173

Electromagnetic Structure of the Nucleon

—Effects of pion correlations on the iso-scalar form factors—

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The three-pion state contribution to the iso-scalar electromagnetic structure of the nucleon is investigated in detail with the inclusion of the effects of the two-pion and three-pion resonances which have recently been discovered in experiments. The effects of these resonances are formulated so as to be represented by respective enhancement factors to the lowest order perturbation calculations on the basis of the unitarity and analyticity requirements.

Numerical calculations have shown that the results of the perturbation calculations with an extended effective $\gamma$-3$\pi$ interaction are greatly modified by including the pion correlations. In particular, the calculated mean-square radius of the iso-scalar charge distribution turns out to be in good agreement with experiment when the effects of both the two-pion and three-pion resonances are taken into account. The charge form factor thus calculated reproduces the experimental curve in the low momentum transfer region $q^2<15\mu^2$.

The anomalous magnetic moment, on the other hand, is positive and large in magnitude even after the inclusion of these resonances.

§ 1. Introduction

Recently much effort has been devoted to the investigation of the electromagnetic structure of the nucleon. In particular, the high energy electron-proton and electron-deuteron scattering experiments carried out by Hofstadter et al.\textsuperscript{1} and by Wilson et al.\textsuperscript{2} have provided a considerable amount of information on the charge and magnetic moment form factors of the nucleon.

The first dispersion-theoretical formulation of this problem was attempted by Chew et al.\textsuperscript{3} and by Federbush et al.\textsuperscript{4} Their formulation has been modified by Frazer and Fulco\textsuperscript{5} to include the effect of pion-pion correlation, and it has been shown that a possible pion-pion resonance of suitable position and width in the $T=1, J=1$ state can bring the calculation of the iso-vector form factors into agreement with experiment. Bowcock et al.\textsuperscript{6} have shown that this resonance is also significant for understanding the small phase shifts in low energy elastic pion-nucleon scattering.

The existence of this $T=1$ $P$-wave pion-pion resonance has been established in recent experiments of inelastic pion-nucleon scattering,\textsuperscript{7} though the resonance energy has turned out to be rather high compared with the prediction of Frazer and Fulco.

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As to the theoretical understanding of the iso-scalar structure of the nucleon we are in a very perplexing situation. Dispersion-theoretically the three-pion intermediate state is expected to be the most important contributor to the iso-scalar charge and moment distributions.

Extensive investigations of this three-pion contribution within the lowest order perturbation theory have been carried out by Hiida, Nakanishi, Nogami and Uehara and by Tamm. Their results suggest that the large mean-square radius of the iso-scalar charge distribution might be explained by the three-pion contribution.

Modified perturbation calculations with an effective $r^{-3}\pi$ interaction were also performed by Hiida and Nakanishi and by Kato. Contrary to the above-mentioned results, their calculations have shown that the mean-square radius of the charge distribution is almost by an order smaller than experiment. But mainly because of technical difficulties of calculation these perturbation results are quantitatively unreliable and, moreover, will be much altered by taking into account the effects of strong pion-pion correlation which is so effective in the iso-vector properties of the nucleon.

As a possibility to explain the mysterious iso-scalar properties a vector meson with $T=0$, $J=1$ and odd parity was once proposed by Nambu. Chew has pointed out that in the particular three-pion state involved in the electromagnetic structure of the nucleon the strong attractive force between $P$-wave pions is likely expected to produce a resonance at about the same total energy as that of the two-pion system.

On the other hand, recent Berkeley analyses of correlations among pions produced in antiproton annihilations have revealed the existence of a $T=0$ three-pion resonance with the total energy about 40 Mev higher than that of the two-pion resonance. The nature of the resonance is not yet fully established but the data suggest that it is of vector type.

The purpose of this paper is to apply dispersion techniques to the estimation of the effects of the newly discovered two-pion and three-pion resonances on the three-pion state contribution to the iso-scalar structure of the nucleon. As these resonance energies are considerably higher than those required from the phenomenological analyses of experiments, it cannot be expected for these resonances alone to explain the whole behaviour of the iso-scalar form factors up to the high momentum transfer region. The three-pion state contribution is more reasonably expected to dominate only the mean-square radii of the charge and magnetic moment distributions, and more massive intermediate states will become important for understanding the behaviours of the form factors in the high momentum transfer region ($q^2 > 15\pi^2$). Then, what should be done to get further insight into the problem is to clarify quantitatively how the results of perturbation calculations of the iso-scalar properties will be modified by including the effects of the strong pion correlations.
In § 2 the formulation of our problem will be discussed on the basis of the unitarity and analyticity. The effects of the two-pion and three-pion resonances will be shown to be represented by respective enhancement factors to the lowest order perturbation calculations. In § 3 numerical results will be presented and discussed in detail. § 4 will summarize our findings and remarks. Appendices consist of discussions of simple models of the three-pion resonance process and final state interaction and details of the lowest order perturbation calculations.

§ 2. Formulation of the three-pion state contribution

2.A. Kinematical Considerations

We begin our detailed discussion by considering kinematical features of the problem. The iso-scalar form factors $F_1^s$ and $F_2^s$ of our interest are defined by the iso-scalar electromagnetic current of the nucleon as follows:

$$J^s_\mu = \left( \frac{p_0 \bar{p}_0}{m^5} \right)^{1/2} \langle 0 | j^s_\mu | \bar{p}, p \text{ in} \rangle,$$

$$= -\bar{v}(\bar{p}) \{ F_1^s[(p+\bar{p})^2]i\gamma_\mu + F_2^s[(p+\bar{p})^2]i\sigma_{\mu\nu}(p+\bar{p})\}u(p).$$

These form factors are assumed to satisfy the following dispersion relations:

$$F_1^s(q^2) = \frac{e}{2} - \frac{q^2}{\pi} \int_{(3\rho)^2} \frac{d\sigma^2}{\sigma^2 + q^2 - i\varepsilon},$$

$$F_2^s(q^2) = \frac{1}{\pi} \int_{(3\rho)^2} \frac{d\sigma^2}{\sigma^2 + q^2 - i\varepsilon}.$$

Their absorptive parts are to be determined from

$$A^s_\mu = -\pi \left( \frac{p_0}{m} \right)^{1/2} \sum_{(3\rho)^2} \bar{v}(\bar{p}) \langle 0 | j_\mu^s | s \rangle \langle s | f | \bar{p} \rangle \delta(p_0 - \bar{p} - p),$$

$$= -\bar{v}(\bar{p}) \{ \text{Im}F_1^s[(p+\bar{p})^2]i\gamma_\mu + \text{Im}F_2^s[(p+\bar{p})^2]i\sigma_{\mu\nu}(p+\bar{p})\}u(p).$$

We retain only the three-pion intermediate state in the sum over a complete set of states $|s\rangle$. Thus the quantity to be calculated is the product $\langle 0 | j_\mu^s | 3\pi \rangle \cdot \langle 3\pi | f | \bar{p} \rangle$.

