Coulomb Interaction in a Plasma with Anisotropic Temperature

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The Coulomb interaction of a charged test particle with an electron plasma whose unperturbed velocity distribution has different temperatures along and across a certain axis is kinetically-theoretically investigated. In the linear approximation, the binary correlation being neglected, the electric potential and energy loss of a charged test particle and the dielectric constant are derived. By introducing an angular-dependent temperature, the dispersion relation and damping constant of plasma oscillations are written in the same form as in the case of an isotropic temperature plasma.

§ 1. Introduction

Since the appearance of Landau's paper concerning plasma oscillations, many authors treated the problem of the Coulomb interaction in a plasma with an isotropic velocity distribution. However, the velocity distribution of plasma particles are actually not always isotropic, especially when an external magnetic field is applied. For example, the distribution of plasma particles is not the Maxwellian during the non-collisional heating of plasma.

The aim of the present paper is to study the influences of anisotropy in the unperturbed velocity distribution on the Coulomb interaction in a plasma. For simplicity, we shall consider the case of an electron plasma without an external magnetic field, and such an anisotropic state may be obtained just after the non-collisional heating. The results obtained in this paper may also be helpful for comparison with the case of a plasma in a magnetic field, and the latter case will be treated in a forthcoming paper.

In the next section, following the approach as that used by Gasiorowicz et al., we solve the linearized equation for the distribution function of plasma electrons where the test particle will be regarded as an external source of force, and derive the expression of the dynamical potential. As a special case of the dynamical potential, the electrostatic one is discussed in § 3. In § 4, the generalized dielectric constant of the plasma is defined, and the dispersion relation and damping constant of plasma oscillations are derived. The formula of the energy loss of the test particle is obtained in the last section.

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§ 2. Fundamental equations and dynamical potential

It is assumed that the unperturbed distribution function of plasma electrons is given by

\[ f_0(v) = N \pi^{-3/2} \beta_{\perp}^2 \beta_{\parallel} \exp \left\{ -\beta_{\perp}^2 (v_x^2 + v_y^2) - \beta_{\parallel}^2 v_z^2 \right\}, \]

(2.1)

where \( N \) is the electron density; \( v_x, v_y \) and \( v_z \) are velocity components in the \( x, y \) and \( z \) directions respectively; \( \beta_{\perp} \) and \( \beta_{\parallel} \) are defined by

\[ \beta_{\perp}^2 = \frac{m}{2T_{\perp}}, \quad \beta_{\parallel}^2 = \frac{m}{2T_{\parallel}}, \]

(2.2)

where \( m \) is the electron mass, \( T_{\perp} \) and \( T_{\parallel} \) are temperatures in the \( xy \)-plane and along the \( z \)-axis respectively.

At time \( t = 0 \), it is assumed that the distribution function is disturbed slightly from Eq. (2.1) by the test particle. For the distribution function at time \( t > 0 \), let us put

\[ f(r, v, t) = f_0(v) + f_1(r, v, t), \]

(2.3)

where \( f_1(r, v, t) \) is a small perturbation. In the integrated Liouville equation, we neglect the two-body correlation and assume that \( f_0 \gg |f_1| \) and the relaxation time between two temperatures is large enough to be \( |\partial f_0 / \partial t| \ll |\partial f_1 / \partial t| \). Then we have the linearized equation, viewed from the system moving with the test particle, as follows:

\[ \frac{\partial f_1}{\partial t} + (v - v_0) \cdot \nabla f_1 - \frac{e}{m} E_0 \cdot \nabla f_0 = -\frac{e}{m} \nabla \phi \cdot \nabla \varphi, \]

(2.4)

where \( E_0 \) is the electric field due to the test particle (charge \( q_0 \), velocity \( v_0 \) ); and \(-e, m\) are the charge and mass of electrons. Here \( \varphi(r, t) \) is the potential due to the polarization:

\[ \varphi(r, t) = \int U(r, r') f_1(r', v', t) \, dr' \, dv', \]

(2.5)

where

\[ U(r, r') = -\frac{e}{|r - r'|}, \]

and it satisfies Poisson's equation:

\[ \nabla^2 \varphi(r, t) = 4\pi e \int f_1(r, v, t) \, dv. \]

(2.6)

