Transport Phenomena in a Nonuniform Slightly Ionized Gas

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The authors have obtained the solution of Boltzmann’s equation for electrons in a nonuniform slightly ionized gas, in the presence of the gradients of electron collision frequency and density as well as the external static and alternating electric field and the static magnetic field. Electrical and thermal currents have been obtained in the integral form as a function of collision frequency and $f_0$, the isotropic part of distribution function of electron velocities. A linear ordinary differential equation for $f_0$ involving collision frequency as well as external fields and gradients of electron density and collision frequency is set up. An important result obtained is the fact that high gradients of electron density and collision frequency lead to a non-Maxwellian distribution of electron velocities. This is applicable to a wide variety of problems, involving plasma sheaths. Another result, clearly brought out by the present vector treatment is the fact that the current in general has a part proportional to the magnetic vector besides those proportional to the electric fields and gradient vectors and their vector product with the magnetic vector. However, this part vanishes when the magnetic vector is perpendicular to electric field and gradient vectors. Applications and limitations of present analysis have been discussed.

§ 1. Introduction

The Boltzmann’s transfer equation, because of its importance in the study of transport phenomena, has been the subject of many investigations. However, most of the investigators have been concerned with uniform gases in which no gradients of electron density and collision frequency of electrons are present. In the few investigations of diffusion of electrons due to gradient of electron density the fact that the gradient of electron density leads to a non-Maxwellian distribution of electron velocities has not been pointed out. No theory of the effects of gradient of collision frequency (due to the variation of composition

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of the gas) on transport phenomena, appears to have been given.

In this communication the authors have investigated the transport phenomena, due to electrons in a slightly ionized gas in the presence of an electric field (static + alternating), a magnetic field and gradients of electron density and electron collision frequency and absence of any gradients of temperature.

§ 2. Solution of Boltzmann's equation

For moderate electric and magnetic fields and gradients of electron density and collision frequency the solution of Boltzmann's transfer equation

\[ \frac{\partial f}{\partial t} + \mathbf{a} \cdot \nabla f + \mathbf{v} \cdot \nabla f = \left( \frac{\partial f}{\partial t} \right)_c \]  \hspace{1cm} (1)

can be expressed as

\[ f = f_0 + \mathbf{v} \cdot f' = f_0 + \mathbf{v} \cdot \left( f_1 + f_2 \sin \omega t + f_3 \cos \omega t \right), \]  \hspace{1cm} (2)

where \( f(\mathbf{v}) \) is the distribution function of electron velocities, \( \mathbf{a} \) is the acceleration of electrons, \( \mathbf{v} \) is the electron velocity, \( t \) is the time, \( (\partial f/\partial t)_c \) is the rate of change of \( f \) due to collisions.

\( f_0 \) and the components of \( f' \) are functions of \( \mathbf{v} \) and the electric and magnetic fields have the form

\[ E = E_0 + E_1 \sin \omega t \]  \hspace{1cm} (3)

and

\[ B = B_0. \]  \hspace{1cm} (4)

From Eq. (2) we have

\[ \frac{\partial f}{\partial t} = \omega \mathbf{v} \cdot \left( f_1 \cos \omega t + f_3 \sin \omega t \right). \]  \hspace{1cm} (5)

A rigorous method to proceed further will be to try for the simultaneous solution of Boltzmann's equations for the electrons and ions and the Poisson's equation. However, such an approach even in much simpler cases leads to serious mathematical difficulties and hence we have proceeded with less rigorous assumptions, viz.

i) The gas is electrically neutral at all points at all times. This assumption is not justified at high values of \( \omega \), when the ions can hardly keep up with the motion of electrons. Thus we can neglect the fields due to nonuniform charge (not electron density) distribution.

ii) The distribution function \( f' \) depends on \( \mathbf{r} \) only on account of the spatial dependence of the electron density \( n \) and the collision frequency, \( \nu \), i.e.
A justification of the above assumptions can only come from a comparison with the results of a more rigorous analysis, as suggested earlier. However, the error made in these assumptions may be of the same order as inherent in considering only the first two terms in the expansion of the distribution function and the derivation of the term \( \partial f/\partial t \).

