \text{Im} A^{(+)}_{\text{int}}(s', t), \quad (2.15) \]

according whether the associated intermediate states involve again one pion (besides one nucleon) or two pions or more \([s' \geq (m+2\mu)^2]\). Thus, for \( \text{Im} A^{(+)}_{\text{el}}(s', t) \), the cut in \( t \) begins only at \( 16\mu^2 \), though the cut in \( s \) does at \((m+\mu)^2\). This suggests that we may quite well expand \( \text{Im} A^{(+)}_{\text{el}}(s', t) \) into partial waves, leaving only up to \( P \)-wave. In \( \text{Im} A^{(+)}_{\text{int}}(s', t) \), the cut in \( s \) begins at \((m+2\mu)^2\), but the cut in \( t \) starts already at \( 4\mu^2 \). We recall the spectral representation for \( \text{Im} A^{(+)}_{\text{int}}(s', t) \):

\[ \text{Im} A^{(+)}_{\text{int}}(s', t) = \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{a^{(+)}_{\text{el}}(s', t')}{t' - t} = \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\delta A^{(+)}(s, t')}{t' - t}. \quad (2.16) \]

where \( N(s') < -4m\mu \). In the dispersion integral of \( \text{Im} A^{(+)}_{\text{int}}(s', t) \) in Eq. (2.14), the second term in Eq. (2.16) gives only weak dependence on \( s \) and \( t \) at the low energy pion-nucleon scattering because \( s' \geq (m+2\mu)^2 \) and \( t' < -4m\mu \). Thus we shall replace the contribution of this term to Eq. (2.14) as a real constant \( C^{(+)} \). On the other hand, since the dispersion integral from the first term in Eq. (2.16) is not expansible in \( t \) in virtue of the low singularity, \( t' \geq 4\mu^2 \), we put simply

\[ \frac{1}{\pi} \int_{(m+\rho)^2}^{\infty} ds' \frac{1}{s' - s} \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{a^{(+)}_{\text{el}}(s', t')}{t' - t} = \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\delta A^{(+)}(s, t')}{t' - t}. \quad (2.17) \]

In this way, the dispersion relation (2.14) takes the following form:

\[ A^{(+)}(s, t) = \frac{1}{\pi} \int_{(m+\rho)^2}^{\infty} ds' \text{Im} A^{(+)}_{\text{el}}(s', t) \left\{ \frac{1}{s' - s} + \frac{1}{s' - \bar{s}} \right\} + C^{(+)} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\delta A^{(+)}(s, t')}{t' - t}. \quad (2.18a) \]

Similarly the dispersion relations for \( A^{(-)}(s, t), B^{(+)}(s, t) \) and \( B^{(-)}(s, t) \) can be written

\[ A^{(-)}(s, t) = \frac{1}{\pi} \int_{(m+\rho)^2}^{\infty} ds' \text{Im} A^{(-)}_{\text{el}}(s', t) \left\{ \frac{1}{s' - s} - \frac{1}{s' - \bar{s}} \right\} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\delta A^{(-)}(s, t')}{t' - t}, \quad (2.18b) \]

\[ B^{(+)}(s, t) = g_r \left\{ \frac{1}{m^2 - s} - \frac{1}{m^2 - \bar{s}} \right\} + \frac{1}{\pi} \int_{(m+\rho)^2}^{\infty} ds' \text{Im} B^{(+)}_{\text{el}}(s', t) \left\{ \frac{1}{s' - s} - \frac{1}{s' - \bar{s}} \right\} \]
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\[ B^{(-)}(s, t) = g_s^2 \left( \frac{1}{m_s^2 - s} + \frac{1}{m_s^2 - \bar{s}} \right) + \frac{1}{\pi} \int ds' \operatorname{Im} B_d^{(-)}(s', t) \left\{ \frac{1}{s' - s} + \frac{1}{s' - \bar{s}} \right\} \]

\[ + C_{A_{d}^{(-)}}(s, t) + \frac{1}{\pi} \int dt' \frac{\partial B_{d}^{(-)}(s, t')}{t' - t}. \]  

In CGLN it is assumed that the absorptive parts \( \operatorname{Im} A_c^{(\pm)} \) and \( \operatorname{Im} B_c^{(\pm)} \) can be expanded in partial wave amplitudes of pion-nucleon scattering, and that the effects of the singularity at \( t = 4\mu^2 \) can be neglected. In this sense, the Born terms plus dispersion integrals involving \( \operatorname{Im} A_{d}^{(\pm)}(s', t) \) and \( \operatorname{Im} B_{d}^{(\pm)}(s', t) \) in Eq. (2·18) are referred to as CGLN terms, denoting as \( A_c^{(\pm)} \) and \( B_c^{(\pm)} \). Then the dispersion relations (2·18) are written

\[ A^{(+)}(s, t) = A_c^{(+)}(s, t) + C_{A_{c}^{(+)}} \left( \frac{1}{\pi} \int dt' \frac{\partial A_{c}^{(+)}(s, t')}{t' - t} \right), \]  

\[ A^{(-)}(s, t) = A_c^{(-)}(s, t) + \frac{1}{\pi} \int dt' \frac{\partial A_{c}^{(-)}(s, t')}{t' - t}, \]  

\[ B^{(+)}(s, t) = B_c^{(+)}(s, t) + \frac{1}{\pi} \int dt' \frac{\partial B_{c}^{(+)}(s, t')}{t' - t}, \]  

\[ B^{(-)}(s, t) = B_c^{(-)}(s, t) + C_{B_{c}^{(-)}} \left( \frac{1}{\pi} \int dt' \frac{\partial B_{c}^{(-)}(s, t')}{t' - t} \right). \]