From standard invariance arguments the first matrix element describing electro-production of three pions has the structure given by

$$\langle 0 | j_\mu^s | k^+, k^-, k^0 \text{ out} \rangle = -i(8\omega_+ \omega_- \omega_0)^{-1/2} \varepsilon_{\mu\lambda\sigma} k^\lambda k^\sigma H^*,$$

where $k^+$, $k^-$ and $k^0$ are, respectively, four momenta of positive, negative and neutral pions. From the boson character of pions $H = H(q^2; s_1, s_2, s_3)$ is a completely symmetric function of the three invariant variables.
\[ s_1 = -(k^+ + k^0)^2, \]
\[ s_2 = -(k^0 + k^+)^2, \]
\[ s_3 = -(k^+ + k^-)^2, \]

and is also dependent on the squared virtual photon mass \( -q^2 \). These four variables are connected by the relation

\[ s_1 + s_2 + s_3 = 3\mu^2 - q^2. \]  

It is to be noted that any pair of pions in (2.4) has isotopic spin 1 and odd relative angular momentum.

The second matrix element, which describes nucleon pair annihilation into three pions, involves only the \( ^4S_1 \) and \( ^3D_1 \) pair states. In the rest frame of the nucleon pair it can be written as

\[ \bar{\psi}(\hat{p}) \langle k^+, k^-, k^0 \rangle \text{ out} |f| \hat{p} \rangle = -\left( \frac{m}{E} \right)^{1/2} (8 \omega_+ \omega_- \omega_0)^{-1/2} \chi_{p^*} \{ \alpha \sigma \cdot k^+ \times k^- \}
\]
\[-\frac{\beta}{\sqrt{2}} [3\sigma \cdot \hat{p} \hat{p} \cdot k^+ \times k^- - \sigma \cdot k^+ \times k^-] \chi_p \]  

where \( \hat{p} = (p, E) \) and \( \hat{p} = p/|p| \).

\( \alpha = \alpha(q^2; s_1, s_2, s_3) \) and \( \beta = \beta(q^2; s_1, s_2, s_3) \) are, respectively, amplitudes for pair annihilation in the \( ^4S_1 \) and \( ^3D_1 \) states and to be regarded as symmetric functions of the variables \( s_i \).

Now inserting the expressions (2.4) and (2.7) into (2.3), we find for the three-pion state contribution, in the center-of-mass system,

\[ A' = \pi (2E) \left( \frac{1}{2\pi} \right)^6 \int \frac{d^3 k^+ d^3 k^- d^3 k^0}{8 \omega_+ \omega_- \omega_0} \delta(k^+ + k^- + k^0) \delta(\omega_+ + \omega_- + \omega_0 - 2E) \]
\[ \times \{ \chi_{p^*} \text{Re} (H^* \alpha) \sigma \cdot k^+ \times k^- \}
\[-\frac{1}{\sqrt{2}} \text{Re} (H^* \beta) [3\sigma \cdot \hat{p} \hat{p} \cdot k^+ \times k^- - \sigma \cdot k^+ \times k^-] \chi_p \} \]  

On the other hand (2.4) is reexpressed, in terms of 2-component Pauli spinors, as follows:

\[ A' = -\frac{E}{m} \chi_{p^*} \left[ \left\{ \frac{2E + m}{3E} \text{Im} F_1^i + \frac{2m + E}{3m} \text{Im} F_2^i \right\} \sigma \right.
\]-\left[ \frac{E - m}{3E} \text{Im} F_1^i - \frac{E - m}{3m} \text{Im} F_2^i \right] (3\sigma \cdot \hat{p} \hat{p} - \sigma) \chi_p \]  

Finally (2.8) is compared with (2.9), some of the integrations are carried out explicitly, and thus we find
Electromagnetic Structure of the Nucleon

\[ \text{Im} F^*_1 = -\frac{1}{12(2\pi)^3} \frac{mE}{E+m} \int d\nu \int du \int \frac{d^3k^+}{(2\pi)^3} (k^+)^2 (k^-)^2 \sin^2\phi \]
\[ \times \left\{ \text{Re}(H^*\alpha) + \frac{2m+E}{E-m} \frac{1}{\sqrt{2}} \text{Re}(H^*\beta) \right\} , \quad (2.10a) \]

\[ \text{Im} F^*_2 = -\frac{1}{24(2\pi)^3} \frac{m}{E+m} \int d\nu \int du \int \frac{d^3k^+}{(2\pi)^3} (k^+)^2 (k^-)^2 \sin^2\phi \]
\[ \times \left\{ \text{Re}(H^*\alpha) - \frac{2E+m}{E-m} \frac{1}{\sqrt{2}} \text{Re}(H^*\beta) \right\} , \quad (2.10b) \]

where we have introduced

\[ u = \omega_+ + \omega_- , \]
\[ v = \omega_+ - \omega_- , \quad (2.11) \]
\[ \cos \phi = \frac{k^+ \cdot k^-}{|k^+||k^-|} = \frac{(2E-\omega_+ - \omega_-)^2 - \omega_+^2 - \omega_-^2 + \mu^2}{2k^+ \cdot k^-} . \]

The limits of integration in (2.10a) and (2.10b) are

\[ u_{\text{max}} = 2E - \mu , \]
\[ u_{\text{min}} = E + \frac{3\mu^2}{4E} , \quad (2.12) \]
\[ v_{\text{max}} = \left\{ (2E-u)^2 - \mu^2 \right\}^{1/2} \left\{ 1 + \frac{4}{(2E-u)^2 - \mu^2 - u^2} \right\}^{1/2} . \]

The invariant variables \( s_i \) are connected with the integration variables through the following relations:

\[ s_1 = \mu^2 + 4E(E-\omega_+) , \]
\[ s_2 = \mu^2 + 4E(E-\omega_-) , \quad (2.13) \]
\[ s_3 = \mu^2 - 4E(E-\omega_+ - \omega_-) . \]

Our next task is to calculate the invariant amplitudes in (2.4) and (2.7).