From Eq. (5A) we obtain

\[
\nabla f = \nabla f_0 + \mathbf{v} \cdot \nabla f' = \gamma f_0 + g \frac{\partial f_0}{\partial \mathbf{v}} + \mathbf{v} \left[ \gamma f' + \frac{g \partial f'}{\partial \mathbf{v}} \right]
\]

where

\[
\gamma = \nabla n/n \\
g = \frac{1}{\frac{\partial n}{\partial \mathbf{v}}} \cdot \nabla \mathbf{v}
\]

and \( n \) is the electron density, since \( \frac{\partial f_0}{\partial \mathbf{v}} = \left( \frac{\partial f_0}{\partial \mathbf{v}} \right) \left( \frac{\partial n}{\partial \mathbf{v}} \right) \) and \( \frac{\partial f'}{\partial \mathbf{v}} = \left( \frac{\partial f'}{\partial \mathbf{v}} \right) \left( \frac{\partial n}{\partial \mathbf{v}} \right) \).

From Eq. (6)

\[
\mathbf{v} \cdot \nabla f = \gamma \mathbf{v} (f_0 + f' \cdot \mathbf{v}) + g \mathbf{v} \left( \frac{\partial f_0}{\partial \mathbf{v}} + \mathbf{v} \cdot \frac{\partial f'}{\partial \mathbf{v}} \right)
\]

The gradient of \( f \) in the velocity space and the acceleration \( \mathbf{a} \) of electrons are respectively given by

\[
\nabla_v f = f' + \frac{\mathbf{v}}{v} \left( \frac{\partial f_0}{\partial \mathbf{v}} + \mathbf{v} \cdot \frac{\partial f'}{\partial \mathbf{v}} \right)
\]

and

\[
\mathbf{a} = \frac{q}{m} (E + \mathbf{v} \times B_0)
\]

\[
= \mathbf{a}_0 + \mathbf{a}_1 \sin \omega t
\]

where \( q \) and \( m \) are the charge and the mass of an electron,

\[
\mathbf{a}_0 = \frac{q}{m} B_0
\]

From Eqs. (10) and (11) we have
For a Lorentzian gas consisting of neutral molecules and electrons (a reasonable assumption for a slightly ionized gas) it can be shown that

\[ \frac{\partial f}{\partial t} = -\mathbf{v} \cdot \mathbf{f}' + \frac{m}{M \nu^3} \frac{\partial}{\partial \nu} \left( \mathbf{f}_0 \nu^3 \right) + \frac{kT}{M \nu^3} \frac{\partial}{\partial \nu} \left( \nu \nu^2 \frac{\partial f_0}{\partial \nu} \right), \]

where \( k \) is the Boltzmann's constant, \( M \) is the mass of the molecules and \( T \) is the temperature of the gas.

Substituting Eqs. (5), (9), (14) and (15) in (1), we obtain

\[
\begin{aligned}
&\left( \frac{1}{\nu} \mathbf{\varepsilon} \cdot \frac{\partial \mathbf{f}'}{\partial \nu} + \gamma \cdot \mathbf{f}' + g \cdot \frac{\partial \mathbf{f}'}{\partial \nu} \right) \cdot \mathbf{v} \\
&+ \left( \omega \mathbf{f}_2 \cos \omega t - \omega \mathbf{f}_1 \sin \omega t + \mathbf{\omega}_0 \times \mathbf{f}' + \frac{1}{\nu} \mathbf{\varepsilon} \cdot \mathbf{f}_0 + \gamma \mathbf{f}_0 + \mathbf{g} \cdot \frac{\partial \mathbf{f}_0}{\partial \nu} + \mathbf{v} \mathbf{f}' \right) \cdot \mathbf{v} \\
&+ \mathbf{\varepsilon} \cdot \mathbf{f}' - \frac{m}{M \nu^3} \frac{\partial}{\partial \nu} \left( \mathbf{f}_0 \nu^3 \right) \\
&- \frac{kT}{M \nu^3} \frac{\partial}{\partial \nu} \left( \nu \nu^2 \frac{\partial f_0}{\partial \nu} \right) = 0.
\end{aligned}
\]