If the singularities at \( t \geq 16\mu^2 \) are neglected in Eq. (2·19), the functions \( \partial A_c^{(\pm)} \) and \( \partial B_c^{(\pm)} \) are equal to the imaginary parts of \( A^{(\pm)}(t, \cos \varphi) \) and \( B^{(\pm)}(t, \cos \varphi) \) in Eq. (2·8) respectively in the region \( t \geq 4\mu^2 \), which corresponds to the physical energy region of pion-pion scattering. We rewrite \( \partial A^{(+)} \), \( \partial A^{(-)} \), \( \partial B^{(+)} \) and \( \partial B^{(-)} \) as \( \Im g_A^{(+)}(t) \), \( (s - \bar{s}) \Im g_A^{(-)}(t) \), \( (s - \bar{s}) \Im g_B^{(+)}(t) \) and \( \Im g_B^{(-)}(t) \) respectively and confine ourselves to the \( S \)- and \( P \)-wave pion-pion scattering. Then, from Eqs. (2·8a) and (2·8b), we get

\[ g_A^{(+)}(t) = (4\pi/p^2) f^{(+)}(t), \]  

\[ g_A^{(-)}(t) = (3\pi/p^2) \{ f^{(-)}(t) - (m/\sqrt{2})f^{(-1)}(t) \}, \]  

\[ g_B^{(+)}(t) = 0, \]  

\[ g_B^{(-)}(t) = (12\pi/\sqrt{2})f^{(-1)}(t). \]
where $p^2 = m^2 - t/4$ and $q^2 = \mu^2 - t/4$. The dispersion relations (2.19) are now written as follows:

\begin{align*}
A^{(+)}(s, t) &= A_0^{(+)}(s, t) + C_A^{(+)} + \frac{1}{\pi} \int_{\mu^2}^{\infty} dt' \frac{\text{Im} g_A^{(+)}(t')}{t' - t}, \\
A^{(-)}(s, t) &= A_0^{(-)}(s, t) + \frac{(s - \bar{s})}{\pi} \int_{\mu^2}^{\infty} dt' \frac{\text{Im} g_A^{(-)}(t')}{t' - t}, \\
B^{(+)}(s, t) &= B_0^{(+)}(s, t), \\
B^{(-)}(s, t) &= B_0^{(-)}(s, t) + C_B^{(-)} + \frac{1}{\pi} \int_{\mu^2}^{\infty} dt' \frac{\text{Im} g_B^{(-)}(t')}{t' - t}.
\end{align*}

(2.21a)

(2.21b)

(2.21c)

(2.21d)

From the analysis given by F-F, we see that as $t \to \infty$, the functions $f_A^{(-)}(t)$, $f_A^{(-)}(t)$ and $f_B^{(-)}(t)/p^2$ goes to zero at least as rapidly as $t^{-1}$, $t^{-1/2}$ and $t^{-1/2}$ respectively. Thus the functions $g_A^{(-)}$ and $g_B^{(-)}$ defined by Eq. (2.20) have the necessary asymptotic behaviors, and the last terms in Eq. (2.21) are well defined in principle. At present, however, we have no information regarding the contribution from the high mass states. To avoid this defect and to obtain reliable results, we introduce one subtraction as follows for $A^{(+)}$, for example,

\begin{align*}
C_A^{(+)} + \frac{1}{\pi} \int_{\mu^2}^{\infty} dt' \frac{\text{Im} g_A^{(+)}(t')}{t' - t} &= C_A^{(+)} + \frac{1}{\pi} \int_{\mu^2}^{\infty} dt' \frac{\text{Im} g_A^{(+)}(t')}{t' - t} + \frac{t}{\pi} \int_{\mu^2}^{\infty} dt' \frac{\text{Im} g_A^{(+)}(t')}{t' - t}, \\
&= C_A^{(+)} + \frac{t}{\pi} \int_{\mu^2}^{\infty} dt' \frac{\text{Im} g_A^{(+)}(t')}{t' - t},
\end{align*}