2.B. Structure of the invariant amplitudes due to pion-pion scattering

To proceed to physical arguments of the problem it is necessary to know how the amplitudes \( H, \alpha, \) and \( \beta \) depend on the variables \( s_i \) and \( q^2 \). In this section we shall develop a detailed investigation of the analytical structure of the amplitude \( H \). \( \alpha \) and \( \beta \) can be treated in the parallel way, so that we shall

* The effects of pion-picon correlation on the three-picon contribution to the iso-scalar nucleon structure have recently been discussed independently by Blankenbecler and Tarski.\(^{17}\)
K. Kawarabayashi and A. Sato give only their results without entering into details.

The approach which we shall follow to analyse the structure of $H$ is based on the unitarity condition

$$\text{Im}\langle 0| j_\mu^*| k^+, k^-, k^0 \rangle = \frac{\pi}{\sqrt{2\alpha_0}} \sum \langle 0| j_\mu^*| 3\pi \rangle \langle 3\pi| J^0| k^+, k^- \rangle \times \delta \left( p^2 - k^+ - k^- - k^0 \right).$$

\[(2.14)\]

The connected scattering part of (2.14) determines the singularity of $H$ across the physical cut in $q^2$, while the disconnected part does the singularities of $H$ across physical cuts in the variables $s_t$. We shall first discuss the structure of $H$ due to the latter singularities.

In this case $H$ is more conveniently regarded as the amplitude for the virtual photo-pion production from pions \(s_0 J' s_1 J \) (Fig. 4). Then, retaining only the \(T=1\), $P$-wave two-pion intermediate state, the unitarity condition reduces approximately to

$$\text{Im} \ h(s_i) = h^*(s_i) e^{i\delta(s_i)} \sin \delta(s_i),$$

\[(2.16)\]

where

$$h(s_i) = \frac{3}{4} \int_{-1}^{+1} d \cos \theta \ sin^2 \theta \ H(q^2; s_i, \cos \theta)$$

\[(2.17)\]

is the $P$-wave projection of $H(q^2; s_1, s_2, s_3)$ in the channel described by $s_t$. We denote by $\delta(s)$ the $T=1$, $P$-wave pion-pion scattering phase shift.

The construction of coupled scattering and production amplitudes satisfying unitarity and analyticity requirements has recently been discussed by several authors with the aids of the so-called ND-1 method \(^{10,19}\) which is now to be applied to our problem. We have the following set of integral equations in the channel described by $s_t$:

$$h(s_i) D_{\ast\ast}(s_i) = \overline{h}_i(s_i),$$

\[(2.18)\]

$$T_{\Pi}(s_i) D_{\ast\ast}(s_i) = N_{\Pi}(s_i),$$

\[(2.19)\]
Electromagnetic Structure of the Nucleon

\[ D_{\pi\pi}(s_i) = 1 - \frac{s_i - \mu^2}{\pi} \int_{(s\phi)^2}^{\infty} ds' \frac{\rho(s') N_{\pi\pi}(s')}{(s' - \mu^2)(s' - s_i - i\epsilon)} \]  

(2.20)

where \( T_{\pi\pi}(s) \) is the pion-pion scattering amplitude in the \( T=1, J=1 \) state and \( N_{\pi\pi} \) has only the left-hand branch point. It is easily seen that \( \bar{h}_t(s_i) \) is, by construction, real on the positive real axis in the variable \( s_i \).

The assumption that the coupled integral equations (2·19) and (2·20) have been solved permits one to write the function \( D_{\pi\pi}(s_i) \) as

\[ D_{\pi\pi}(s_i) = \exp \left\{ -U(s_i) \right\} \]

\[ = \exp \left\{ -\frac{s_i - \mu^2}{\pi} \int_{(s\phi)^2}^{\infty} ds' \frac{\delta(s')}{(s' - \mu^2)(s' - s_i - i\epsilon)} \right\} \]  

(2·21)*

Then (2·18) immediately leads to the following form of the solution for \( H \),

\[ H(q^2; s_1, s_2, s_3) = \frac{\bar{H}_t(q^2; s_1, s_2, s_3)}{D_{\pi\pi}(s_i)} \]  

(2·22)

The function \( \bar{H}_t(q^2; s_1, s_2, s_3) \) corresponding to \( \bar{h}_t(s_i) \) is real on the positive real axis in the variable \( s_i \). Since (2·22) should be valid for any \( i \) we have

\[ H(q^2; s_i) = \frac{\bar{H}_t(q^2; s_i)}{D_{\pi\pi}(s_i)} = \frac{\bar{H}_t(q^2; s_1)}{D_{\pi\pi}(s_1)} = \frac{\bar{H}_t(q^2; s_2)}{D_{\pi\pi}(s_2)} \]  

(2·23)

Thus the function \( H \) which satisfies (2·23) achieves the form

\[ H(q^2; s_i) = \bar{H}(q^2; s_i) \frac{D_{\pi\pi}(s_1)}{D_{\pi\pi}(s_2)} \]  

\[ = \bar{H}(q^2; s_i) \exp \{ U(s_1) + U(s_2) + U(s_3) \} \]  

(2·24)

The quantity \( \bar{H}(q^2; s_i) \) now does not involve the two-pion elastic cut singularities in any variable \( s_i \).