Since Eq. (2) is an approximation of the first order, we have to consider only the first two terms in polyadic expansion, namely the terms with \( \nu \mathbf{v} \) and \( \mathbf{v} \). In order to insure the best use of the term with \( \nu \mathbf{v} \) we may, following previous workers,\(^{1,2}\) average it. Remembering that \( \langle \mathbf{A} \cdot \mathbf{v} \mathbf{B} \cdot \mathbf{v} \rangle = \mathbf{v}^2 / 3 \cdot \mathbf{A} \cdot \mathbf{B} \) if \( \mathbf{A} \) and \( \mathbf{B} \) depend only on the magnitude of \( \mathbf{v} \), Eq. (16) can be expressed as

\[
\begin{aligned}
&\left[ \omega \left( \mathbf{f}_2 \cos \omega t - \mathbf{f}_1 \sin \omega t \right) + \mathbf{\omega}_0 \times \mathbf{f}' + \frac{1}{\nu} \mathbf{\varepsilon} \cdot \mathbf{f}_0 + \gamma \mathbf{f}_0 + \mathbf{g} \cdot \frac{\partial \mathbf{f}_0}{\partial \nu} + \mathbf{v} \mathbf{f}' \right] \cdot \mathbf{v} \\
&+ \mathbf{\varepsilon} \cdot \mathbf{f}' - \frac{m}{M \nu^3} \frac{\partial}{\partial \nu} \left( \mathbf{f}_0 \nu^3 \right) - \frac{kT}{M \nu^3} \frac{\partial}{\partial \nu} \left( \nu \nu^2 \frac{\partial f_0}{\partial \nu} \right) + \frac{\mathbf{v}^2}{3} \left( \frac{1}{\nu} \mathbf{\varepsilon} \cdot \frac{\partial \mathbf{f}'}{\partial \nu} + \gamma \cdot \mathbf{f}' + g \cdot \frac{\partial \mathbf{f}'}{\partial \nu} \right) = 0.
\end{aligned}
\]

Since \( \mathbf{v} \) is an arbitrary vector, the coefficients of \( \mathbf{v} \) and the remaining term in (17) must be independently equal to zero, i.e.

\[
\begin{aligned}
&\left( \omega \mathbf{f}_2 \cos \omega t - \mathbf{f}_1 \sin \omega t \right) + \mathbf{\omega}_0 \times \mathbf{f}' + \frac{1}{\nu} \mathbf{\varepsilon} \cdot \mathbf{f}_0 + \gamma \mathbf{f}_0 + \mathbf{g} \cdot \frac{\partial \mathbf{f}_0}{\partial \nu} + \mathbf{v} \mathbf{f}' \\
&+ \mathbf{\omega}_0 \times \mathbf{f}_1 + \frac{1}{\nu} \mathbf{\varepsilon} \cdot \mathbf{f}_0 + \gamma \mathbf{f}_0 + \mathbf{g} \cdot \frac{\partial \mathbf{f}_0}{\partial \nu} + \mathbf{f}_1 = 0
\end{aligned}
\]

and

\[
\begin{aligned}
&\frac{m}{M} \frac{\partial}{\partial \nu} \left( \mathbf{f}_0 \nu^3 \right) + \frac{kT}{M} \frac{\partial}{\partial \nu} \left( \nu \nu^2 \frac{\partial f_0}{\partial \nu} \right) = \mathbf{v}^2 \mathbf{\varepsilon} \cdot \mathbf{f}' + \frac{\mathbf{v}^4}{3} \left( \frac{1}{\nu} \mathbf{\varepsilon} \cdot \frac{\partial \mathbf{f}'}{\partial \nu} + \gamma \cdot \mathbf{f}' + g \cdot \frac{\partial \mathbf{f}'}{\partial \nu} \right).
\end{aligned}
\]
Because Eq. (18) is valid for any instant of time, coefficients of \( \sin \omega t \), \( \cos \omega t \) and the time independent term must be equal to zero, i.e.

