The Observables and Localized States of a Dirac Particle

P.M. MATHEWS and A. SANKARANARAYANAN

Department of Physics, University of Madras
Madras 25, India

(Received December 25, 1961)

It is shown that for a non-zero mass Dirac particle, only one of the four position operators recently discovered by the authors has eigenstates (describing the localized states of the particle) which obey the regularity condition imposed by Newton and Wigner. However, this condition is not applicable to the states of a particle with zero rest mass, and in such a case the multiplicity of possible position (and other) operators persists.

§ 1. Introduction

In a recent publication the authors showed that there are four different possibilities consistent with reasonable physical requirements for defining the position operator (and hence the operators for the other dynamical variables too) of a Dirac particle. It was found there that in each of these four sets, the operator representatives for the dynamical variables were such that all the usual commutation relations were obeyed. This invariance of the commutation relations in passing from one set to another suggests that the different sets are related by unitary transformations. In the following we obtain the actual unitary transformations involved. We then establish the relationship between the Newton-Wigner (N.W.) position operator and the position operator found by Foldy and Wouthuysen (F.W.) and Pryce, the latter of which is the simplest of our four operators. This relationship enables us to deduce the localized eigenstates for these four operators. An examination of these in the light of the requirements laid down by Newton and Wigner shows that all the states (and hence the corresponding operators) are admissible in the case of a zero mass particle, but only the F.W. operator remains in the finite mass case.

§ 2. The unitary transformations

In reference 1), we showed that there are four possible sets of choices for the operator representatives of the dynamical variables of a Dirac particle, and that in particular, in the C representation in which the Hamiltonian takes the form

\[ H' = \beta E, \quad E = + (p^2 + m^2)^{1/2}, \]  

For a definition of the various representations, see, for instance, reference 5).
the possibilities for the position operator are \( \mathbf{x}', \mathbf{x}, \mathbf{y}' \) and \( \mathbf{y} \), obtained by giving, respectively, the sets of values

\[
P = B = 0; \quad A = -1, B = 0; \quad A = B = -1/2; \quad A = -B = -1/2
\]

to the constants \( A \) and \( B \) in the general formula

\[
\mathbf{X} = \mathbf{x} + (A + B\beta) (\sigma \times \mathbf{p})/\rho^2.
\]

\( \mathbf{x}' = \mathbf{x} \) is identical with the F.W. operator and yields, in the \( D \) representation, the operator \( \tilde{q} \) of Pryce.\(^4\) It may now be verified that the other three are related to this by the following unitary transformations:

\[
\begin{align*}
\mathbf{y}' &= \mathbf{x} - (\sigma \times \mathbf{p})/\rho^2 = U_1 \mathbf{x} U_1^\dagger \text{ where } U_1 = (\sigma \cdot \mathbf{p})/\rho, \quad (4.1) \\
\mathbf{y} &= \mathbf{x} - (1 + \beta) (\sigma \times \mathbf{p})/(2\rho^2) = U_2 \mathbf{x} U_2^\dagger \text{ where } U_2 = 1/2 \{(1 - \beta) + (1 + \beta)(\sigma \cdot \mathbf{p})/\rho\}, \quad (4.2) \\
\mathbf{y}' &= \mathbf{x} - (1 - \beta) (\sigma \times \mathbf{p})/(2\rho^2) = U_3 \mathbf{x} U_3^\dagger \text{ where } U_3 = 1/2 \{(1 + \beta) + (1 - \beta)(\sigma \cdot \mathbf{p})/\rho\}. \quad (4.3)
\end{align*}
\]

The corresponding transformation matrices in the \( D \) or \( E \) representation may of course be obtained by replacing \( \beta \) in \( U_2 \) and \( U_3 \) by \( H^0/E \) or \( H^\infty/E \) respectively. The matrices \( U_1, U_2, U_3 \) are arbitrary to the extent of a phase factor, as usual. Any of them may also be multiplied by a factor of \( \beta \) without affecting the results, because \( \beta (=H^0/E) \) commutes with all the operators for the particle; indeed, this was one of the requirements made of all one-particle observables in reference 1).