(2.22)

and introduce the new parameter $C_A^{(+)}$ instead of $C_A^{(+)}$. Then we obtain

\begin{align*}
A^{(+)}(s, t) &= A_0^{(+)}(s, t) + C_A^{(+)} + \frac{t}{\pi} \int_{\mu^2}^{\infty} dt' \frac{\text{Im} g_A^{(+)}(t')}{t' - t}, \\
A^{(-)}(s, t) &= A_0^{(-)}(s, t) + (s - \bar{s}) \left[ C_A^{(-)} + \frac{t}{\pi} \int_{\mu^2}^{\infty} dt' \frac{\text{Im} g_A^{(-)}(t')}{t' - t} \right], \\
B^{(+)}(s, t) &= B_0^{(+)}(s, t), \\
B^{(-)}(s, t) &= B_0^{(-)}(s, t) + C_B^{(-)} + \frac{t}{\pi} \int_{\mu^2}^{\infty} dt' \frac{\text{Im} g_B^{(-)}(t')}{t' - t}.
\end{align*}

(IA)

(IIb)

(Id)

Eqs. (I a, b, c, d) are our basic relations to be used for the analysis of pion-nucleon scattering as well as electromagnetic structure of nucleon. Note
that the parameters \( C_A^{(+)} \) and \( C_B^{(-)} \) also contain a part of contributions from pion-pion interaction to be added to the high mass state contributions \( C_A^{(+\prime)} \) and \( C_B^{(-\prime)} \). In what follows, we shall, however, call the integrals over \( t \) from \( 4\mu^2 \) to \( \infty \) pion-pion terms. It is noted that the dispersion relation for \( B^{(+)} \) does not contain the pion-pion term in the approximation, in which \( D_- \) and higher partial waves are neglected. As the analysis by Ishida\(^{10}\) shows, this is consistent with the pion-nucleon scattering data.

The parameters \( C_A^{(\pm)} \) and \( C_B^{(\pm)} \) can be inferred directly from the experimental data on pion-nucleon scattering, using the dispersion relations (I) at forward angle, as is shown in § 4. It is important to decide how many parameters are really needed at present. In this paper, we analyse the property of the \( S \)-wave pion-pion interaction by using the dispersion relations (Ia) and (Ic), and briefly compare the results also with the information obtained from the experiment \( p + d \rightarrow He + \pi + \pi \) quite recently reported by Booth et al.\(^9\) From the comparison, we can infer the number of parameters to be required for consistency of the dispersion relations with the experimental data; these points will be mentioned in § 4 and § 5.

### § 3. Analysis of pion-pion term

In this section, the pion-pion terms (the integral terms in (I)) are analysed by a method similar to that of F-F, but using the subtracted form for \( g_A^{(+)}(t) \), etc. This is, from the analytic properties of \( g_A^{(+)}(t) \), given by

\[
g_A^{(+)}(t) = g_A^{(+)}(0) + \frac{t}{\pi} \int_{-\infty}^{a} dt' \frac{\text{Im} g_A^{(+)}(t')}{t' (t' - t)} + \frac{t}{\pi} \int_{4\mu^2}^{a} dt' \frac{\text{Im} g_A^{(+)}(t')}{t' (t' - t)}, \tag{3.1}
\]

where \( a = 4\mu^2 (1 - \mu^2/4m^2) \). In the approximation of neglecting all but two-pion intermediate state, \( g_A^{(+)}(t) \) can be written as

\[
g_A^{(+)}(t) = g_A^{(+)}(0) F_{\pi}^{\delta}(t) + \frac{t}{\pi} \int_{-\infty}^{a} dt' \frac{F_{\pi}^{\delta}(t') \text{Im} g_A^{(+)}(t')}{t' (t' - t)}, \tag{3.2}
\]

where \( F_{\pi}^{\delta}(t) \), the form factor of pion, is

\[
F_{\pi}^{\delta}(t) = \exp \left[ \frac{t}{\pi} \int_{4\mu^2}^{a} dt' \frac{\delta_{\delta}(t')}{t' (t' - t)} \right],
\]

\( \delta_{\delta}(t) \) being the \( I=0 \) (\( S \)-wave) pion-pion scattering phase shift. From Eqs. (3.1) and (3.2), we obtain for the pion-pion term

\[
\frac{t}{\pi} \int_{4\mu^2}^{a} dt' \frac{\text{Im} g_A^{(+)}(t')}{t' (t' - t)} = g_A^{(+)}(0) \left[ F_{\pi}^{\delta}(t) - 1 \right] + \frac{t}{\pi} \int_{-\infty}^{a} dt' \left[ \frac{F_{\pi}^{\delta}(t)}{F_{\pi}^{\delta}(t')} - 1 \right] \frac{\text{Im} g_A^{(+)}(t')}{t' (t' - t)}. \tag{3.3a}
\]
Applying the same method for $g_A(t)$ and $g_B(t)$, we find

$$
\int_{\pi}^{2\pi} dt' \frac{\text{Im} g_A'(t')}{t' (t' - t)} = g_A'(0) \left[ F_{\pi}^s(t) - 1 \right] + \int_{-\infty}^{t} dt' \left[ F_{\pi}^s(t') - 1 \right] \frac{\text{Im} g_A'(t')}{t' (t' - t)}
$$