After the Fourier and Laplace transformation, we get from Eq. (2.4)

\[ (s - ik \cdot v_0 + i k \cdot v) g(k, v, s) = -i \frac{e}{m} \left\{ \frac{4\pi q_0}{sk^2} + \Phi(k, s) \right\} k \cdot v \, f_0, \quad \text{Re}(s) > 0, \]

(2.7)

the initial value of the Fourier transform of \( f_1(r, v, t) \) being taken to be zero.
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according to our assumption, and from Eq. (2.6) we have

\[ k^2 \Phi(k, s) = -4\pi e \int g(k, v, s) \, dv, \quad (2.8) \]

where \( g(k, v, s) \) and \( \Phi(k, s) \) are the Fourier and Laplace transform of \( f_1(r, v, t) \) and \( \varphi(r, t) \) respectively:

\[
\begin{align*}
\left\{ g(k, v, s) \right\} &= \int e^{-ikr} \, dr \int e^{-st} \left\{ f_1(r, v, t) \right\} \, dt, \\
\left\{ \Phi(k, s) \right\} &= \int e^{-ikr} \, dr \int e^{-st} \left\{ \varphi(r, t) \right\} \, dt.
\end{align*}
\]

(2.9)

The solution of Eqs. (2.7) and (2.8) is written as

\[ \Phi(k, s) = -\frac{4\pi q_0}{sk^2} \frac{\chi(k, s)}{k^2 + \chi(k, s)}, \]

(2.10)

where

\[ \chi(k, s) = -i \frac{4\pi e^2}{m} \int \frac{k \cdot v_0 \, f_0}{s - ik \cdot v + ik \cdot v} \, dv. \]

(2.11)

If we add the Fourier and Laplace transform of the self-potential, \( \phi_0(k, s) = \frac{4\pi q_0}{sk^2} \), to \( \Phi(k, s) \), we get the Fourier and Laplace transform of the total potential due to the test particle in the plasma

\[ \phi_t(k, s) = \frac{4\pi q_0}{s} \frac{1}{k^2 + \chi(k, s)}. \]

(2.12)

Taking \( k \) in the \( xz \)-plane as shown in Fig. 1, and integrating Eq. (2.11) with \( v_y \), we have

\[ \chi(k, s) = 2\omega^2 \pi \beta_1 \beta_2 \int_{-\infty}^{\infty} \exp \left(-\beta_1^2 v_x^2 - \beta_2^2 v_z^2\right) \frac{v_x^2 k_1 v_x + \beta_0^2 k_z v_z}{k_1 v_x + k_z v_z - k \cdot v_0 - is} \, dv_x \, dv_z, \]

(2.13)

Fig. 1.

where \( k_1 \) and \( k_z \) are components of the wave vector \( k \) along the \( x \)- and \( z \)-axis, and \( \omega^2 = 4\pi Ne^2/m \). Transforming the coordinate from the system \((x, y, z)\) to \((x', y, z')\) (see Fig. 2), and integrating with \( v_x \), we have

\[ \chi(k, s) = 2\omega^2 \beta^2 \pi^{1/2} \int_{-\infty}^{\infty} \exp \left(-\beta^2 v_x'^2\right) \frac{k v_x'}{kv_x' - k \cdot v_0 - is} \, dv_x', \]

(2.14)

Fig. 2.
where $\beta$ is the inverse of the root mean square velocity in the $k$-direction, defined as

$$\beta^2(\theta) = \frac{\beta_1^2 \beta_2^2}{\beta_1^2 \cos^2 \theta + \beta_2^2 \sin^2 \theta} = \frac{m}{2T(\theta)}. \quad (2.15)$$

Here, the temperature in the direction of $k$, $T(\theta)$, is given by

$$T(\theta) = T_1 \sin^2 \theta + T_2 \cos^2 \theta. \quad (2.16)$$

In Eq. (2.14) we consider a limiting case, $\text{Re}(s) = s_1 \to 0$. As is well known, if necessary, it is possible to consider the case of finite $s_1$, and also to consider the case of $s_1 < 0$ by the analytic continuation.