The unitary operators given above can be deduced by noting that they should commute with \( H \), since all the four position operators are possibilities in the same representation, i.e. with the same expression for \( H \). Hence the \( U \)'s must be made up of the only two non-trivial operators which commute with \( H \), namely \( H/E \) (\( =\beta \) in the \( C \) representation) and \( (\sigma \cdot \mathbf{p})/\rho \). The particular linear combinations of these two quantities that are necessary are then easily determined by actual calculation.

Incidentally, it may be observed that if the \( C \) representation is defined merely by the form of the Hamiltonian as \( H^0 = \beta E \), the transition from the \( D \) to the \( C \) representation may be achieved not only by the F.W. transformation \( U = \sqrt{E + m + \beta (\mathbf{a} \cdot \mathbf{p})} [2E(E + m)]^{-1/2} \), but by any of the transformations \( U_1 U \), \( U_2 U \) and \( U_3 U \) also. The forms of the other observables will of course depend on the particular transformation employed.

\section*{§ 3. The localized states corresponding to the four position operators}

We shall now examine the localized states represented by eigenfunctions of the different position operators. From the relations (4) between the operators, it is clear that the positive and negative energy eigenfunctions \( \phi \) and \( \chi \) of the
operator $\sigma X$ determines those of the other operators as in the following table:

<table>
<thead>
<tr>
<th>Operator</th>
<th>$\sigma X$</th>
<th>$\lambda X$</th>
<th>$\tau X$</th>
<th>$\rho X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pos. Energy</td>
<td>$\phi$</td>
<td>$(\sigma \cdot p/p) \phi$</td>
<td>$(\sigma \cdot p/p) \phi$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>Neg. Energy</td>
<td>$\chi$</td>
<td>$(\sigma \cdot p/p) \chi$</td>
<td>$\chi$</td>
<td>$(\sigma \cdot p/p) \chi$.</td>
</tr>
</tbody>
</table>

It will be noted that as long as states corresponding to a definite sign of the energy are considered, only two operators, say, $\sigma X$ and $\lambda X$, are independent. The remaining two operators juxtapose the positive energy eigenfunctions of one of the operators $\sigma X$ and $\lambda X$ with negative energy eigenfunctions of the other. The explicit determination of $\phi$ and $\chi$ themselves is best accomplished by establishing the relationship between the Newton-Wigner operator $q$ (whose eigenfunctions are known) and $\sigma X$, or rather, the positive energy projection of $\sigma X$ since the N.W. operator has been constructed to operate on positive energy states only. The comparison between the two operators will be made in the $D$ representation in which the expression for $q$ is given. In the following we will denote the operator $\sigma X^D$ ($D$ standing for the $D$ representation) by $\tilde{q}$ in the notation of Pryce.\(^4\) We have

$$\tilde{q} = x + i\beta \alpha \frac{E}{2E} - i\beta (\alpha \cdot p) \frac{p + (\sigma \times p) E}{2E^3(E + m)}$$  \(5\)

and

$$q = A_+ (1 + \beta) E^{3/2} (E + m)^{-1/2} E^{-1/2} (E + m)^{-1/2} A_+.$$  \(6\)

The operator $x$ in these expressions is of course equivalent to $i\nabla_x$, and $A_+ = 1/2 (1 + H/E)$ is the projection operator to positive energies. The positive energy projection of $\tilde{q}$, to be compared with $q$, is $\tilde{q} = A_+ \tilde{q} A_+$, which, on actual evaluation is found to be

$$\tilde{q} = q + (ip/E^3) A_+.$$  \(7\)

It is now easily shown that the eigenfunctions of (7) describing localization at the origin of co-ordinates are

$$\phi_m = E^{-1} \phi_m, \quad (m = 1/2, -1/2)$$  \(8\)

where the $\phi_m$ are the corresponding eigenfunctions of the N.W. operator $q$:

$$\phi_m = (2\pi)^{-3/2} 2^{1/2} E^{3/2} (E + m)^{-1/2} A_+ v_m,$$  \(9\)

the $v_m$ being pure spin functions satisfying

$$\beta v_m = v_m.$$  \(10\)

It is now clear that the relation (7) may be written as

$$\tilde{q} = E^{-1} q E$$  \(11\)
and this may also be verified directly by making use of the identity

\[ A_\pm (1 + \beta) A_\pm = \left[ (E + m)/E \right] A_\pm. \]

The negative energy eigenfunctions \( \chi \) of \( \tilde{q} \) can be shown to be

\[ \chi_m = (2\pi)^{-3/2} 2^{1/2} E^{1/2} (E + m)^{-1/2} A_- \nu_m. \]

Indeed, this is almost evident from the forms of (8) and (9).