where

$$
F_{\pi}^s(t) = \exp \left[ \frac{t}{\pi} \int_{4\pi^2}^{\infty} dt' \frac{\delta_P(t')}{t' (t' - t)} \right],
$$

and $\delta_P(t)$ is the $P$-wave pion-pion scattering phase shift. Using these solutions the dispersion relations (I) becomes finally*)

$$
A(\pm 1, s, t) = A_0(\pm 1, s, t) + C_A(\pm 1, 0) \left[ F_{\pi}^s(t) - 1 \right] + \int_{-\infty}^{t} dt' \left[ F_{\pi}^s(t') - 1 \right] \frac{\text{Im} g_A'(t')}{t' (t' - t)}
$$

$$
B(\pm 1, s, t) = B_0(\pm 1, s, t),
$$

$$
B(\pm 1, s, t) = B_0(\pm 1, s, t) + C_B(\pm 1, 0) \left[ F_{\pi}^s(t) - 1 \right] + \int_{-\infty}^{t} dt' \left[ F_{\pi}^s(t') - 1 \right] \frac{\text{Im} g_B'(t')}{t' (t' - t)}
$$

To manipulate (3·4), we observe that for the region $t \leq a$, the quantities $\text{Im} g_A(\pm 1)(t)$ and $\text{Im} g_B(\pm 1)(t)$ are given in terms of pion-nucleon scattering amplitudes and can be written as

$$
\text{Im} g_A'(\pm 1)(t) = \frac{2\pi m^2 f^2 z_0}{\mu^2 p^2 - 1} - \frac{\theta(-t)}{8\pi^2} \int_{(m+\rho)^2}^{L(0)} ds' \left[ \frac{p}{q} \text{Im} A'(s', t) + mz \text{Im} B'(s', t) \right],
$$

$$
\text{Im} g_A'(\pm 1)(t) = \frac{3\pi m^2 f^2 (3z_0^2 - 1)}{4\mu^2 p^2 - 1} - \frac{\theta(-t)}{32\pi} \int_{(m+\rho)^2}^{L(0)} ds' \left[ \frac{p}{q} \text{Im} A'(s', t) + m \frac{3z_0^2 - 1}{2} \text{Im} B'(s', t) \right],
$$

$$
\text{Im} g_B'(\pm 1)(t) = \frac{3\pi m^2 f^2 (z_0^2 - 1)}{\mu^2 p^2 - 1} + \frac{3\theta(-t)}{8\pi} \int_{(m+\rho)^2}^{L(0)} ds' \left[ \frac{p}{q} \text{Im} A'(s', t) + m \frac{3z_0^2 - 1}{2} \text{Im} B'(s', t) \right]
$$

* The author is indebted to Dr. K. Ishida for the derivation of these solutions.
where \( z_0 = (m^2 - p_\perp^2 - q^2) / 2p_\perp q_\perp \), \( z = (s' - p_\perp^2 - q^2) / 2p_\perp q_\perp \) and \( L(t) = m^2 + \mu^2 + 2p_\perp q_\perp - t/2 \). In Eq. (3.5), although the energy variable \( s' \) is in the physical region for pion-nucleon scattering, the upper limit \( L(t) \) is such that \( \cos \theta \leq -1 \), where \( \theta \) is the pion-nucleon scattering angle, as was shown by F-F. According to the F-F method, we shall make an analytic continuation by expanding \( \text{Im} A^{(\pm)}(s, t) \) and \( \text{Im} B^{(\pm)}(s, t) \) in Legendre polynomials:

\[
\text{Im} A^{(\pm)}(s, t) = \frac{1}{4\pi} \left[ \text{Im} f_i^{(\pm)} P_{i+1}^{(\pm)}(\cos \theta) - \text{Im} f_{i-1}^{(\pm)} P_i^{(\pm)}(\cos \theta) \right],
\]

(3.6)

where \( W^2 = s, E = (s + m^2 - \mu^2) / 2W, k^2 = E^2 - m^2 \) and \( \cos \theta = 1 + t + 2k^2 \) (cf. (2.5) and (2.6)).

We retain in Eq. (3.6) the terms only up to \( P \)-wave amplitudes. Moreover we neglect all small partial wave amplitudes except for the \( (3^{-3}) \) state, and approximate the imaginary part of the \( (3^{-3}) \) amplitude by a delta function. Then the last integrals in Eq. (3.4) are convergent, so that we can safely cut off these integrals at \( t = -26\mu^2 \). If we do not subtract, then the integrals over the left-hand cut \( t < a \) may be divergent and we can not obtain any reliable results by the cutoff procedure. It is for this reason that we use the subtraction technique in the above.

