Real and imaginary Kirkwood gaps

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ABSTRACT
The evidence for resonant structure in the distribution of the semimajor axes of members of asteroid families and meteoroid streams is reviewed. It is shown that, because families and streams are identified by clustering of orbital elements, great care must be taken to distinguish between actual removal of objects ('real' gaps) and the failure to identify members of families or streams because of resonant perturbations ('imaginary' gaps). If it can be shown that gaps are real then this can lead to important constraints on the time-scale for the removal of objects. Sample variations of semimajor axis and eccentricity are calculated for resonant locations in the Themis asteroid family and the Quadrantid meteoroid stream using high-order expansions of the planetary disturbing function. Evidence is presented suggesting that a prominent gap in the Themis family associated with the 15:7 jovian resonance is real, whereas that associated with the 3:1 resonance in the Quadrantid meteors is imaginary. Recent claims concerning resonant structure in the Perseid meteor stream are examined in a similar context.

Key words: methods: analytical – meteors, meteoroids – minor planets, asteroids.

1 INTRODUCTION

Complex resonant phenomena pervade the Solar system. The effects of resonance are most obvious where one finds perturbers and a spatially extended supply of material to be perturbed. One such location is the outer part of the A ring of Saturn where mean motion resonances of the form \( p + 1 : p \) between ring particles and the small satellites orbiting just outside the main ring system result in features which are clearly detectable in Voyager images (see, for example, fig. 1 of Murray 1994).

Although it has a smaller spatial density than the ring system of Saturn, the main belt of asteroids extends over 2 \( \text{au} \) in semimajor axis and undergoes substantial perturbations from Jupiter. There is resonant structure (see, for example, Dermott & Murray 1981, 1983) which is clearly associated with mean motion resonances with Jupiter. The first hints of this structure were recognized by Kirkwood (1867) with a sample of less than 100 asteroids, and the gaps in the distribution are now referred to as the Kirkwood gaps.

The IAU Meteor Data Center at Lund Observatory (Lindblad 1991) has orbital data on \( \geq 69,000 \) meteors covering a much wider range of semimajor axis and eccentricity than the asteroid distribution. The resonant structure of the meteor distribution was discovered by Lindblad (1973) using a sample of 1822 orbits of photometric meteors.

Although the resonant structure of asteroid and meteoroid orbits has now been demonstrated, an understanding of the mechanism that has produced this structure has been more difficult to discover. In this paper we review our knowledge of the formation of Kirkwood gaps. In particular we concentrate on the evidence for resonant structure within groups of asteroids and meteoroids which are thought to be younger samples of their respective populations. We highlight the need to distinguish between gaps that are formed by the genuine removal of material ('real' gaps) and those that could result from a failure to identify some material as belonging to a particular group because of the resonant perturbations ('imaginary' gaps). The resonant structures of the Themis family of asteroids and the Quadrantid meteoroid stream are taken as examples. Using high-order expansions of the planetary disturbing function and elementary perturbation theory, we develop some analytical tools which, in certain circumstances, can allow us to distinguish between real and imaginary gaps. We also comment on the recent paper by Wu & Williams (1995) on the resonant structure of the Perseid meteoroid stream.

2 KIRKWOOD GAPS

The prominent Kirkwood gaps in the asteroid belt are located at the major mean motion resonances with Jupiter (see Fig. 1). Note that, although there are gaps associated with most of the resonances, there are actually concentrations of material at the 3:2, 4:3 and 1:1 resonances. The asteroids at the 1:1 resonance are the Trojan group of asteroids which librate about the \( L_4 \) or \( L_5 \) Lagrangian equilibrium points co-orbital with Jupiter. The asteroids at the 3:2 resonance (the Hilda group) and the 4:3 resonance are known to be undergoing stable libration. Note that only those resonances associated with well-established gaps or groups are indicated in Fig. 1.
Figure 1. The distribution of asteroids in semimajor axis showing the location of gaps and groupings associated with jovian resonances.

A good indication of the significance of any fluctuation in the distribution in such a histogram is that typical errors in a local, normal distribution would be $\sim \sqrt{N}$ where $N$ is the number of objects in each bin. A more quantitative approach using the same principle was employed by Dermott & Murray (1981, 1983) in their statistical studies of the Kirkwood gaps.