It will now be observed that the \( \phi_m \) and \( \chi_m \) differ in their normalization from the \( \phi_m \). While the latter are normalized so that \( \phi_m^* \phi_m = (2\pi)^{-3} E^2 \), we have, by virtue of (8), \( \phi_m^* \phi_m = (2\pi)^{-3} \) and similarly \( \chi_m^* \chi_m = (2\pi)^{-3} \). This difference in the definition of the scalar products in the F.W. and N.W. cases accounts for the transformation (11) being non-unitary and for the non-Hermitian nature of the difference between \( \phi \) and \( \chi \).

Having determined the eigenfunctions \( \phi \) and \( \chi \) of \( \tilde{q} = \sigma \cdot p/p \), let us now turn to the consideration of the eigenfunctions of the remaining three operators. As may be seen from the table, the positive energy and/or the negative energy eigenfunctions in these cases differ from \( \phi \) and \( \chi \) by a factor of \( (\sigma \cdot p/p) \). A comparison with the work of Newton and Wigner shows that the set of functions \( E(\sigma \cdot p/p) \phi_m \) does satisfy all their conditions regarding localizability and is indeed identical with the term corresponding to \( l=1, s=1/2 \) in their general formula (21). Nevertheless it does not finally appear as a possibility because it fails to satisfy the regularity condition which is essentially a condition that the states be uniquely defined in the rest system of the particle: \( (\sigma \cdot p/p) \) does not tend to a unique limit as \( p \to 0 \). It therefore follows that our operators \( \sigma X, \tau X \) and \( \tau X \) whose eigenfunctions include \( (\sigma \cdot p/p) \phi \) or \( (\sigma \cdot p/p) \chi \) (or both) are not admissible as position operators as long as a rest system exists, i.e. for finite mass particles. But in the case of spin-1/2 particles of zero mass, (specifically, the neutrino), this consideration does not apply, and wave functions involving \( (\sigma \cdot p/p) \) explicitly are admissible and, indeed, such wave functions have been used in the literature. We are thus left with all the four possibilities (of which only two are distinct as far as operation on states of definite energy are concerned) for the position and other operators for the neutrino.

It may seem strange that we have proceeded by working out the the eigenfunctions in order to draw the conclusions of the last paragraph, when they could after all have been arrived at directly from the relations (4) between the different position operators (granting that \( \sigma X \) itself is an admissible operator). There is a good reason, however, for adopting the present procedure, apart from the intrinsic interest of the results derived in the earlier part of this section. In the Newton-Wigner treatment it is the regularity condition that reduces the choice of localized eigenstates, from the infinite variety encompassed in their general formula (21) to a unique set which, in the case of spin-1/2 particles, is given by (9) above. However, we have just seen that the regu-
larity condition does not appear in the zero mass case. The question then arises: Why is our choice of positive energy eigenstates in the zero-mass case limited to just two possible sets ($\phi_m$ and $(\sigma \cdot p/p)\phi_m$) instead of our having an infinite variety of them? Could it be that the conditions we have placed on admissible position operators from physical considerations, in reference 1), are more restrictive than the invariance requirements laid down by Newton and Wigner? It turns out that this is not so. In fact, the normalization condition $\phi_m^*\phi_m = (2\pi)^{-3}E^2$, given above, which results from the requirement that a localized state be orthogonal to states obtained from it by translation, is by itself sufficient to reduce the choice of localized wave functions to just two sets: $\phi_m$, of even parity, and $(\sigma \cdot p/p)\phi_m$, of odd parity. Invoking the regularity condition then merely serves to eliminate the second of these when the mass of the particle is non-zero.

References