Next, we must calculate the subtraction constants \( g_A^{(+)}(0), g_A^{(-)}(0) \) and \( g_B^{(-)}(0) \). These constants are related to the forward pion-nucleon scattering amplitudes, and can be calculated directly from experimental information on pion-nucleon scattering, using the dispersion relations in the forward direction. From the definition of \( g_A^{(+)}(t) \) (cf. Eqs. (2.20a) and (2.9a)), \( g_A^{(+)}(0) \) are written

\[
g_A^{(+)}(0) = \frac{1}{f} \left[ A_0^{(+)}(t) - \mu B_1^{(+)}(t) \right]_{t=0},
\]

(3.7)

where

\[
A_0^{(+)}(0) = \int_{-1}^{1} dx P_0(x) A^{(+)}(s(x), 0), \quad B_1^{(+)}(0) = \int_{-1}^{1} dx P_1(x) B^{(+)}(s(x), 0).
\]

(3.8)

The variables \( s(x) \) and \( \bar{s}(x) \) with \( x = \cos \varphi \) are given by Eq. (2.4). The amplitudes \( A^{(+)}(s(x), 0) \) and \( B^{(+)}(s(x), 0) \) are, in turn, obtained from the dispersion relation (1) at \( t = 0 \). Integration over the angle and insertion into Eq. (3.7) give for \( g_A^{(+)}(0) \)

\[
g_A^{(+)}(0) = -\frac{8\pi mf^3}{\mu^2} \left\{ 2 - a \log \left| \frac{1+a}{1-a} \right| \right\}.
\]
Similarly, we get

\[ g_A^{(+)}(0) = \frac{3\pi f^2}{\mu^2} \left\{ 6a + (1 - 3a^2) \log \left| \frac{1 + a}{1 - a} \right| \right\} \]

\[ + \frac{3}{8m\mu} \int_{(m + \rho)^2}^{\infty} ds' \left\{ \mu \text{Im} B(s', 0) Y_B^{(-)}(s') - \text{Im} A^{(-)}(s', 0) Y_A^{(-)}(s') \right\} + C_A^{(-)}, \]

\[ g_B^{(-)}(0) = -\frac{12\pi mf^2}{\mu^2} \left\{ 2a + (1 - a^2) \log \left| \frac{1 + a}{1 - a} \right| \right\} \]

\[ + \frac{1}{\pi} \int_{(m + \rho)^2}^{\infty} ds' \text{Im} B^{(-)}(s', 0) Y_B(s') + C_B^{(-)}, \]

where

\[ Y_A^{(+)}(s') = (1/2m\mu)\alpha, \]

\[ Y_B^{(+)}(s') = (1/2m\mu) (2 - b\alpha) = (1/2) Y_A^{(-)}(s'), \]

\[ Y_B^{(-)}(s') = (1/2m\mu) \{ -6b + (3b^2 - 1)\alpha \}, \]

\[ Y_B(s') = (3/4m\mu) \{ 2b - (b^2 - 1)\alpha \}, \]

and

\[ a = \mu/2m, b = (s' - m^2 - \mu^2)/2m\mu \quad \text{and} \quad \alpha = \log \left| \frac{1 + b}{1 - b} \right|. \]

Recently, Ball and Wong attempted to modify the F-F method. That is, the unsubtracted integrals over the left-hand cut were also cut off at \( t = -26p^2 \), but the effect of the singularity in region \( t \leq -26p^2 \) was taken into account by adding one term corresponding to one simple pole. And its position and residue were determined by the values of amplitude and its derivative at \( t = 0 \), which correspond to our \( g_A^{(+)}(0) \) and \( g_A^{(+)}(0) \). However, the properties of such a simple pole could not be determined definitely owing to the present experimental inaccuracy of the pion-nucleon scattering data.

§ 4. Numerical results and the analysis of \( I=0 \) (S-wave) pion-pion scattering amplitude

We now evaluate the constant parameters \( C_A^{(+)} \), \( C_A^{(-)} \) and \( C_B^{(-)} \) introduced in § 2, and the subtraction constants \( g_A^{(+)}(0) \), \( g_A^{(-)}(0) \) and \( g_B^{(-)}(0) \) in § 3, and the properties of \( I=0 \) (S-wave) pion-pion scattering amplitude will be derived.

(i) The parameters \( C_A^{(+)} \), \( C_A^{(-)} \) and \( C_B^{(-)} \). By applying the dispersion relations (I) at \( s = (m + \rho)^2 \) and \( t = 0 \), these constants are written as
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\[ C_A^{(+)}/4\pi = \text{Re}(A^{(+)}((m + \mu)^2, 0) / 4\pi) - A_0^{(+)}((m + \mu)^2, 0) / 4\pi, \]
\[ 4m\mu C_A^{(-)}/4\pi = \text{Re}(A^{(-)}((m + \mu)^2, 0) / 4\pi) - A_0^{(-)}((m + \mu)^2, 0) / 4\pi, \]
\[ \mu C_B^{(-)}/4\pi = \mu \text{Re}(B^{(-)}((m + \mu)^2, 0) / 4\pi) - \mu B_0^{(-)}((m + \mu)^2, 0) / 4\pi. \] (4.1)

We use the experimental values reported by Puppi at CERN Conference for the pion-nucleon scattering lengths for Re \( A^{(+)} \) and Re \( B^{(-)} \), and the empirical formula given by Anderson for the (3·3) phase shift. The pion-nucleon coupling constant \( f^2 \) appearing in the Born term in \( B_0^{(-)} \) is taken as \( f^2 = 0.08 \). The numerical values of \( C_A^{(+)} \), \( C_A^{(-)} \) and \( C_B^{(-)} \) thus determined are shown in Table I.\(^*l\)

Table I. Numerical values of \( C_A^{(+)} \), \( C_A^{(-)} \) and \( C_B^{(-)} \). All values are given in units \( \mu^{-1} \).