Dermott & Murray (1981) concluded that the gaps could be understood on the basis of the restricted three-body problem involving the gravitational perturbations of Jupiter on individual asteroids. They pointed out that the width of the gaps corresponds to the maximum libration width associated with the dominant resonant argument for each resonance. Wisdom (1982, 1983) developed an algebraic mapping technique to study asteroid motion at the 3:1 resonance. He showed that there was a sizeable chaotic region at the centre of the resonance and that test particles started in this zone would undergo large variations in their eccentricities causing them to cross the orbit of Mars and be removed. The extent of the chaotic zone was in good agreement with the cleared region. Murray & Fox (1984) showed that Wisdom’s mapping was an accurate reflection of the motion based on the averaged disturbing function truncated to second order. By separating the three different time-scales (orbital, resonant and secular) involved in the problem, Wisdom (1985) provided an analytical basis for the observed motion and the origin of the chaos.

Murray (1986) derived mappings for the 2:1 and 3:2 resonances using a second-order expansion of the planetary disturbing function. On this basis he showed that there were large chaotic regions at the centres of both resonances, although he was aware of the limitations of low-order expansions for studying asteroid motion in a regime of slow convergence such as the location of the 3:2 resonance. His suspicions were subsequently verified by Wisdom (1987) who showed by numerical integration that the central region of the 3:2 resonance is not chaotic, thereby providing the first numerical evidence for a qualitative difference between the 2:1 and 3:2 resonances.

Regions of resonance overlap give rise to chaotic motion (see, for example, Chirikov 1979), and it had been thought for some time that the outer part of the asteroid belt (semimajor axis, $a$, between 4 au and 5 au) had been depleted by adjacent resonances overlapping at successively lower values of eccentricity, $e$, as $a$ approaches the orbit of Jupiter (Wisdom 1980; Dermott & Murray 1983; Milani & Nobili 1985).

In particular, Wisdom (1980) had shown that adjacent resonances of the form $p + 1 : p$ would overlap at all $e$ for $p > 4$ in the Sun–Jupiter system, in good agreement with observations. This implies that the width, $W$, of the cleared zone due to these first-order resonances around any planet would vary with the planet–Sun mass ratio, $\mu$, according to

$$ W \propto \mu^{2/7}. $$

(1)

This result has now been verified by a variety of numerical investigations of test particle motion in the outer Solar system (see, for example, Duncan, Quinn & Tremaine 1989; Holman & Wisdom 1993).

One would also expect orbits that take asteroids sufficiently close to Jupiter (not just its orbit) also to cause removal on time-scales of less than an orbital period due to direct interaction. However, the one exception to this is material on orbits of relatively low eccentricity and inclination in the immediate vicinity of Jupiter. Such material would be in the 1:1 resonance where stable motion about Lagrangian equilibrium points is possible.

Thus, for the purposes of this paper we define a Kirkwood gap to be an absence of material at the location of a first-, or higher-order, resonance due to perturbations which can be modelled in the context of a restricted three-body problem. Hence we do not consider gaps coincident with the perturber’s orbit or in its immediate vicinity to be Kirkwood gaps.

3 GAPS IN ASTEROID FAMILIES

Each of the Hirayama families of asteroids is thought to be composed of the collisional fragments of a single asteroid which broke up after impact with another body. This explains why the clustering in $a$–$e$–$I$ space, where $I$ is the orbital inclination, is more apparent when proper rather than osculating elements are used. Note that information on the pericentres and nodes of asteroid orbits is not used in the determination of family members, since their free components are expected to have been randomized by planetary perturbations.

Since each family contains members of the same age, the existence of Kirkwood gaps in the distribution of a family would place a limit on the time-scale for the formation of the Kirkwood gaps. Thus, provided it is possible to date the families, an upper bound on the time-scale of the gap formation process can be obtained. Unfortunately the age of the larger families is still uncertain; estimates range from $\sim 10^6$ yr (Steins 1956) to $\sim 10^7$ yr (Dermott et al. 1984). Nevertheless, it is important to establish the time-scales for relative spreading of family members and the formation of resonant gaps if we are to be able to develop realistic models of the long-term evolution of the asteroid belt.

Fig. 2 shows the distribution in proper semimajor axis and eccentricity of those asteroids identified as family members in the study by Zappala et al. (1994), together with the location of all jovian resonances of the form $p + q : p$ for $q \leq 9$ and $q \leq 5$ that lie within the semimajor axis range covered by the families. It is clear that, with the possible exception of the 7:2 resonance and Flora family, no major resonances lie within family groupings. However, the Flora family has some unusual characteristics which may indicate that its origin is different from other families (Gradie, Chapman & William 1979).