From the solution (2·24) for \( H \) and the reality condition of the absorptive parts of the form factors we have expressions for the amplitudes \( \alpha \) and \( \beta \) as follows:

\[ \alpha(q^2; s_i) = \bar{\alpha}(q^2; s_i) \exp \{ U(s_1) + U(s_2) + U(s_3) \} \]  

(2·25a)

\[ \beta(q^2; s_i) = \bar{\beta}(q^2; s_i) \exp \{ U(s_1) + U(s_2) + U(s_3) \} \]  

(2·25b)

The quantity \( \exp \{ U(s) \} \) which has been introduced above is essentially equivalent to the pion electromagnetic form factor. Following Frazer and Fulco, we express it in terms of an approximate pion-pion scattering amplitude \( f_{\pi\pi}(s) \) as

\[ \ast \] Note that we have normalized the quantity \( U(s) \) as \( \exp U(\mu^2) = 1 \). The problem of normalizations of the invariant amplitudes will be discussed in some detail at the end of this section.
where

\[ f_{\pi\pi}(s) = \frac{\Gamma}{\nu - \nu \left[ 1 - \Gamma \alpha(\nu) \right] - i \theta(\nu) \Gamma \nu / (\nu + \mu^2)^{1/2}}, \]

\[ \nu = \frac{1}{4} s - \mu^2, \]

\[ \nu_r = \frac{1}{4} s_r - \mu^2, \]

\[ \alpha(\nu) = \frac{2}{\pi} \left( \frac{\nu}{\nu + \mu^2} \right)^{1/2} \log \left( \frac{\sqrt{\nu + \sqrt{\nu + \mu^2}}}{\mu} \right), \]

and \(-s_0\) is the position of the left-hand pole of \(f_{\pi\pi}\). The values of parameters \(\nu_r = 4.87\mu^2\) and \(\Gamma = 0.30\)** are chosen so as to fit recent data\(^7\) on the pion-pion scattering resonance. For these parameters the behaviour of \(|f_{\pi\pi}(s)|^3\) is illustrated in Fig. 5.

**2.C. Structure of the invariant amplitudes due to three-pion scattering**

We shall next investigate the singularity of \(H\) across the physical cut in \(q^2\) which arises from the connected three-pion scattering. As to the matrix element \(\langle 2\pi|J|2\pi\rangle_{\text{conn.}}\) we assume that the three-pion scattering process occurs dominantly through the newly discovered \(T=0\) three-pion resonance. The mechanism which leads to this resonance is not yet clear but, for simplicity, let us assume that the resonance can be treated in a chain approximation illustrated in Fig. 6.**

![Fig. 6. Chain approximation for the (connected) three-pion scattering process.](image-url)

**Since \(s_0\ll s\) for all values of \(s\) considered, we have safely taken \((s + s_0)/(\mu^2 + s_0) = 1\) in numerical calculations in §3.**

**These values of parameters give the resonance energy of 5.2\(\mu\) and the total width of 170 Mev.**

**A model of the \(T=0\) three-pion resonance has been discussed by the present authors.\(^2\)**
The corresponding matrix element for the three-pion scattering process is evaluated in Appendix A. We find

$$\langle k', k, k'' \text{ out} | J^0 | k, k' \text{ in} \rangle = -\left(\frac{1}{32\omega_+\omega_-\omega_{0}'}\right)^{1/2} \lambda$$

$$\times \varepsilon_{\mu
u'\lambda'\tau'} k_{\mu}' k_{\lambda}' k_{\tau}' \varepsilon_{\mu
u\lambda\tau} k_{\mu} k_{\lambda} k_{\tau} \exp \{U(s_1') + U(s_2') + U(s_3')\}$$

$$\times \exp \{U(s_1) + U(s_2) + U(s_3)\} D^{-1}(q^2), \quad (2.28)$$

where $q^2 = (k' + k - k')^2$ and the function $D(q^2)$ is defined by (A.9).

Substituting (2.28) in (2.14) the unitarity condition for the connected three-pion scattering is formulated as

$$\Im H(q^2; s) = \frac{\pi\lambda}{3(2\pi)^6} (-q^2) \left[ \frac{d^3k'}{8\omega_+} \frac{d^3k}{8\omega_-} \frac{d^3k''}{8\omega_{0}'} \delta(k' + k - k'' - q) \right]$$

$$\times (k' \times k')^2 H^*(q^2; s) \exp \{U(s_1') + U(s_2') + U(s_3')\}$$

$$\times \exp \{U(s_1) + U(s_2) + U(s_3)\} D^{-1}(q^2). \quad (2.29)$$

If we neglect in (2.29) the dependence of $D(q^2)$ on the variables $s_i'$, we have approximately

$$\Im H(q^2; s) = \frac{\pi\lambda}{3(2\pi)^6} (-q^2) \left[ \frac{d^3k'}{8\omega_+} \frac{d^3k}{8\omega_-} \frac{d^3k''}{8\omega_{0}'} \delta(k' + k - k'' - q) \right]$$

$$\times (k' \times k')^2 H^*(q^2; s) \exp \{U(s_1') + U(s_2') + U(s_3')\} \right|^8$$

$$= -H^*(q^2; s) \Im \frac{D(q^2)}{D(q^2)}. \quad (2.30)$$

It is obvious from this result that the function

$$H(q^2; s) = \overline{H}(q^2; s) \cdot \frac{D(0)}{D(q^2)} \exp \{U(s_1) + U(s_2) + U(s_3)\} \quad (2.31)^*)$$

satisfies the unitarity condition (2.30), assuming that the function $\overline{H}(q^2; s)$ does not involve the two-pion and three-pion elastic cut singularities and is weakly dependent on the variables $s_i$.