By using a relation

$$\lim_{s_1 \to +0} \frac{1}{\nu_s' - \frac{k \cdot v_0 + is}{k}} = P \frac{1}{\nu_s' - \frac{k \cdot v_0 + \omega'}{k}} + i\pi \delta\left(v_s' - \frac{k \cdot v_0 + \omega'}{k}\right),$$

where $\text{Im}(s) = -\omega'$, Eq. (2.14) becomes

$$\chi(k, s) = k_d^2(\theta) \left[1 - 2\beta\left(v_0\mu + \frac{\omega'}{k}\right) \exp\{-\beta^2\left(v_0\mu + \frac{\omega'}{k}\right)^2\} K\left(\beta\left(v_0\mu + \frac{\omega'}{k}\right)\right)\right. \right. \right.$$

$$\left. + i\sqrt{\pi} \beta\left(v_0\mu + \frac{\omega'}{k}\right) \exp\{-\beta^2\left(v_0\mu + \frac{\omega'}{k}\right)^2\}\right], \quad (2.17)$$

where

$$k_d^2(\theta) = 4\pi N e^2 / T(\theta), \quad (2.18)$$

$$\mu = \cos(k v_0), \quad (2.19)$$

and

$$K(x) = \int_0^\infty \exp(-tx) \, dt. \quad (2.20)$$

Here, $k_d(\theta)$ is a generalization of the Debye wave number to the anisotropic temperature electron plasma. $\omega'$ is the frequency in the system moving with the test particle. Thus, due to the Doppler effect, there exists a relation

$$\omega' = \omega - k \cdot v_0$$

where $\omega$ and $k$ are the frequency and wave vector in the rest system of the plasma and $k$ is approximately equal to the wave vector, used before, viewed from the moving system.

### § 3. Electrostatic potential

The Fourier and Laplace transform of the electrostatic potential due to the test particle is obtained from Eq. (2.12), by putting $v_0 = 0$ and $s \to 0$, as follows:
Therefore, the electrostatic potential is symmetric about the $z$-axis. Introducing the cylindrical coordinate $r = (\rho, 0, z)$ and $k = (k_\perp, \phi, k_z)$, we obtain, for the inverse transform,

$$
\varphi_i(r) = q_0 \int_0^\infty \frac{J_0(k_\perp \rho) k_\perp dk_\perp}{(k_\parallel^2 - k_\perp^2) k_\perp^2 + k_\perp^2} \left[ k_\perp (k_\parallel^2 - k_\perp^2) \exp(-|z|k_\perp) \right. \\
\left. + \frac{k_\parallel^2 k_\perp^2 \exp(-|z|\sqrt{(k_\parallel/ k_\perp)^2 k_\perp^2 + k_\parallel^2})}{\sqrt{(k_\parallel/ k_\perp)^2 k_\perp^2 + k_\parallel^2}} \right],
$$

(3.2)

where

$$
k_\parallel^2 = 4\pi Ne^2/T_\perp,
$$

(3.3)

and $J_0(k_\perp \rho)$ is the Bessel function of order zero.

**§ 4. Dielectric constant and dispersion equation**

According to their definitions,\textsuperscript{9)\textsuperscript{10)} Fourier and Laplace transforms of the electric field intensity and the electric displacement are given, respectively, as follows:

$$
E(k, s) = -ik\varphi_i(k, s),
$$

(4.1)

$$
D(k, s) = -ik\varphi_o(k, s).
$$

(4.2)

Thus we get the generalized dielectric constant

$$
\varepsilon(k, s) = \phi_o(k, s)/\phi_i(k, s) = 1 + k^{-2} \chi(k, s).
$$

(4.3)

It must be noted that Eq. (4.3) can also be defined in the left half-plane of the $s$-plane by the analytic continuation.

In a limiting case $s \to +0$, considering the Doppler effect as stated in § 2, we obtain from Eqs. (2.17) and (4.3)

$$
\varepsilon(k, \omega) = \varepsilon_i(k, \omega) + i\varepsilon_s(k, \omega),
$$

$$
= 1 + \frac{k_\parallel^2(\theta)}{k^2} \left[ 1 - \frac{2\beta\omega}{k} \exp\left(-\frac{\beta^2\omega^2}{k^2}\right) K\left(\frac{\beta\omega}{k}\right) + i\sqrt{\pi} \frac{\beta\omega}{k} \exp\left(-\frac{\beta^2\omega^2}{k^2}\right) \right],
$$