It could be argued that the family members shown in Fig. 2 already exhibit a marked preference for avoiding the
major resonances. The reason for this is that resonances such as the 3:1, 5:2 and 2:1 are those that are known to be associated with (real) Kirkwood gaps (cf. Fig. 1) and correspond to genuine removal of asteroids, whether family members or not. It is likely that resonance has played a role in determining the limits in semimajor axis of some asteroid families: for example, the sharp outer edge of the Eos family is close to the 9:4 resonance. However, here we are concerned with the existence of gaps within asteroid families.

3.1 The Themis family

Dermott & Murray (1981) showed that there are gaps in the distribution in semimajor axis of members of the Themis family of asteroids ($a \approx 3.14$ au, $e \approx 0.15, I \approx 1^\circ.42$). These gaps are associated with the location of the 13:6 and 15:7 resonances with Jupiter (a seventh- and eighth-order resonance respectively). Dermott & Murray used a running box technique on data for the 62 Themis family members. In Fig. 3 we show a similar plot but using a more recent sample of 307 asteroids identified by Zappalà et al. (1994) as Themis members. The data were plotted using a box size of 0.01 au and a step size of 0.001 au. Again it is clear that there is structure in the distribution and that the gap at the 15:7 resonance remains in the enlarged sample. Superimposed on Fig. 3 we have plotted the locations of all resonances of the form $p+q:p$, for $q < 13$, where $q$ is the order of the resonance. The correlation of other possible gaps with high-order mean motion resonances (for example, Fig. 3 shows evidence of a gap at the 25:12 jovian resonance) is noted here but not investigated further. It is important to note that there are an infinite number of mean motion resonances of the form $p+q:p$ within any finite range of semimajor axis. However, as we show in Section 5 below, the strength of a given resonance varies approximately as $\sqrt{e^2}$ or $\sin f^4$ and hence falls off dramatically with increasing $q$. Therefore, although there may well be additional resonant gaps in the family, we restrict ourselves to those for which the evidence and strength are greatest.

Comparing Fig. 3 with fig. 5 in Dermott & Murray (1981) we note that there is no longer a clearly defined gap at the 13:6 resonance. There are a number of possible explanations for this. The five-fold increase in family membership due to the discovery of new asteroids has added fainter, and hence smaller, objects to the data set. It is possible that structure within a family may be partly determined by the collisional disruption of the several larger fragments at some time after the initial event, causing the addition of fainter material to the new data set. It is also possible that the 13:6 gap detected by Dermott & Murray, the less prominent of the two gaps discussed by the authors, may not have been a real gap. Consequently we confine the perturbation analysis in Section 5 to a study of the 15:7 resonance alone.

Dermott & Murray (1981) pointed out that the gaps in the Themis family might not be real and that resonant perturbations could cause some asteroids not to be recognized as genuine family members. One possible test of their hypothesis is to investigate the distribution of asteroids that are not Themis members but which lie in the same range of semimajor axis. Since these are not family members (see Fig. 2), there can be no suggestion that a detectable gap is imaginary. Such a plot is shown in Fig. 4. An inspection of the asteroid distribution shows evidence of a gap associated with the 13:6 resonance, but there are gaps of similar width nearby with no comparable strength resonance to have produced them. Furthermore the evidence for a gap at the 15:7 resonance is marginal in this case. This suggests that the 15:7 gap shown in Fig. 3 could be imaginary. However, there may be other subtle effects at work. For example, it is conceivable that there may be dynamical reasons why a clustering of $a, e$ and $I$ in a family is more conducive to gap formation. Therefore, we must resort to a more sophisticated analysis in order to determine if the 15:7 Kirkwood gap is real or imaginary. Such an analysis is presented in Section 5 below. First we consider equivalent phenomena in meteoroid streams.

4 GAPS IN METEOROID STREAMS

The first evidence for gaps in the distribution of meteoroid orbits was given by Lindblad (1973) in his study of 1822 precisely reduced photographic meteors from the Harvard, Dushanbe and Odessa programmes. In histograms of the distribution of $1/a$ he noticed considerable fine structure which appeared to correlate with the location of the principal jovian resonances. After investigating a variety of possible causes he concluded that the structure and correlation were real. Using the $D$-criterion of Southworth & Hawkins (1963), he divided the data sample into stream members and sporadics, showing that the Geminid meteoroids produced a significant local maximum in the distribution of stream members. When local enhancements of structure due to streams were removed there was little evidence that the two distributions were significantly different. Lindblad (1973) argued that sporadics may be older and hence "more gravitationally perturbed" than stream meteors, and so there may be theoretical reasons why the gaps should be different in each data set. Although he searched for such differences, Lindblad appears not to have recognized the possibility of imaginary gaps, nor to have realized the constraints that real gaps in streams would place on time-scales for a gap formation mechanism.