The amplitudes $\alpha$ and $\beta$ are assumed to have similar structures:

$$\alpha(q^2; s) = \tilde{\alpha}(q^2; s) \frac{D(0)}{D(q^2)} \exp \{U(s_1) + U(s_2) + U(s_3)\}, \quad (2.32a)$$

*) There is a well-known multiplicity of solutions to dispersion relations. Our solution (2.31) is constructed so that it reduces to the chain approximation of the $\gamma$-3$\pi$ vertex when $H$ is taken as a constant (see Appendix A).
\[ \beta (q^2; s_i) = \beta (q^2; s_i) \frac{D(0)}{D(q^2)} \cdot \exp \{ U(s_i) + U(s_2) + U(s_3) \}. \] (2.32b)

When the function \( D(0)/D(q^2) \) shows a resonance behaviour it can be rewritten as derived in Appendix A:

\[
\begin{align*}
\frac{D(0)}{D(q^2)} &= \frac{m^2}{(q^2 + m_a^2)^2} \int_{m_a^2}^{\infty} \frac{m^2 \text{Im} \sum (-m^2, i)}{m^2 (m^2 - m_a^2)} dm^2 \\
&= \int_{m_a^2}^{\infty} \frac{m^2 \text{Im} \sum (-m^2, i) - i \theta (-q^2 - 9 \mu^2) \text{Im} \sum (q^2, i)}{(q^2 + m_a^2) (m^2 + q^2) (m^2 - m_a^2)} dm^2 \\
&\approx \frac{m^2}{q^2 + m_a^2} - i \theta (-q^2 - 9 \mu^2) \text{Im} \sum (q^2, g_a^2). \tag{2.33}
\end{align*}
\]

The effective coupling constant \( g_a \) is related to the observed width \( (\Gamma_a \lesssim 30 \text{ MeV}) \) of the three-pion resonance as

\[
m_a \Gamma_a = \text{Im} \sum (-m^2, g_a^2) = g_a \cdot \frac{\pi m_a^2}{3 (2\pi)^3} \int \frac{d^3 k^+ d^3 k^- d^3 k^0}{8 \omega_+ \omega_- \omega_0} \delta(k^+ + k^- + k^0) \delta(\omega_+ + \omega_- + \omega_0 - m_a) \\
\times |(k^+ \times k^-)^2| \exp \{ U(s_i) + U(s_2) + U(s_3) \} \|^2. \tag{2.34}
\]

Performing numerical integrations we get

\[
\frac{g_a^2}{4\pi} \lesssim 0.62 \mu^- 6. \tag{2.35}
\]

With this value of the coupling constant \( g_a \) the resonance behaviour of the function \( |D(0)/D(q^2)|^2 \) is illustrated in Fig. 7. Hereafter we shall denote \( D(0)/D(q^2) \) simply as \( D^{-1}(q^2) \).

2.D. Remaining structures of the invariant amplitudes

The final task of our formulation of the three-pion contribution is to determine remaining structures of the functions \( \bar{\alpha}, \bar{\beta} \) and \( \bar{H} \). Even if we neglect more massive intermediate states than the three-pion state, \( \bar{\alpha} \) and \( \bar{\beta} \) have singularities from crossed cuts and complex singularities present in the production amplitudes.\(^{17,20}\) In this
paper, however, we do not enter into detailed discussions of them but restrict ourselves to the lowest order perturbation theory. This is because the main objective of the present paper is to see how the perturbation results are modified by including the effects of the two-pion and three-pion scattering resonances.

Let us begin with the calculation of $\bar{H}$. In this case we have contributions from three Feynman diagrams shown in Fig. 8.* The calculations are straightforward and given in Appendix B. We have

$$\bar{H}(q^2; \sigma_i) = \frac{e^2 g^3}{12\pi^2 m^2} \{ I(q^2; k^+, k^-) + I(q^2; k^0, k^-) + I(q^2; k^+, k^0) \}, \quad (2.36)$$

where

$$I(q^2; k^+, k^-) = 6 \int_0^1 dx \int_0^1 dy \int_0^1 z (1-z) dz$$

$$\times \left\{ 1 - [xz - x^2 z^2 + y(1-z) - y^2 (1-z)^2] \frac{\mu^2}{m^2} + z(1-z) [q^2 - 2xk^+ \cdot q - 2yk^- \cdot q + 2xyk^+ \cdot k^-] \frac{1}{m^2} \right\}^{-2} \quad (2.37)$$

The first, second and third terms in (2.36) correspond respectively to Figs. 8a, 8b and 8c.

For ease of numerical calculations the function $I(q^2; k^+, k^-)$ is approximated in § 3 by

$$I(q^2) = I(q; k^+, k^-) |_{k^+ = |k^-| = |k^0|} \quad (2.38)$$

The dependence of $\bar{H}$ on the squared photon mass $-q^2$, on the contrary, is essentially the same as the case of the nucleon-loop current as far as $-q^2 < m^2_{\text{Baryon}}$.

---

* Contributions from hyperon-loop currents are neglected. They are not always additive, since they are proportional to $g^3$. The magnitude of $H$, therefore, is not reliably estimated by (2.36). The dependence of $\bar{H}$ on the squared photon mass $-q^2$, on the contrary, is essentially the same as the case of the nucleon-loop current as far as $-q^2 < m^2_{\text{Baryon}}$. 

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Fig. 8. The lowest order perturbation diagrams for electro-production of three pions.

Fig. 9. The lowest order perturbation diagrams for pair annihilation into three pions.
the explicit expression of which is given by (B·15).

We next calculate the nucleon pair annihilation amplitudes from Feynman diagrams shown in Fig. 9. From the results obtained in Appendix B we find

\[
\bar{\alpha}(q^2; s_i) = -\frac{g^4}{2\pi} \int d\Omega_\rho \left( 1 + \frac{E-m}{m} \hat{p}_\rho^2 \right)
\times \left\{ F(\rho; k^+, k^-) + F(\rho; k^0, k^-) + F(\rho, k^+, k^0) \right\},
\]

\[
\frac{1}{\sqrt{2}} \beta(q^2; s_i) = -\frac{g^4}{2\pi} \int d\Omega_\rho \left( \frac{3\hat{p}_\rho^2-1}{2} + \frac{E-m}{m} \hat{p}_\rho^2 \right)
\times \left\{ F(\rho; k^+, k^-) + F(\rho; k^0, k^-) + F(\rho, k^+, k^0) \right\},
\]

where

\[
F(\rho; k^+, k^-) = \frac{1}{-2p\cdot k^- - 2E_{\omega^+} + \mu^2} \cdot \frac{1}{2pk^+ - 2E_{\omega^+} + \mu^2}
\]

(2·40)
corresponds to Figs. 9a and 9a', while the second and third terms in curly brackets come from Figs. 9b, b' and Figs. 9c, c' respectively.