(4.4)

\textsuperscript{9) For isotropic case, the connection between the generalized dielectric constant and the dispersion equation was clearly explained by S. Nakajima.\textsuperscript{9)}
where \( \varepsilon_1 \) and \( \varepsilon_2 \) are the real and imaginary parts respectively. By taking \( s_1 \) as a finite quantity, namely, considering complex \( \omega \), we have the dispersion equation

\[
\varepsilon(k, \omega) = \varepsilon_1(k, \omega) + i\varepsilon_2(k, \omega) = 0, \quad (4.5)
\]

where \( \varepsilon(k, \omega) \) is also defined in the lower half-plane of the complex \( \omega \) plane by the analytic continuation.

Let us write the solution of (4.5) as \( \omega - i\gamma \), where \( \omega \) and \( \gamma \geq 0 \) are real and \( \gamma/\omega \) is assumed to be much less than unity. Then, provided that \( \beta \omega > k \), we find the dispersion equation of plasma waves, to the first order of \( \gamma/\omega \), as follows:

\[
1 - \frac{\omega_p^2}{\omega^2} \left( 1 + \frac{3}{2} \frac{k^2}{\omega^2} + \cdots \right) = 0,
\]

in which the following asymptotic expansion has been used

\[
2xe^{-x^2}K(x) = 1 + \frac{1}{2x^2} + \frac{3}{4x^4} + \cdots, \quad (x > 1).
\]

Thus we obtain from Eq. (4.6)

\[
\omega^2 = \omega_p^2 + \frac{3T(\theta)}{m} k^2,
\]

which is the generalization of the well-known dispersion relation. It is found from Eq. (4.7) that the frequency and phase velocity of plasma waves are larger in the direction of higher temperature than those in the direction of lower temperature.

Also, the generalization of the Landau damping is given by

\[
\gamma = \omega_p \sqrt{\frac{\pi}{8}} \left( \frac{k_d(\theta)}{k} \right)^3 \exp \left( -\frac{k_d^2(\theta)}{2k^2} \right) \exp \left( -\frac{3}{2} \right).
\]

The damping constant of plasma waves depends upon the direction of propagations and is larger along the direction of higher temperature, because the damping arises from the interaction between thermal motions of electrons and plasma waves.

§ 5. Energy loss

The energy loss of the test particle moving through the plasma is derived by making use of Eqs. (4.1) and (2.17), where we should take a limit \( s \to +0 \). The energy loss per unit path length is

\[
-\frac{dW}{dl} = -q_0 E(0) \cdot \mathbf{v}_0 / v_0
\]

\[
= \frac{4\pi q_0^2}{v_0} \frac{1}{(2\pi)^3} \int \frac{(k \cdot \mathbf{v}_0) \chi_1(k)}{\left( k^2 + \chi_1(k) \right)^2 + \chi_2^2(k)} dk,
\]

(5.1)
where $dl$ is a line element along the path of the test particle and

$$\zeta(k) = \lim_{s \to k^+} \zeta(k, s) = \chi_1(k) + i\chi_2(k), \quad (\chi_1, \chi_2: \text{real}).$$

Of course, Eq. (5·1) may be rewritten by making use of the dielectric constant.

The integral of Eq. (5·1) is approximately splitted into two parts, namely, the contributions from the region $k > k_d$ (short-range interaction) and from the region $k < k_d$ (long-range interaction),* where $k_d$ is the smaller one of $k_{d\perp}$ and $k_{dz}$.

(1) $k > k_d$

When $|\beta v_0 \mu| \leq 1$, the contribution from this region is negligibly small compared with that from the short-range interaction. While, if $|\beta v_0 \mu| \gg 1$, the test particle can emit the plasma waves by the Cerenkov-like effect, whose contribution for the energy loss is easily found as

$$- \frac{dW}{dl} \bigg|_{k < k_d} = \frac{q_0^2 \omega_p^2}{v_0^2} \log \left( \frac{k_1 v_0}{\omega_p} \right), \quad (5·2)$$

in which $k_1$ may be nearly equal to $k_d$ approximately. Here, the next asymptotic formula has been used:

$$k^3 + \chi_1(k) = k^3 \left( 1 - \frac{\omega_p^2}{k^2 v_0^2 \mu^2} \right),$$

from which the Cerenkov relation for the plasmon emission is obtained:

$$\mu = \pm \frac{\omega_p}{k v_0}. \quad (5·3)$$

The anisotropy of electron temperature hardly has an effect upon the energy loss due to plasmon emissions, because the plasmon emission is possible only for the fast test particle and not affected actually by the thermal motion of plasma electrons.