\[
\omega f_2 + \omega_0 \times f_2 + v f_3 = 0,
\]

(20)

\[
v f_2 - \omega f_3 + \omega_0 \times f_3 = - \frac{1}{\nu} \epsilon_1 \frac{\partial f_0}{\partial \nu}
\]

(21)

and

\[
\omega_0 \times f_1 + v f_1 = A,
\]

(22)

where

\[
A = \frac{1}{\nu} \epsilon_0 \frac{\partial f_0}{\partial \nu} - \gamma f_0 - g \frac{\partial f_0}{\partial \nu}.
\]

(23)

Solving for \( f_1 \) from Eq. (22) we have

\[
f_1 = \frac{\nu A + (1/\nu) A \cdot \omega_0, \omega_0 + A \times \omega_0}{\nu^2 + \omega_0^2}.
\]

(24)

Solving for \( f_2 \) and \( f_3 \) from (20) and (21), we have

\[
f_2 = - \frac{(1/\nu) (\partial f_0/\partial \nu)}{\nu^2 + \omega^2 - \omega_0^2 + 4 \nu^2 \omega_0^2} \left[ \nu (\nu^2 + \omega_0^2 + \omega^2) E_1 + (\omega^2 - \omega_0^2 - \nu^2) E_1 \times \omega_0
\]

\[
\quad + \nu \left( 1 + \frac{\omega_0^2 + 4 \nu^2}{\omega^2 + \nu^2} \right) E_1 \cdot \omega_0 \right]
\]

(25)

and

\[
f_3 = - \frac{(1/\nu) (\partial f_0/\partial \nu)}{\nu^2 + \omega^2 - \omega_0^2 + 4 \nu^2 \omega_0^2} \left[ \omega (\nu^2 + \omega_0^2 + \omega^2) E_1 + 2 \nu E_1 \times \omega_0
\]

\[
\quad + \omega \left( 1 + \frac{\omega_0^2 + 4 \nu^2}{\omega^2 + \nu^2} - 1 \right) E_1 \cdot \omega_0 \right].
\]

(26)

From Eqs. (24) and (25) one can see that when the magnetic field is zero, both \( f_2 \) and \( f_3 \) are in the direction of the applied electric field. However, when the magnetic field is present, then not only a component of \( f' \) in the direction \( E_1 \times \omega_0 \) or \( E_1 \times B_0 \) appears, but also a component in the direction of the magnetic fields, giving rise to part of the current in that direction.

Substituting for \( f' \) from Eqs. (24), (25) and (26) in Eq. (19), averaging over the cycle and rearranging the terms, we obtain the following ordinary differential equation for \( f_0 \),

\[
\frac{\partial}{\partial \nu} \left\{ \frac{m}{M} \nu \nu^3 f_0 + \frac{kT}{M} \nu \nu^3 \frac{\partial f_0}{\partial \nu} - \frac{\nu^3}{3 (\nu^2 + \omega_0^2)} \left( \nu A \cdot E_0 + \frac{1}{\nu} A \cdot \omega_0 E_0 \cdot \omega_0 + A \times \omega_0 \cdot \omega_0 \right) \right\}
\]

\[
+ \frac{v^3}{6} \frac{\partial f_0}{\partial \nu} \left( \nu^2 + \omega^2 - \omega_0^2 \right) + 4 \nu^2 \omega_0^2 + \nu \left( 1 + \frac{\omega_0^2 + 4 \nu^2}{\omega^2 + \nu^2} \right) E_1 \cdot \omega_0 \right]
\]

\[
= \frac{\nu^4}{3 (\nu^2 + \omega_0^2)} \left( \nu \gamma \cdot A + \frac{1}{\nu} A \cdot \omega_0 \gamma \cdot \omega_0 + A \times \omega_0 \cdot \gamma \right) + \frac{v^4}{3 g} \frac{\partial}{\partial \nu} \left( \nu A + (1/\nu) A \cdot \omega_0 E_0 + A \times \omega_0 \right).
\]