Using orbital data on the $\sim$40 000 meteors recorded in the Harvard radio meteor survey (see, for example, Sekanina 1970), Murray (1987) carried out a preliminary study of the
Figure 3. The number of Themis asteroids per au as a function of proper semimajor axis. The locations of all jovian mean motion resonances of the form \( p + q : p \) for \( p < 10 \) and \( q < 13 \) are shown as dashed lines. The data are plotted using the same running box technique as Dermott & Murray (1981).

Figure 4. The number of non-Themis asteroids per au as a function of osculating semimajor axis for those asteroids lying between 3.05 and 3.25 au. The data were derived using the same technique as in Fig. 3.

possibility of resonant structure in the principal meteoroid streams. By analogy with possible gaps in the Themis family of asteroids, he pointed out the importance of (i) distinguishing real gaps from artificial ones and (ii) using real gaps to establish time-scales of formation. Since the radio data are known to have larger errors than the best photographic data, his results were inconclusive. However, Murray did find some evidence for resonant structure in the Quadrantid stream.

Fig. 5 shows a histogram of the distribution of \( a \) of the precisely reduced photographic orbits given in the IAU Meteor Data Center catalogue (Lindblad 1991). Here we are only considering the orbital data of those meteors with \( e < 1 \) and \( 1.5 < a < 5.5 \) au (cf. Fig. 1). The sample contains 1760 orbits. The locations of a selection of strong jovian mean motion resonances are indicated by dashed lines. Although a more detailed statistical analysis can be carried out, a comparison with Fig. 1 shows that a correlation between gaps and resonances is much harder to establish in the case of meteors. Application of the \( \sqrt{N} \) rule (see Section 2 above) confirms the weakness of the meteor result. However, Fig. 5 does suggest that there may be some evidence for gaps associated with the 5:2 and 2:1 jovian resonances.

At this stage it is important to note a fundamental difference between the cataloguing of asteroid and meteor orbits. Unlike asteroid orbits, a meteoroid orbit can only be determined because a particle collides with the Earth's atmosphere and produces a meteor that alerts an observer to its existence and imminent demise. Therefore, all meteor orbits suffer from an extreme observational selection effect whereby only the subset of objects that collide with Earth is observed. Furthermore, intersection of meteoroid orbits with the Earth must occur at the ascending or descending node of orbits that are out of the ecliptic. Another important difference occurs in the identification of clusterings. Members of an asteroid family are identified by a grouping in proper \( a \), \( e \) and \( i \); indeed observations show that family members have randomized free pericentres and nodes. On the other hand, meteoroid stream members are identified on the basis of clustering in the angular orbital elements, particularly the ascending or descending node since this is related to the time of observation at Earth.

In Fig. 6 we show the \( a-e \) distribution of all the photographic orbits with \( 2 < a < 3.5 \) au that have been identified as stream members. Comparing Fig. 6 with Fig. 2 we note two qualitative differences: (i) there is no obvious clustering corresponding to the individual streams, and (ii) there appears to be a lower cut-off in \( e \) which increases with increasing \( a \). The solid curve in Fig. 6 is defined by \( q = a(1 - e) = 1 \) au. Since \( q \) is the perihelion distance, this condition corresponds to a perihelion close to the Earth's orbit. Those orbits with \( a \) and \( e \) values lying close to this curve tend to have inclinations close to 90° with one node close to perihelion.

A plot of the \( a-\iota \) distribution for the same set of stream members as given in Fig. 6 illustrates groupings in inclination that show up as a number of quasi-parallel lines (Fig. 7). The Quadrantid stream (\( \iota \approx 70° \)) and the Perseid stream (\( \iota \approx 110° \)) are clearly discernible.

An inspection of Figs 6 and 7 shows no obvious correlation between the location of jovian resonances and gaps in the distribution of meteors. However, there is some evidence that such gaps exist, particularly in the case of the Quadrantid and Perseid meteor streams. We do not regard the evidence...
from meteoroid streams. The dashed lines denote the locations of the 4:1, 7:2, 3:1, 8:3, 5:2, 7:3, 9:4 and 2:1 jovian resonances.