<table>
<thead>
<tr>
<th>&quot;Real part&quot;</th>
<th>&quot;Born term&quot;</th>
<th>&quot;Dispersion integral&quot;</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.47±0.26</td>
<td>3.42</td>
<td></td>
<td>( C_A^{(+)}/4\pi = -0.95±0.26 )</td>
</tr>
<tr>
<td>-1.16±0.22</td>
<td>0</td>
<td>-0.71</td>
<td>( 4m\mu C_A^{(-)}/4\pi = -0.45±0.22 )</td>
</tr>
<tr>
<td>1.26±0.22</td>
<td>0.16</td>
<td>0.59</td>
<td>( \mu C_B^{(-)}/4\pi = 0.51±0.22 )</td>
</tr>
</tbody>
</table>

(ii) The numerical values of \( g_A^{(+)}(0) \), \( g_A^{(-)}(0) \) and \( g_B^{(-)}(0) \). They are estimated from Eq. (3·9), using the same experimental values as adopted in (i),

\[ g_A^{(+)}(0)/4\pi = -0.44±0.26\mu^{-1}, \] (4·2a)
\[ \mu^2 g_A^{(-)}(0)/4\pi = -0.0012±0.0082\mu^{-1}, \] (4·2b)
\[ w g_B^{(-)}(0)/4\pi = 0.50±0.22\mu^{-1}. \] (4·2c)

Owing to the mutual cancellation between the Born term and dispersion integral and \( C_A^{(+)} \) in \( g_A^{(+)}(0) \), there is a very large error in \( g_A^{(-)}(0) \); this circumstance is an obstacle to the analysis of nucleon structure, as will be shown in a separate paper.

(iii) The \( I=0 \) (S-wave) pion-pion scattering amplitude. According to CGLN, \( \text{Re}[f p_{0\pi}/k^3]_k\rightarrow_0 \) can be written as follows:

\[ \text{Re}(f p_{0\pi}/k^3)_{k\rightarrow 0} = (2/3)(1 + \mu/m)^{-1}\{\text{Re}(A^{(+)}((m + \mu)^2, 0) + \mu \text{Re}(B^{(-)}((m + \mu)^2, 0))/4\pi. \] (4·3)

The right-hand side of Eq. (4·3) are rewritten, using the dispersion relations (3·4) and we obtain

\(^*l\) It is remarked that the dispersion relation for \( B^{(+)}(s, t) \) is consistent with the experimental data, if we take \( f^2 = 0.075 \).
\[ \text{Re} \left( \frac{f_{\delta\delta}^{\text{e}}(2\pi)}{F_{\delta\delta}^{\text{e}}} \right)_{k=2} = \frac{2}{3}(1 + \mu/m)^{-1} \left\{ A_{\delta}^{(+)1}((m + \mu)^{\delta}, 0) + \mu B_{\delta}^{(+)1}((m + \mu)^{\delta}, 0) \right\} / 4\pi \]

\[ + \frac{2}{3}(1 + \mu/m)^{-1} \left\{ (g_{\delta}^{(+)1}(0)) / 4\pi \right\} F_{\delta\delta}^{\text{e}}(0) + \frac{P}{\pi} \int_{-\infty}^{\infty} dt' \left[ \frac{1}{F_{\delta\delta}^{\text{e}}(t')} - 1 \right] \]

\[ \times \text{Im} g_{\delta}^{(+)1}(t') / 4\pi \right\} \right. \]

This reduces to the result given by CGLN, if we neglect the contribution from pion-pion interaction, the last term in Eq. (4.4). From the analysis by Ishida, however, this pion term is expected to be \(0.022 \pm 0.013 \mu^{-3}\). From this, we try to obtain some conclusion on the \(I=0\) (S-wave) pion-pion interaction. On evaluation of the pion-pion term in Eq. (4.4), some approximations are done: Owing to the energy denominator, the behavior of pion-pion scattering phase shift at high energy is expected to be insensitive to the value of pion form factor, \(F_{\delta\delta}(\tau)\). Thus we assume the scattering length approximation

\[ \tan \delta_{\delta}(\tau) = \alpha_{\delta}\tau. \]