Thus obtained results (2·36), (2·39a) and (2·39b) are now inserted into (2·10) and we obtain the final expressions for the absorptive parts as follows:

\[
\text{Im } F_1^s = -\frac{1}{12(2\pi)^3} \frac{E_m}{E+m} \int d\omega_+ d\omega_- (k^+) s(k^-) \sin^2 \phi 
\times \left\{ \bar{\alpha}(q^2; s_i) + \frac{2m+E}{E-m} \cdot \frac{1}{\sqrt{2}} \bar{\beta}(q^2; s_i) \right\} \bar{H}(q^2; s_i)
\times \left\{ D^{-1}(q^2) \exp \{ U(s_1) + U(s_2) + U(s_3) \} \right\}^2,
\]

\[
\text{Im } F_2^s = -\frac{1}{24(2\pi)^3} \frac{m}{E+m} \int d\omega_+ d\omega_- (k^+) s(k^-) \sin^2 \phi 
\times \left\{ \bar{\alpha}(q^2; s_i) - \frac{2E+m}{E-m} \cdot \frac{1}{\sqrt{2}} \bar{\beta}(q^2; s_i) \right\} \bar{H}(q^2; s_i)
\times \left\{ D^{-1}(q^2) \exp \{ U(s_1) + U(s_2) + U(s_3) \} \right\}^2.
\]

(2·41a) and (2·41b)

It will be easily seen that each of the three terms in curly brackets in (2·39a) and (2·39b) have an equal contribution in (2·41a) and (2·41b). The D-wave amplitude \(\bar{\beta}\) which goes to zero as \(p^2\) for \(E\rightarrow m\) cancels out the kinematical pole of the factor \((E-m)^{-1}\) in (2·41), leaving the finite contribution as

\[
-\frac{1}{E-m} \frac{\bar{\beta}}{\sqrt{2}} \frac{g^4}{2\pi m} \int d\Omega_\rho \hat{p}_\rho^2 \left\{ F(\rho; k^+, k^-) + F(\rho; k^0, k^-) \right. \\
+ \left. F(\rho; k^+, k^0) \right\} \bigg|_{E=m}.
\]

(2·42)

(2·41a) and (2·41b) together with (2·26), (2·33), (B·11a), (B·11b) and
(B·14) are basic formulas for the numerical calculation of the form factors. We insert (2·41a) and (2·41b) into the dispersion relations (2·2) with a cut-off at \( \sigma^2 = (2m)^3 \). For the contribution from the physical region of nucleon pair annihilation \((\sigma^2 > 4m^2)\), quite different approximations will be possible since in this region the unitarity condition allows one to estimate the upper limits for the contributions of the amplitudes \( \alpha \) and \( \beta \). It is our feeling, however, that the most dominant contribution will surely come from the unphysical region \((\sigma^2 \leq 4m^2)\) when pion correlations are included.

Before closing this section we have to point out an ambiguity which still remains in our formulation, that is, the ambiguity of the normalization of the amplitudes \( H, \alpha \) and \( \beta \). The corresponding problem of normalization does not occur in the formulation of the iso-vector form factors of the nucleon since we know the magnitudes of the renormalized charge of the pion and the Born term in the pion-nucleon scattering amplitude.

As to the amplitude \( H \) we have normalized the pion correlation factors \( \exp\{U\} \) and \( D \) so that \( H \) reduces to \( H \) at the symmetric point \( s_1 = s_2 = s_3 = \mu^2 \) and \( q^2 = 0 \). The reason for this choice is that the strength of the amplitude \( H \) at the symmetric point may be determined experimentally by performing measurements of the photo-pion production from pions and by extrapolating them to the symmetric point.\(^*\) In other words, an effective \( \gamma - 3\pi \) coupling constant is defined by

\[
    g_{\text{eff}} = H(q^2 = 0; \ s_1 = s_2 = s_3 = \mu^2).
\]

The magnitude of \( g_{\text{eff}} \) is studied by several authors in connection with the problems of photoproduction of neutral pions from hydrogen,\(^{10} \) positive to negative ratio of the photopion production at threshold,\(^{24} \) and iso-scalar current properties of the deuteron.\(^{26} \) These investigations have shown that the amplitude at the symmetric point should be at most of the same order of magnitude as that predicted by the perturbation calculation. So the normalization adopted in this paper and the estimation of \( H \) by the perturbation calculation will be reasonable.

A trouble arises from our ignorance of the magnitudes of the amplitudes \( \alpha \) and \( \beta \), which we have constructed so as to satisfy the unitarity and analyticity requirements and the reality condition of the absorptive parts of the form factors. Our tentative estimate of \( \bar{\alpha} \) and \( \bar{\beta} \) by the perturbation theory will be modified, for example, by taking into account the rescattering effect in pion-nucleon scattering.

\section*{§ 3. Numerical results\(^{**}\)}

In this section numerical calculations are performed for the three-pion state

\(^*\) Since \( s_i (i = 1, 2, 3) \) are related to each other by the relation (2·6), this normalization is equivalent to the subtraction method in the usual sense only for \( q^2 = 0 \), i.e. the real photon. We assume the analytic continuation in \( q^2 \) of the form given by (2·24) and (2·21).

\(^{**}\) Numerical calculations were carried out with the aid of the electronic computer PC-I at the University of Tokyo and the Mark 4-A at the Electro-Technical Laboratory.
K. Kawarabayashi and A. Sato

contribution to the iso-scalar form factors on the basis of the formulation derived in the preceding section. In addition to the form factors we evaluate the mean-square radii \( \langle r^2 \rangle_{i_z} \) of the charge and magnetic moment distribution, the anomalous magnetic moment \( \mu^a \) and the total amount of charge \( Q^a \) which spreads out in the three-pion configuration. A particular interest thereby is taken in the change of the shapes of the absorptive parts of the form factors by including the effects of the two-pion and three-pion resonances.