(27)
In general Eq. (27) can be solved numerically for $f_o$ (providing one knows the dependence of collision frequency on velocity) with the boundary conditions $f_o = 0$ at $v = \infty$ and $\int_0^\infty 4\pi v^3 f_o dv = n$.

§ 3. Electric current in the plasma

The current density due to electrons is given by

$$i = q n \langle v \rangle,$$

where the average velocity $\langle v \rangle$ is given by

$$\langle v \rangle = \frac{1}{n} \int f v df.$$

and $dA$ is the volume element in velocity space. Expressing explicitly from Eq. (2) and integrating over all directions, Eq. (28) reduces to

$$\langle v \rangle = \frac{4\pi}{3n} \int_0^\infty v^4 f d\nu.$$

The direct current density is

$$i_d = \frac{4\pi q}{3} \int_0^\infty v^4 f_0 dv$$

and the alternating current density is

$$i_a = \frac{4\pi q}{3} \int_0^\infty v^4 f_0 dv + \frac{4\pi q}{3} \cos \omega t \int_0^\infty v^4 f_0 dv.$$

Using for $f_i$ expression (25), Eq. (30) becomes

$$i_d = \frac{4\pi q}{3} \int_0^\infty \nu A + \nu^{-1} A \cdot \omega_0 \omega_0 + A \times \omega_0 \nu^4 d\nu.$$

Similary, using Eqs. (24) and (26) in Eq. (31) we have

$$i_a = -\left[ \frac{4\pi q}{3} \int_0^\infty \nu (\nu^2 + \omega_0^2 + \omega_0^2) \mathcal{E}_1 + (\omega_0^2 - \omega_0^2 - \nu^2) \mathcal{E}_1 \times \omega_0 + \nu (1 + (\omega_0^2 + 4\nu^2/\nu^2 + \omega_0^2)) \mathcal{E}_1 \cdot \omega_0 \omega_0 \right.$$

$$\times \left. \frac{(\nu^2 + \omega_0^2 - \omega_0^2)^2 + 4\nu^2 \omega_0^2}{(\nu^2 + \omega_0^2 + \omega_0^2)^2 + 4\nu^2 \omega_0^2} \frac{\partial f_0}{\partial \nu} d\nu \right] \sin \omega t +$$

$$\times \nu^3 \frac{\partial f_0}{\partial \nu} d\nu \right].$$
Thermal current density due to the electrons is given by
\[
c = -\frac{nm}{2} \langle v^2 \rangle.
\] (34)

or, in the integral form,
\[
c = \frac{m}{2} \int_\mathcal{A} v^2 v f dA.
\] (35)

Integrating over all directions, (35) reduces to
\[
c = \frac{2m\pi}{3} \int_0^{\infty} v^6 f' dv.
\] (36)

Direct thermal current is
\[
c_d = \frac{2m\pi}{3} \int_0^{\infty} v A + v^{-1} A \cdot \omega_0 \omega_0 + A \times \omega_0 d\nu
\] (37)

and the alternating component is
\[
c_a = -\frac{2\pi m}{3} \int_0^{\infty} v (v^2 + \omega_0^2) \mathbf{E}_1 + (\omega_0^2 - \omega_0^2 - v^2) \mathbf{E}_1 \times \omega_0 + \nu (1 + (\omega_0^2 + 4v^2) / (v^2 + \omega_0^2)) \mathbf{E}_1 \cdot \omega_0 \omega_0
\]
\[
\times v^6 \frac{\partial f_0}{\partial v} dv \sin \omega t
\]
\[
+ \frac{2\pi m}{3} \int_0^{\infty} (v^2 + \omega_0^2) \mathbf{E}_1 + 2v \mathbf{E}_1 \times \omega_0 + ((\omega_0^2 + 4v^2) / (v^2 + \omega_0^2) - 1) \mathbf{E}_1 \cdot \omega_0 \omega_0
\]
\[
\times v^6 \frac{\partial f_0}{\partial v} dv \cos \omega t.
\] (38)