Then, as in A, we have for \(t<4\mu^{2}\)

\[ F_{\delta\delta}(\tau) = \left\{ \begin{array}{ll}
(1 + \mu\alpha_{\delta}) (1 + \alpha_{\delta} \tau)^{-1} & \text{for } \alpha_{\delta}>0, \\
(1 + |\alpha_{\delta}| \tau) (1 + \mu|\alpha_{\delta}|)^{-1} & \text{for } \alpha_{\delta}<0.
\end{array} \right. \quad (4.6) \]

Thus the last term of Eq. (4.4) becomes

\(\alpha_{\delta}>0: \)

\[ (2/3)(1 + \mu/m)^{-1} \left\{ \frac{g_{\delta}^{(+)1}(0)}{4\pi} \frac{\alpha_{\delta}}{8\mu(1 + \alpha_{\delta}\mu)} - \frac{\alpha_{\delta}/4}{1 + \mu\alpha_{\delta}} \right\} + \frac{P}{\pi} \int_{-\infty}^{\infty} dt' \left[ \text{Im} \frac{g_{\delta}^{(+)1}(t') / 4\pi}{t'(q_{-} + \mu)} \right] \]

\(\alpha_{\delta}<0: \)

\[ (2/3)(1 + \mu/m)^{-1} \left\{ -\frac{g_{\delta}^{(+)1}(0)}{4\pi} \frac{|\alpha_{\delta}|}{8\mu(1 - |\alpha_{\delta}|\mu)} + \frac{|\alpha_{\delta}|}{4} \right\} \]

\[ + \frac{P}{\pi} \int_{-\infty}^{\infty} dt' \left[ \frac{\text{Im} \frac{g_{\delta}^{(+)1}(t') / 4\pi}{t'(q_{-} + \mu)} \right] \]

In Fig. 1, the value of (4.7a, b) for various \(\alpha_{\delta}\) is plotted and compared with the experimental value \(0.022 \pm 0.013 \mu^{-3}\). We may conclude that the \(I=0\) (S-wave) pion-pion interaction is attractive with scattering length \(0.05\sim0.2 \mu^{-1}\), which

\(\text{*) This is considerably smaller than the value used in A. In A, however, we did not analyse the most probable value of this quantity. On the other hand, in the analysis by Ishida, all P-wave pion-nucleon scattering phase shifts were fitted by (CGLN)\(+(\text{pion-pion term})\), so the value } 0.022\pm0.013 \mu^{-3}\text{ is accepted in this paper.}\)
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![Diagram](https://example.com/diagram.png)

**Fig. 1.** The value of Eqs. (4.7a) and (4.7b) for various $\alpha_s$. The shaded area represents the value required from the experimental data, $0.22 \pm 0.013 \, \mu^{-3}$.

$I=0$ (S-wave) pion-pion interaction is attractive, and this is consistent with our result. The magnitude of scattering length depends on the $\lambda$; $\alpha_s = 0.35 \mu^{-1}$ for $\lambda = 0.01$ and $2.81 \mu^{-1}$ for $\lambda = -0.20$.

As summarized in the Introduction, we have used many approximations to obtain the above result. In these approximations, the treatment of left-hand cut is most questionable. In this paper, the remedy of the method of $F-F$ has been attempted by introducing one subtraction. However, only with the assumption that the partial $\pi-N$ scattering amplitudes other than $S$- and $P$-waves are negligible on the cut, the effect of rescattering may reasonably be included in the integral. Here, comparing with the experimental data, $\alpha_s = 2 \sim 3 \mu^{-1}$, we shall suggest under the approximations used in this paper the possibility of introducing one more subtraction in the dispersion relation for $A^{(+)}(s, t)$ as follows:

$$A^{(+)}(s, t) = A_c^{(+)}(s, t) + C_A^{(+)} + C_A^{(+)}t + \frac{t^2}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\text{Im} g_A^{(+)}(t')}{t'^2(t' - t)}.$$ (Ia')

Here, the new parameter $C_A^{(+)}$ should be responsible for the disagreement of the value of $\text{Re} \left[ f_{\pi N}^{(+)}(k^2) \right]_{k^2 > 0}$ calculated by $A_c^{(+)}$ and $B_c^{(+)}$ and the one given by experimental data on scattering lengths of $P$-wave pion-nucleon scattering amplitudes. That is, if we use the dispersion relation (Ia') for $A^{(+)}(s, t)$, $\text{Re} \left[ f_{\pi N}^{(+)}(k^2) \right]_{k^2 > 0}$ becomes

$$\text{Re} \left[ f_{\pi N}^{(+)}(k^2) \right]_{k^2 > 0} = (2/3) (1 + \mu/m)^{-1} \left\{ A_c^{(+)} ((m + \mu)^2, 0) + \mu B_c^{(+)} ((m + \mu)^2, 0) \right\} / 4\pi$$

$$+ (2/3) (1 + \mu/m)^{-1} C_A^{(+)} / 4\pi$$ (4.4')

