A Reflection on *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* by J. Guckenheimer and P. J. Holmes

In the autumn of 1983, I was just embarking on a Ph.D. in structural dynamics at the University College London (UCL) under the guidance of Professor Michael Thompson, FRS. I had spent the previous year as an exchange student at the University of Illinois and was enjoying being back in the center of London. At this time the cold war was at its height, Margaret Thatcher won reelection (thanks to the Falklands war), CDs were the latest thing, *Dallas* and *Dynasty* were the most popular shows on TV, John McEnroe ruled Wimbledon, Liverpool dominated English football, and Michael Jackson was known for his music.

The main reason I had chosen UCL for graduate school (other than its history and location) was the chance to study how (engineering) systems lose stability with someone who had built a reputation for enthusiastically embracing the interdisciplinary nature of dynamical systems. During the previous 20 or so years, UCL had developed a world-renowned reputation for research in buckling. However, despite the practical impetus provided by buckling problems associated with North Sea oil development, it was clear that the most exciting research avenues in engineering mechanics would involve dynamics.

Also at this time, a number of separate strands were developing in the world of scientific research that were profoundly affecting the way dynamical systems could be studied. Digital computation was becoming accessible. Large mainframe computers were being supplemented by early versions of today's PCs, and this enabled a good deal of numerical experimentation (especially including graphics) to become routine. The concept that even simple nonlinear dynamical systems could exhibit *extreme* sensitivity to initial conditions was starting to fascinate a wide spectrum of people trying to predict physical behavior, with the implications of chaos stretching (and folding) far and wide. Furthermore, the classification and framework of bifurcation theory was leading to a deeper understanding of generic behavior and this, of course, was crucial in helping to decipher the large amounts of data becoming available, as well as providing a guiding path for experimental work. These factors together were simultaneously expanding the horizon on the range of problems that could be attacked as well as highlighting the limits of what could be done (even under practically perfect conditions and relatively precise knowledge of a physical system).

Thus, like most starting postgraduate students, I set about searching the literature (in the cozy and warrenlike library at UCL, rather than the Web of Science). I felt that I had a pretty good handle on linear vibration theory, as well as an inkling of the potential complexity of (static) nonlinear structural behavior, but the idea of studying *nonlinear* dynamics in the context of structural mechanics started to focus my attention. Linear theory had served engineers very well. I had graduated believing that pretty much all differential equations could be solved analytically (with Laplace transforms, what else did we need), and that it was a highly unusual situation (or more likely an error) if a small change to an experiment led to a completely different response.

Two of the first books I came across at this time were *Nonlinear Ordinary Differential Equations* by Jordan and Smith [1], and *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* by Guckenheimer and Holmes (G&H) [2]. Perhaps, like many engineering students, I felt that my mathematical background was not especially strong, and books toward the theoretical end of the mathematical spectrum rapidly delved into material that was difficult to follow, even in the first chapter. Without wanting to generalize, it seemed to me that authors of these types of book did not necessarily have pedagogic intent as a high priority, which, coupled with my limited mathematical background, made hard work.

However, these two books resonated with me straight away. Their clear, concise explanations were a breath of fresh air. Jordan and Smith [1] had been published a few years earlier and largely focused on the role of approximate analytical methods. G&H [2] came hot off the press and felt strongly *current*. Both of these books made very effective use of examples, diagrams and sketches, and explanations and details that were exactly at the level I needed to learn about the behavior (especially the instability) of nonlinear dynamical systems. Although I always had the goal of conducting physical experiments in the back of my mind, these fine examples of the power of analysis to explain physical phenomena both became permanent fixtures on my desk.

The first example in G&H [2] is a mass attached to a nonlinear spring. Now this was something I could use. Concepts of linearization and stability came next, and I started to become immersed in the geometrical view of trajectories evolving in phase space with equilibrium *attracting* local transients, etc. Then, closed orbits extended this view from point to periodic attractors, and opened the door to the potentially bewildering array of possible behavior in higher-order phase space. In engineering terms, we had gone from free to forced vibrations, but now it was becoming apparent how restrictive the conventional restriction to linearity had been. For a forced *linear* oscillator, resonance was the new feature when one passed from free (two-dimensional (2D) phase space) to forced (three-dimensional (3D) phase space) vibrations, with long-term recurrent behavior unique, and thus independent, of the initial conditions. But for a forced *nonlinear* oscillator the range of possible behavior was daunting to say the least (let alone for high-order systems)—fertile ground for research though.

Perhaps the key tool that I learned early from G&H was the notion of a Poincaré map. I suppose I had had a little exposure to discrete dynamical systems primarily via a brief review of the z transform in digital circuit analysis, but that, again, was restricted to linear systems. G&H used the context of forced mechanical oscillators to introduce a Poincaré map, which, for a periodically forced linear oscillator, could be written analytically. This effectively shifted my understanding from the conventional amplitude and phase response diagram to the stroboscopic sampling of a periodic orbit and the reduction of a continuous flow in 3D to a discrete map in 2D. For nonlinear systems, the Poincaré map could only be obtained in an approximate analytic sense or numerically. But I could now study stability, which was not really an issue for linear systems, in terms of characteristic multipliers, and using basic linear algebra, with which I did have adequate experience. It also became a goal of mine to develop some of these concepts in an experimental context.

The second chapter proceeded to focus on a number of archetypal equations, with the first two having a strong foundation in engineering applications: Van der Pol and Duffing. It was especially the latter example that I started to scrutinize in my research with G&H as my guide. The fact that G&H were able to relate behavior back to a well-defined example in engineering mechanics, the vibrations of a buckled beam, was especially useful to me.

The third chapter focused on local bifurcations, and having some background in buckling helped, with, again, an extension to discrete systems (and all within a dynamics context) illustrating the general nature of bifurcation theory. The ability to reduce the order of a system in the vicinity of a bifurcation using the tools of center manifold theory and normal forms was valuable. This material was then followed by a chapter on perturbation theory and averaging, laying the groundwork for the development of Melnikov theory, which provided a useful lower bound for the bifurcation of homoclinic orbits and the appearance of complex behavior. The final chapters focused on strange attractors and global bifurcations. Examining these features in an experimental context would motivate me well beyond my Ph.D. work.

In thinking back over 20 years, it is apparent that G&H [2] has stood the test of time. I think for many graduate students, in the natural course of events, there are often one or two books and papers that become key references. G&H [2] served this purpose for me, and it is interesting to speculate on the extent to which later books, for example, Thompson and Stewart's *Nonlinear Dynamics and Chaos* [3] (largely focusing on geometric behavior) and Moon's *Chaotic and Fractal Dynamics* [4] (largely focusing on physical experiments), were influenced, or even motivated by G&H. The influence of G&H [2] on the writer's *Introduction to Experimental Nonlinear Dynamics* [5] is unambiguous. I recently noted that the copy of G&H in Duke's library was especially tatty and actually falling apart from overuse. That says it all, really.

Thus, I am very grateful for G&H [2]. It still has a current feel to it and, despite my one small quibble (their slightly pretentious inclusion of a poem in French in the preface), still occupies a prominent place on my bookshelf—when it is not being loaned out to a new postgraduate student needing some background and motivation before diving into the world of research. Sounds familiar, does it not?

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