For convenience we reproduce here the expressions for the absorptive parts of the form factors and various quantities to be discussed in this section.

\[
\text{Im} \ F_{1,\alpha}^a(-\sigma^a) = e \left( \frac{g^2}{4\pi} \right)^3 \frac{1}{24\pi^3} \frac{Em}{E+m} J^a(-\sigma^a), \quad (3.1a)
\]

\[
\text{Im} \ F_{1,\beta}^a(-\sigma^a) = e \left( \frac{g^2}{4\pi} \right)^3 \frac{1}{24\pi^3} \frac{Em(E+2m)}{E^3-m^2} J^a(-\sigma^a), \quad (3.1b)
\]

\[
\text{Im} \ F_{2,\alpha}^a(-\sigma^a) = \frac{e}{2m} \left( \frac{g^2}{4\pi} \right)^3 \frac{1}{24\pi^3} \frac{m^2}{E+m} J^a(-\sigma^a), \quad (3.1c)
\]

\[
\text{Im} \ F_{2,\beta}^a(-\sigma^a) = \frac{e}{2m} \left( \frac{g^2}{4\pi} \right)^3 \frac{1}{24\pi^3} \frac{m^4(2E+m)}{m^2-E^2} J^a(-\sigma^a), \quad (3.1d)
\]

where, with \( \alpha = -\frac{g^2}{4\pi} \hat{\alpha} \), \( \beta = -\frac{g^2}{4\pi} \hat{\beta} \) and \( \hat{H} = \frac{ef^2}{4\pi} \hat{H} \),

\[
J^a(-\sigma^a) = \int d\omega_+ \int d\omega_- (k^+)^2 (k^-)^2 \sin^2 \phi \hat{\alpha} (q^a; s_1) \hat{\hat{H}}(q^a; s_1) \times |D^{-1}(q^a)|^2 \exp \{U(s_1) + U(s_2) + U(s_3)\} |^2, \quad (3.2a)
\]

\[
J^\beta(-\sigma^a) = \int d\omega_+ \int d\omega_- (k^+)^2 (k^-)^2 \sin^2 \phi \frac{1}{\sqrt{2}} \hat{\beta} (q^a; s_1) \hat{\hat{H}}(q^a; s_1) \times |D^{-1}(q^a)|^2 \exp \{U(s_1) + U(s_2) + U(s_3)\} |^2. \quad (3.2b)
\]

The superscripts \( \alpha \) and \( \beta \) stand for the contributions from the \( ^3S_1 \) and \( ^3D_1 \) nucleon pair states respectively.

\[
\langle r^2 \rangle_{i_z} = 6 \frac{1}{F_{1,\sigma}^a(0)} \frac{1}{\pi} \int_{(\omega=)^a}^{(\omega=)^a} d\sigma^3 \text{Im} \ F_{1,\sigma}^a(-\sigma^a), \quad (3.3)
\]

\[
Q^a = \frac{1}{\pi} \int_{(\omega=)^a}^{(\omega=)^a} d\sigma^3 \frac{\text{Im} \ F_{2,\sigma}^a(-\sigma^a)}{\sigma^3} \quad (3.4)
\]

\[
\mu^a = \frac{1}{\pi} \int_{(\omega=)^a}^{(\omega=)^a} d\sigma^3 \frac{\text{Im} \ F_{3,\sigma}^a(-\sigma^a)}{\sigma^3} \quad (3.5)
\]

We have introduced the cutoff to neglect the contribution from the inner part
(r<1/2m) of the nucleon structure.

3.A. Perturbation calculations \((U(s) = 0, D(q^2) = 1)\)

We shall begin with the lowest order perturbation calculations which are the starting point of our numerical discussions.

i) \(\overline{H}(q^2; s_i)\): Because of a computational restriction we approximate \(\overline{H}\) throughout this section by

\[
\overline{H}(q^2; s_1, s_2, s_3) \approx \overline{H}(q^2; \tilde{s}, \tilde{s}, \tilde{s}),
\]

where \(\tilde{s} = \mu^2 - q^2/3\) (see appendix B).

This approximation corresponds to an extended effective \(\gamma - 3\pi\) interaction, which coincides for \(q^2 = 0\) with the point \(\gamma - 3\pi\) interaction adopted by Kato\(^{20}\) and by Fujii, Kawaguchi and Miyamoto.\(^{24,25}\) The function \(\overline{H}(q^2; \tilde{s})\) is now calculated from (B·14) and the result is shown in Fig. 10.

![Fig. 10. The \(\gamma - 3\pi\) amplitude \(\overline{H}\) in the perturbation theory as a function of \(q^2\) when \(s_1 = s_2 = s_3 = \mu^2 - q^2/3\).](image)

The dependence of \(\overline{H}(q^2; \tilde{s})\) on \(q^2\) shown in Fig. 10 is probably characteristic of the lowest order perturbation calculation and higher order corrections will considerably modify the high frequency \((-q^2\sim4m^2)\) behaviour of \(\overline{H}\).

ii) \(\text{Im} F_{1,\ast}^\gamma(-\sigma^2)\): The absorptive parts of the charge and magnetic moment form factors are calculated from (3·1a) \(- (3·1d)\) with \(\overline{H}(q^2; \tilde{s})\) substituted for \(\overline{H}(q^2; s_i)\) and the resonance factors put equal to unity. Figs. 11a and 11b show the results. For comparison we give in Fig. 12 the phase space integral defined by

\[
J_{\text{phase}}(-\sigma^2) = \int \int d\omega_+ d\omega_- (k^+)^2 (k^-)^2 \sin^2 \phi.
\]

From Figs. 11a and 11b we see that \(\text{Im} F_{1,\ast}^\gamma(-\sigma^2)\) and \(\text{Im} F_{2,\ast}^\gamma(-\sigma^2)\) are both positive, while \(\text{Im} F_{1,\ast}^\gamma(-\sigma^2)\) is negative and has an opposite sign to \(\text{Im} F_{2,\ast}^\gamma(-\sigma^2)\). This property together with the relation

\[
\text{Im} F_{1,\ast}^\gamma(-\sigma^2) \text{ (in unit of } \frac{e}{2m}) > \text{Im} F_{1,\ast}^\gamma(-\sigma^2) \text{ (in unit of } e) > 0
\]
Fig. 11. Absorptive parts of the iso-scalar form factors in the perturbation theory.
Fig. 12. Variation of the phase space integral (3.8) with $\sigma^2$.