![Figure 6. The distribution of semimajor axis (a) and eccentricity (e) of all photographic meteors that have been identified as originating from meteoroid streams. The dashed lines denote the locations of the 4:1, 7:2, 3:1, 8:3, 5:2, 7:3, 9:4 and 2:1 jovian resonances.](image)

The Quadrantid meteor shower occurs in early January and is characterized by a short, high-activity period suggesting a young (several $\times 10^2$ yr) stream which has yet to undergo significant dispersion due to planetary and non-gravitational perturbations. For many years there was no plausible parent comet but, with the recent work of Wu (1992), attention has focused on Comet 1491 I and the newly discovered Comet Machholz 1 as possible candidates.

Whereas the IAU photographic meteor data contain 3078 orbits with $e < 1$, only 983 have been assigned to meteor streams. Of these the majority are in either the Geminid or Perseid streams, with only 42 having been identified as Quadrantids. However, Dermott & Murray (1981) employed a running box technique to investigate the resonant structure of the Themis family using a total sample of only 62 asteroids. Their identification of a gap at the 15:7 jovian resonance has now been confirmed using a more complete sample of 307 asteroids (see Fig. 3). Here we adopt the same technique in our analysis of structure within the Quadrantid meteor stream.

There are 32 identified Quadrantid members in the range of interest in the precisely reduced photographic meteor catalogue. In order to increase our sample, even at the expense of reducing its accuracy, we have added the orbits of 38 radio Quadrantids identified by Sekanina in the Harvard data (Sekanina 1970), now part of the IAU data set (Lindblad 1991). Fig. 8 shows the resulting number density histogram produced using a box length of 0.1 au stepped through the selected Quadrantid population at increments of 0.01 au. The stark nature of the plot is a result of the small number of orbits in the sample. On the basis of this plot there is slight evidence for gaps associated with the 3:1, 8:3, 5:2 and 2:1 resonances. These gaps appear in both data sets, although curiously the 3:1 gap is actually more obvious in the radio data. The equivalent distribution of sporadic (i.e. non-stream) meteors is shown in Fig. 9 for comparison. In this case the most prominent gap appears to be associated with the 7:2 jovian resonance. There is other structure in the distribution and even a suggestion of small clusterings of sporadic meteors at other jovian resonances. Although a comparison of Figs 8 and 9 does not provide enough evidence to allow us to distinguish between real and imaginary gaps in the Quadrantid stream, the results are more conclusive than those from a similar comparison using the Themis family of asteroids (see above). As before, a perturbation analysis should help to clarify the situation; such an analysis for the 3:1 resonance is carried out in Section 5 below.

### 4.2 The Perseid stream

In the context of resonant gaps in meteoroid streams, it is appropriate to make some remarks about the recent work by Wu & Williams (1995) on gaps in the semimajor axis distribution of the Perseid meteorotes. Apparently unaware of the work of Lindblad (1973), they produced plots of the distribution of 1/a for the meteors identified as Perseids in the latest IAU photographic catalogue, claiming the existence of gaps where ‘the number of meteoroids is very significantly smaller than in surrounding bins’. Wu & Williams used half the bin size of Lindblad and so, even though the number of known orbits has increased by a factor of $\sim 2$, the number in each bin is comparable. They plotted the positions of the 1:1, 1:2, and 2:1 resonances for the meteors identified as Perseids in the latest IAU photographic catalogue, claiming the existence of gaps where ‘the number of meteoroids is very significantly smaller than in surrounding bins’. Wu & Williams claimed that in their chosen range of 1/a the only locations where no gap was evident were the 2:3 Saturn resonance and the 2:1 Uranus resonance. However, inspection of their fig. 1 shows that the drop in Perseid numbers is at best comparable to $\sqrt{N}$ where $N \leq 3$ for all orbits within 6 au; the mean value of $N$ in their chosen range is $3.6 \pm 3.1$. Therefore the gaps discussed by Wu & Williams cannot be described as ‘very significant’ in the same sense as those for the Kirkwood gaps in the asteroid belt. Indeed the gaps are on a par with those discovered by Lindblad and those discussed above for the Quadrantids – their actual existence has to be regarded as problematic until more data become available.

Wu & Williams (1995) carried out a series of numerical integrations of 11000 test meteoroids all started at perihelion on orbits similar to that of P/Swift–Tuttle (the suspected
Dermott & Murray (1983) showed that the extent of the Kirkwood gaps in the asteroid belt corresponded well with expected maximum libration widths derived from perturbation theory. Here we use a similar theoretical approach and extend their analysis to calculate the changes in $e$ and $I$ associated with the maximum libration. The reasoning behind our method of distinguishing between real and imaginary gaps is as follows.