Equations (32), (33), (37) and (38) may be expressed in the form of averages by using the relation
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\[ \int_{v=0}^{\infty} \xi(v) \frac{\partial f_0}{\partial v} dv = \xi(0) f_0(0) - \frac{n}{4\pi} \langle 1 - v^2 \frac{\partial \xi}{\partial v} \rangle. \] (39)

§ 5. Some special cases

When there is no electric or magnetic field present, but only the density gradient and collision frequency gradient, then only direct component of the current density exists, viz.

\[ i_d = -\frac{4\pi q}{3} \int_{0}^{\infty} \gamma f_0 + g \frac{\partial f_0}{\partial v} \nu d\nu. \] (40)

It is important to note that \( f_0 \) is not Maxwellian, but the solution of the differential equation:

\[ \frac{\partial}{\partial v} \left( \frac{m}{M} v^3 f_0 + \frac{kT}{M} v^2 \frac{\partial f_0}{\partial v} \right) = -v^4 \left( \gamma f_0 + g \frac{\partial f_0}{\partial v} \right) - \frac{v^4}{3} \gamma \frac{\partial}{\partial v} \left( \frac{\gamma f_0 + g}{\nu} \frac{\partial f_0}{\partial v} \right) \] (41)

obtained from Eq. (27).

If the only source of current is the electron density gradient,

\[ i_d = -\frac{4\pi q}{3} \int_{0}^{\infty} \gamma f_0 \nu^4 d\nu = -\frac{qn}{3} \langle v^2 \rangle \] (40A)

\[ \frac{\partial}{\partial v} \left( \frac{m}{M} v^3 f_0 + \frac{kT}{M} v^2 \frac{\partial f_0}{\partial v} \right) = -\frac{v^4}{3} \gamma f_0. \] (41A)

For a uniform gas (\( \gamma = 0 \) and \( g = 0 \)) Eqs. (32), (33) (37) and (38) reduce to the results obtained by Sodha.\(^3\)

§ 6. Discussion

An interesting result which the present analysis, using the vector notation, clearly brings out is the fact that the current in general has a part proportional to the magnetic field vector \( B_0 \) besides those proportional to the electric field and gradient vectors, and the vector product of the electric field and gradient vectors with the magnetic field vector. When the magnetic vector is perpendicular to the electric field and gradient vector, the part of current proportional to the magnetic vector vanishes.

Another interesting result, which may be hard to arrive at intuitively is the fact that high gradients of electron density and collision frequency lead to non-Maxwellian distribution of electron velocities. This is of importance in problems involving plasma sheaths, where high gradients of electron density and collision frequency are present. Under such conditions the use of usual theories (based
on Maxwellian distribution of electron velocities) may lead to large errors and
the present theory, though tedious may be the only way for getting satisfactory
solutions.

Sodha\textsuperscript{1} has discussed the application of a previous similar analyses to the
investigation of propagation of electromagnetic waves in and transport properties
of ionized gases. The discussion is equally valid for the present analysis.

A remark about the validity of averaging over a cycle, which is the basis
for Eq. (27) is in order. The averaging is meaningful only when the period
of the cycle $2\pi/\omega$ is much less than the period of observation, which is of
course finite. Hence Eq. (27) is not valid as $\omega \to 0$ and gives an incorrect
result when we put $\omega = 0$.

References

   \textbf{21}, 383.

2) S. Chapman and T. G. Cowling, \textit{Mathematical Theory of Nonuniform Gases} (Cambridge