(2) $k < k_d$

As $|\chi_1|$ and $|\chi_2|$ in this region are usually smaller than $k^3$, the interaction is considered to be non-resonant and we have

$$- \frac{dW}{dl} \bigg|_{k > k_d} = \frac{4\pi q_0^2}{v_0} \left( \frac{1}{2\pi} \right)^3 \int \frac{(k \cdot v_0) \chi_2(k)}{k^4} dk
\frac{-\omega_p^2}{v_0^2} \log \left( \frac{k_2}{k_1} \right) \left( \frac{2\pi}{k_1} \right)^{3/2} \int_0^k (\beta v_0)^3 \mu^2 \exp \left\{ - (\beta v_0 \mu)^2 \right\} \sin \theta d\theta d\phi, \quad (5·4)$$

where $k_2$ is a maximum wave number allowed in our treatment and $\mu$ is given by

$$\mu = \sin \theta \sin \theta \cos \phi + \cos \theta \cos \phi,$$

where $\theta = \tilde{v}_0 e_z$, $\theta = \tilde{k} e_z$ and $\phi$ is the angle between the planes $(v_0 e_z)$ and $(k e_z)$ as shown in Fig. 3, $e_z$ being the unit vector along the $z$-axis. The angular dependence of $k_1$ has been neglected in Eq. (5·4).

* In reference 3), Gasiorowicz et al. missed the contributions from the long-range interaction.
In general, it is impossible to perform the integration in Eq. (5·4) analytically. Let us consider some special cases.

When \( T_\perp=T_z=T \), Eq. (5·4) naturally becomes the familiar expression:

\[
-\frac{dW}{dl} \bigg|_{\kappa>\kappa_d} = \left\{ \begin{array}{cl}
\frac{q_0^2 \omega_p^2}{v_0^2} \log \left( \frac{k_2}{k_1} \right), & \beta v_0 > 1, \\
\frac{4q_0^2 \omega_p^2}{3v_0} \beta^2 v_0 \log \left( \frac{k_2}{k_1} \right), & \beta v_0 < 1,
\end{array} \right.
\]

where \( \beta = m/2T \).

When \( \mathbf{v}_0 \) is parallel to the \( z \)-axis, \((\mu=\cos \theta)\), Eq. (5·4) becomes

\[
-\frac{dW}{dz} \bigg|_{\kappa>\kappa_d} = \frac{q_0^2 \omega_p^2}{v_0^2} \log \left( \frac{k_2}{k_1} \right) 2\pi^{-1/2} \int_1^\infty (\beta v_0)^3 \mu^2 \exp \left[ - (\beta v_0 \mu)^2 \right] d\mu. \tag{5·7}
\]

If \( \beta v_0 \gg 1 \), Eq. (5·7) roughly equals Eq. (5·6A).

When \( \mathbf{v}_0 \) is perpendicular to the \( z \)-axis, \((\mu=\sin \theta \cos \phi)\), we have

\[
-\frac{dW}{dx} \bigg|_{\kappa>\kappa_d} = \frac{q_0^2 \omega_p^2}{v_0^2} \log \left( \frac{k_2}{k_1} \right) \pi^{-1/3} \times \int_0^\pi (\beta v_0 \sin \theta)^2 \exp \left( - \frac{\beta^2 v_0^2 \sin^2 \theta}{2} \right) \left\{ I_0 \left( \frac{3\beta v_0^2 \sin^2 \theta}{2} \right) - I_1 \left( \frac{3\beta v_0^2 \sin^2 \theta}{2} \right) \right\} d\theta,
\]

where \( I_0 \) and \( I_1 \) are the modified Bessel functions. If \( \beta v_0 \gg 1 \) it is easy to show that Eq. (5·8) nearly equals Eq. (5·6A).

From the above results, we may conclude that if \( \beta v_0 \gg 1 \) the anisotropy of electron temperature has not large effect on the energy loss of the test particle.

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References