Using orbital information about the known objects in a family or stream we can calculate the standard deviation, $\sigma$, in the distribution of any of the orbital elements that are used in assigning the object to a grouping in the first place. We then use perturbation theory to estimate the upper limits on the changes to be expected in these elements due to a given resonance and compare these values with the observed $\sigma$. If the predicted changes are significantly less than the $\sigma$ of the group then we consider it likely that the gap is real. On the other hand, if the changes are comparable to or greater than the observed $\sigma$ we consider it likely that the gap is imaginary. The method has to be regarded as approximate but it should at least provide an indication of the nature of a resonant gap.

In order to calculate the amplitude of expected variations in orbital elements due to resonance, we make use of a new algorithm for obtaining high-order expansions of the planetary disturbing function (Ellis & Murray 1996). The disturbing function, $\mathcal{R}$, is the gravitational potential per unit mass experienced by one orbiting body due to another. We consider only the effect of an external perturber and so in a heliocentric frame $\mathcal{R}$ can be written as

$$\mathcal{R} = \frac{\mathcal{F}_m'}{|r - r'|} - \mathcal{F}_m' \frac{r - r'}{r'^3}$$

where $r$ and $r'$ are the position vectors of the perturbed and perturbing bodies respectively, $\mathcal{F}$ is the universal gravitational constant and $m'$ is the mass of the perturber. The two terms on the right hand side of equation (2) are called the direct and indirect parts of the disturbing function respectively.

The function $\mathcal{R}$ can be expanded as a Fourier series of the form

$$\mathcal{R} = \frac{\mathcal{F}_m'}{a} \sum S(a, a', e, e', I, I') \cos \varphi$$

where $a$, $e$ and $I$ denote the semimajor axis, eccentricity and inclination of the perturbed body respectively with similar primed quantities for the perturbing body; the angle $\varphi$ has the general form

$$\varphi = j_1 \lambda' + j_2 \lambda + j_3 \omega + j_4 \Omega + j_5 \Omega$$

where $\lambda$, $\omega$ and $\Omega$ denote the mean longitude, longitude of perihelion and longitude of ascending node respectively with determining stable and unstable configurations with respect to perturbing orbits, further studies should be carried out to discover its role (if any) in determining the extent of a gap.

We note from Fig. 7 that there appears to be a gap in the Perseid distribution associated with the 3:1 resonance; this gap was not mentioned by Wu & Williams since it was outside their region of interest. Although we do not investigate this further in this paper, we have used a running box technique and confirmed that the gap is more pronounced and wider than that detected in the Quadrantid distribution at the same resonance.

5 Resonant Perturbations

Parent comet (parent comet of the Perseids) but with $0 \leq a < 0.22$ au$^{-1}$. These integrations showed that gaps formed at some of the resonant locations in the $1/a$ distribution on time-scales of $< 150$ yr. In this case most of the gaps can be regarded as significant and certainly not imaginary in the sense defined above. However, it is debatable whether or not the removal of meteoroids at these locations can be ascribed to resonant effects. This is particularly true for semimajor axes in the outer part of the Solar system where the orbital period of the test particle is greater than or comparable to that of the resonant perturber. Resonant effects take place over several orbital periods and therefore cannot play a role in removing material on time-scales which are comparable to that of the orbital period of the perturber. A more likely removal mechanism is direct perturbation due to a close approach to a planet, particularly since all the particle orbits had high eccentricities. In the integrations by Wu & Williams the test particles were all started at perihelion since this is the most likely ejection point from a cometary nucleus. However, their choice of orbital elements means that all the test particles started with similar values of the resonant argument (see Section 5 below) for any given resonance. Since the initial argument is crucial in

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similar primed quantities for the perturbing body. There is a
diversity of conditions on the permissible values of the integers,
j, (see Ellis & Murray 1996 for a more complete discussion of
the disturbing function and its properties); one condition, the
d'Alembert relation, requires that 2\cdot or < 1
which can be expressed in terms of Laplace coefficients and
their derivatives (see, for example, Brouwer & Clemence
S including terms up to order 8 in the eccentricities and
inclinations. The new algorithm by Ellis & Murray (1996)
enables expressions for higher order terms to be calculated for
a given value of \varphi.

Note that the d'Alembert relation and the form of the
expansion shown in equation (5) imply a relationship between
the coefficients in \varphi and the lowest powers of the eccentricities and inclinations. If we consider an
argument of the form
\varphi = (p + q)\lambda - p\lambda + j_1\sigma' + j_2\sigma + j_1\Omega + j_2\Omega

then the sum of the lowest powers of e, e', I and I' in S will be q. This type of argument is appropriate for a p + q : p resonance where p + q times the orbital period of the inner
body is approximately equal to p times the orbital period of the outer body. In this case \{j_1 + j_2\} = q is termed the order
of the resonance. This relationship implies that any study of
a resonance of order q requires terms in the expansion of the
disturbing function that are complete to at least order q. Given
the convergence properties of S, high-order expansions may also
be required to deal with low-order resonances in cases where e and I are large.