and the value of $C_A^{(+)} / 4\pi$ should be $0.029 \pm 0.017 \mu^{-3}$. In this case, the effect of $I=0$ (S-wave) pion-pion interaction to pion-nucleon scattering at low energies is masked by $C_A^{(+)}$, so that the dynamical effect of the interaction to $\text{Re} \left[ f_{\pi N}^{(+)}(k^2) \right]_{k^2 > 0}$ seems to be inconsistent with the value $1 \mu^{-1}$ in $A$. This is really not inconsistent, because the scattering length in $A$ turns out to be $0.2 \sim 0.3 \mu^{-1}$ for the smaller value of pion-pion term adopted here. Recently, Desai\(^{(10)}\) has investigated the properties of $I=0$ and 2 (S-wave) scattering amplitudes from the crossing relations, assuming the resonance in $P$-wave pion-pion scattering. He obtained that for the positive and negative values of $\lambda$ defined by Chew and Mandelstam\(^{(6)}\), the
should be analysed at higher energies. Thus, we may conclude that if the large value of $\alpha_s = 2^{-3} \mu^{-1}$ is accepted, there is a possibility to use the dispersion relation (Ia') for $A^{(+)}(s, t)$ instead of (Ia).

§ 5. Discussions

As there is not much information on the contribution to the dispersion integral from high mass states, three parameters ($C_{A^{(+)}}$, $C_{A^{(-)}}$, and $C_{B^{(-)}}$) are introduced in the dispersion relations by the subtraction. And the pion-pion terms are treated in a subtracted form to improve the bad convergence of the integral. Using the dispersion relations obtained in this way, Eqs. (3·4a, c), we analyse the property of the $I=0$ (S-wave) pion-pion interaction, and obtain the result that the interaction is attractive and the magnitude of the scattering length $\alpha_s$ is $0.05^{-0.2} \mu^{-1}$, which is consistent with A.

Here, we shall give some consideration on the treatment of the integral over the left-hand cut. If we use, instead of Eq. (3·3), the unsubtracted relation for $g_{A^{(+)}}(t)$, the following equation is to hold:

$$
\frac{g_{A^{(+)}(0)}}{4\pi} = \frac{1}{\pi} \int_{-\infty}^{\infty} dt' \frac{\text{Im} g_{A^{(+)}(t')}}{F_\pi(t')}.
$$

For our approximate treatment of $\text{Im} g_{A^{(+)}(t)}$, the integral in the right-hand side, however, does not converge, so that the cutoff must be introduced to get any definiteness. By cutting off the integral at $t = -26 \mu^2$, the integral amounts to $-1.35 \mu^{-1}$ for $\alpha_s = 0.1 \mu^{-1}$. This is to be compared to the value of $g_{A^{(+)}(0)}/4\pi$ for which we expect $-0.44 \pm 0.26 \mu^{-1}$ from Eq. (4·2a). The disagreement should be attributed to inadequateness of the unsubtracted form or bad convergence of the integral over the left-hand cut.\(^(*)\)

If the parameters $C_{A^{(+)}}$, $C_{A^{(-)}}$, and $C_{B^{(-)}}$ in this paper are all necessary at present for the analysis of low energy pion-nucleon scattering, the scattering lengths of S-wave and some combinations of scattering lengths of P-wave pion-nucleon scattering can not be determined from the theoretical calculation only. For instance, the disagreement of the value of S-wave scattering lengths $(1/3) \times (a_1 + 2a_2)$ calculated from $A_{\sigma^{(+)}}$ and $B_{\sigma^{(+)}}$, and the one obtained from the experimental data should be attributed to $C_{A^{(+)}}$, and the dynamical effect of $I=0$ (S-wave) pion-pion interaction to the S-wave pion-nucleon scattering will appear only in the behavior of $\text{Re} f_\pi^{(+)}(s)$ at higher energies. As shown in § 4, in order to explain the large scattering length $\alpha_s = 2^{-3} \mu^{-1}$, a parameter $C_{A^{(s)}}$ should be introduced by one more subtraction in the dispersion relation for $A^{(+)}(s, t)$.

\(^(*)\) If we use this unsubtracted form, the $I=0$ interaction turns out to be repulsive. As the integral over the left-hand cut is divergent in this case and does not produce the normalization value $g_{A^{(+)}(0)}/4\pi$, this result is unreliable.
and then the dynamical effect of $I=0$ (S-wave) pion-pion interaction to $\text{Re} \left( f_{\rho \delta}^{2d}/k^2 \right)$ must be analysed likewise at higher energies. It will be shown in another paper that the parameter $C_{\rho}^{(-)}$ may be substituted for $B^{(-)}(s, t)$ from the analysis of the effect of pion-pion scattering resonance on pion-nucleon scattering.

Thus, we may say that we are in a situation to analyse the pion-nucleon scattering at low and higher energies using the dispersion relations (Ia, b, c, d) (or (Ia', b, c, d)). Recently, using the crossing relation for pion-nucleon scattering amplitudes, the subtraction method has been reported by Frazer at the Rochester Conference. In his report, the dispersion relation (Ia') in stead of (Ia) has been proposed for $A^{(+)}(s, t)$.

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