An object is actually in resonance when \varphi is such that
it changes sign and the resonant angle \varphi librates about some
mean value rather than circulating. Exact resonance occurs when
\varphi = 0. Dermott & Murray (1983) showed that the
maximum change of the mean motion at a resonance is related
to S by the equation
\delta n_{\max} = \pm \left[12(m'/M)|S|\alpha\right]^{1/2} n

where n is the unperturbed mean motion at the resonance and M is the mass of the central body. Equation (7) can be re-written in terms of the maximum change in semimajor axis
using Kepler's third law. This gives
\delta a_{\max} = \pm \left[\left(16/3\right)(m'/M)|S|\alpha\right]^{1/2} a.

A simple application of Lagrange's planetary equations gives the corresponding approximate expressions for the maximum
changes in e and I at the resonance. These are
\delta e_{\max} = -\frac{j_e}{j_e} \delta a_{\max} \frac{j_e}{j_e} 2ae
and
\delta I_{\max} = -\frac{j_i}{j_i} \delta a_{\max} \frac{j_i}{j_i} 4as.

Note that these equations imply that the maximum changes increase with increasing e and I.

We now turn our attention to specific resonances located in
the Themis family of asteroids and the Quadrantid meteor
stream.

5.1 The 15:7 resonance

Fig. 10 shows the distribution of Themis family asteroids in
a-e space with the location of the 15:7 resonance indicated by
the dashed line. There is some evidence of a drop in asteroid
numbers at this location, confirming the result from the running
box analysis shown in Fig. 3. Because of the high order
of the resonance we would not expect a wide gap in a-e space.

The number of possible 15:7 resonant arguments is a function of the maximum order of the expansion. For example, if
order 8, 10, 12 and 14 expansions are considered then
the number of resonant arguments is 85, 185, 371 and 567
respectively. Since we are considering the Themis family of
asteroids we have e ≈ 0.15 ± 0.048 and I ≈ 14° ± 1°. Therefore,
from equation (5) we expect the argument associated with the
strongest (i.e. widest) resonance to be
\varphi = 15\lambda' - 7\lambda - 8\pi.

In the classification of Murray & Harper (1993) this is argument BD.1 with j = 15; there are no contributions from the
indirect terms in S in expansions up to and including order 14.
 Hence j_1 = 15, j_2 = -7 and j_3 = -8 with j_1 = j_2 = j_3 = 0. Taking e' = 0.048, I' = 2.48 and I = 14° we can calculate an
expression for S using the formula given by Ellis & Murray
(1996). Using an expansion to order 14 we carried out a num-
ber of tests on the convergence properties of the expression
for S. These indicated that the approximate error for e = 0.15
was ≈ 10 per cent. We obtained \delta a_{\max} = 5 \times 10^{-4} au, con-
firming the prediction of a small width, and \delta \delta a_{\max} = 6 \times 10^{-4}
for e = 0.15, the mean value for the Themis family. Since the
standard deviation in e of the current distribution of Themis
members is 0.012 it seems unlikely that resonant perturbations
at the 15:7 resonance could be strong enough to prevent a
Themis family member from being recognized in a statistical
analysis. Therefore our results suggest that the 15:7 resonance
gap in the Themis family is real and is caused by the actual
removal of objects.

5.2 The 3:1 resonance

Fig. 11 shows the distribution of Quadrantid meteors in a-I
space with the location of the 3:1 resonance indicated by
the dashed line. Note that the scatter in the distribution of the
radio meteor orbits (I = 71.4° ± 3.8°) is larger than in the case of
the photographic meteors (I = 71.4° ± 1°); this may be due to
the poorer accuracy of the radio orbits. Furthermore there is a
clear gap in the radio meteor distribution at the 3:1 resonance
but there are several photographic meteors in the vicinity of
the resonance. Because of the different mass distributions of
each population it may be that this difference is real and may
indicate dynamical differences between the two populations.
Note that we would expect a gap (real or imaginary) to be
large because of the low order of the resonance and the large
values of e and I.

If expansions of order 8, 10, 12 and 14 are considered for
the 3:1 resonance the number of possible arguments is 140,
290, 434 and 728 respectively. Since we are considering the
Quadrantid meteor stream at the 3:1 resonance we have

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In the classification of Murray, strong resonance with argument similar analysis for inclined orbits and estimate that expansion to order 12 with \( \epsilon' = 0.048, \) \( \lambda = \) at least comparable to the width of the gap in the radio distance. Inspection of Fig. 11 shows that given the errors this is apparent in Fig. 6 we choose to examine the strong resonance with argument
\[
\rho = 3\lambda - 2\Omega. \tag{12}
\]
In the classification of Murray & Harper (1993) this is argument 8D2.4 with \( j = 3; \) there are no indirect terms associated with this argument in expansions up to and including order 14. We have \( j_1 = 3, j_2 = -1 \) and \( j_6 = -2 \) with \( j_4 = j_5 = j_6 = 0. \) Ellis & Murray (1996) showed that because of the low value of \( a \) the convergence properties of the expansion at the 3:1 resonance are good in the planar case. We have carried out a similar analysis for inclined orbits and estimate that an expansion to order 12 with \( \epsilon' = 0.048, \) \( \lambda' = 2.48 \) and \( e = 0.6 \) gives an error of \( \approx 50 \) per cent for \( I = 71^\circ. \) Using this expansion with \( s = 0.574 \) we find \( \delta a_{\text{max}} = 0.2 \) au and \( \delta I_{\text{max}} = 9^\circ. \) Therefore the predicted width of the resonance in \( a-I \) space is \( 2\delta a_{\text{max}} = 0.4 \) au. Inspection of Fig. 11 shows that given the errors this is at least comparable to the width of the gap in the radio distribution. Furthermore, \( \delta I_{\text{max}} \) exceeds that of those observed meteors that have already been identified as Quadrantid members. Therefore we reach the tentative conclusion that the 3:1 resonance gap in the Quadrantid stream is imaginary and is caused by the failure to identify other Quadrantid members because of the large perturbations produced by the resonant effects. We have to stress that this result is much less certain than that for the 15:7 resonance in the Themis family which was discussed above.

6 CONCLUSIONS AND DISCUSSION

Having introduced the concepts of real and imaginary Kirkwood gaps, we have developed a new method for estimating the expected maximum changes in \( e \) and \( I \) at a given resonance. We have presented evidence which suggests that the 15:7 gap in the \( a-e \) distribution of members of the Themis family of asteroids is real and represents a genuine removal of material. In the case of the 3:1 gap in the \( a-I \) distribution of the Quadrantid meteoroid stream the evidence is weaker but it suggests that the gap is imaginary and an artefact of the way in which stream members are identified.

A number of issues still need to be resolved before more definitive statements can be made about the real or imaginary nature of these and other gaps. For example, we have taken no account of the effect of nearby, strong resonances on changes in \( e \) and \( I, \) yet it is known that overlap of adjacent resonances can lead to chaotic motion and possibly even larger changes in orbital elements. In the cases of both asteroid families and meteoroid streams, full numerical integrations of the equations of motion need to be carried out to look for genuine removal of objects by resonant perturbations; such calculations, by their very nature, would include all the relevant perturbations. However, great care needs to be exercised in the choice of starting conditions for such integrations. This is because the existence of resonant phenomena implies that, even for given \( a, e, \) and \( \Omega, \) the initial choice of \( \lambda \) and \( \lambda' \) can determine whether the resonant argument librates or circulates, and also whether or not close approaches to the perturber can occur.

In order to be realistic, such integrations also need to model accurately the original distribution of elements, and not just assume the current values. For example, the coherent nature of an initial family grouping may help explain why the 15:7 gap is detectable in the Themis family but not in the background non-family distribution. The whole subject becomes much more problematic when the modelling of initial distributions of meteoroids is undertaken.

It is also possible that a gap may have real and imaginary characteristics. Such a ‘complex’ gap could be produced in an asteroid family or meteoroid stream by a combination of genuine removal of material and a failure to identify perturbed material as belonging to the original grouping. In this case a numerical simulation would have to include an on-going analysis of the group identifications.

Particular difficulties arise in the case of gaps in meteoroid streams. First, meteoroid orbits tend to have large values of \( e \) and \( I \) which preclude a perturbative analysis for some resonances. For example, there does seem to be evidence for a gap at the 2:1 resonant location in the Quadrantid stream but we have been unable to verify its true nature using the techniques developed in this paper. Secondly, by their very nature the orbital data on meteoroids suffer from extreme observational selection effects. One obvious way of overcoming this problem is to find new, non-terrestrial points of observation, and a meteor detector on Mars (Adolfsson, Gustafson & Murray 1995) would be an ideal solution.
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