Fig. 13. Pair annihilation amplitudes $\bar{a}$ and $\bar{b}$ in the perturbation theory as functions of $q^2$ when $s_1 = s_2 = s_3 = \mu^2 - q^2/3$. 
K. Kawarabayashi and A. Sato

is to be noticed.

iii) $\bar{\alpha}(q^2; s_i)$ and $\bar{\beta}(q^2; s_i)$: Let us see the behaviours of $\bar{\alpha}$ and $\bar{\beta}$ in more detail by making the same approximation for them as done for $H$ (See Appendix B):

$$\bar{\alpha}(q^2; s_1, s_2, s_3) \approx \bar{\alpha}(q^2; \bar{s}, \bar{s}, \bar{s}), \quad (3.10a)$$

$$\bar{\beta}(q^2; s_1, s_2, s_3) \approx \bar{\beta}(q^2; \bar{s}, \bar{s}, \bar{s}). \quad (3.10b)$$

The dependences of $\bar{\alpha}(q^2; \bar{s})$ and $\bar{\beta}(q^2; \bar{s})$ on $q^2$ are shown in Figs. 13a and 13b. We can see that $\bar{\alpha}(q^2; \bar{s})$ is a monotonically decreasing function of $-q^2$ while $\bar{\beta}(q^2; \bar{s})$ decreases very rapidly to zero at $-q^2 = 4m^2$.

iv) $\langle r^2 \rangle_{1,2}, Q^l$, and $\mu^l$: These quantities are now straightforwardly calculated from (3·3), (3·4) and (3·5). We give them in Table I together with their experimental values.*

| Table I. Perturbation estimates of $\langle r^2 \rangle_{1,2}, Q^l$, and $\mu^l$. |
|-----------------|-----------------|
| Perturbation    | Experiment      |
| $\langle r^2 \rangle_{1,2}$ (\(\mu^2\)) | 0.041           | 0.32           |
| $\mu^l(r^2)_{23}$ (\(\mu^2 e/2m\)) | 0.026           | \(\ll \mu^l(r^2)_{23} \approx 0.63\) |
| $Q^l$           (\(e\)) | 0.43            |                |
| $\mu^l$         (\(e/2m\)) | 0.54            | -0.06          |

The magnitude of the renormalized pion-nucleon coupling constant is taken to be $g^2/4\pi = 14.5$.

The calculated mean-square radius of the charge distribution turns out to be small compared with experiment. The anomalous magnetic moment is large in magnitude and has a wrong sign, which is a general feature of the three-pion contribution. (See also § 4.)

3.B. Inclusion of the two-pion resonance ($D(q^2) = 1$)

In order to include the effects of the two-pion correlation an assumption must be made that the pion-pion scattering amplitude is well approximated with the $P$-wave one-level formula (2·27) up to rather high energy region ($s \leq 4m^2$).

i) $\text{Im} F_{1,2}^l(-\sigma^2)$: Calculations are carried out using the formulas (3·1a) (3·1d) with $D=1$. The shapes of the absorptive parts are greatly modified as illustrated in Figs. 14a and 14b. The effects of the two-pion resonance are seen to be reflected in broad maxima of $\text{Im} F_{1,2}^l(-\sigma^2)$ at $\sigma^2 \approx 80\mu^2$ which are by an order higher than the perturbation curves.

The relative behaviour between the charge and magnetic moment form

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* On account of the smallness of $\mu^l$ it is difficult to obtain from experiment some definite information on the magnitude of the mean-square radius of the magnetic moment distribution $\langle r^2 \rangle_{23}$. In this paper, therefore, detailed discussions on the behaviour of the magnetic moment form factor $F_{23}^l(q^2)$ are not attempted. Only the calculated values of $\mu^l(r^2)_{23}$ as well as $Q^l$ will be presented.
Electromagnetic Structure of the Nucleon

Fig. 14. Absorptive parts of the iso-scalar form factors modified by including the two-pion resonance factors. For comparison \( \text{Im} \ F_{1,2} \text{pert} \) are shown by broken lines.

Factors, on the other hand, remains almost unchanged from the perturbation results. This implies that an enhancement of the charge radius \( \langle r^2 \rangle \), for example, leads at the same time to a resultant enlargement of the anomalous magnetic moment \( \mu' \).

ii) \( \langle r^2 \rangle, Q' \) and \( \mu' \): We show in Table II the values of these quantities calculated with \( D(q^2) = 1 \). Table II contains, for comparison, their values calculated with the approximate pair annihilation amplitudes (3.10a) and (3.10b).

These values are very large in both cases though the absolute values are not to be taken seriously as stated at the end of § 2. It is interesting to remark that if the unsubtracted dispersion relation is used for \( F'_1 \) the ambiguity of the normalization of the absorptive part does not affect the evaluation of the mean-
Table II. Inclusion of the two-pion resonance.

<table>
<thead>
<tr>
<th></th>
<th>perturbation + 2π resonance</th>
<th>perturbation + 2π resonance (with approximate $\alpha$ and $\beta$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle r^2 \rangle$</td>
<td>1.3</td>
<td>1.5</td>
</tr>
<tr>
<td>$\mu^2 \langle r^4 \rangle$</td>
<td>1.3</td>
<td>1.0</td>
</tr>
<tr>
<td>$Q^2$</td>
<td>9.3</td>
<td>11.7</td>
</tr>
<tr>
<td>$\mu^2$</td>
<td>18.5</td>
<td>14.4</td>
</tr>
</tbody>
</table>

square radius of the charge distribution and we get $\langle r^2 \rangle \approx 0.07 \mu^{-2}$.

iii) $F_1^*(q^2)$: One way for fixing the normalization of the absorptive part is to adjust it so as to reproduce the observed charge radius. With the normalization thus fixed the charge form factor $F_1^*(q^2)$ is shown in Fig. 15. Figure 16 shows

![Graph](image)

**Fig. 15.** The iso-scalar charge form factor modified by including the two-pion resonance factors. The absorptive part is renormalized so as to reproduce the observed charge radius $\langle r^2 \rangle = 0.32 \mu^2$.

![Graph](image)

**Fig. 16.** A comparison of the calculated charge form factor of the proton with experiments under the assumption that $F_1^*$ is well described by (3·11). Two-pion resonance factors are included. The normalization of the absorptive part is the same as in Fig. 15.
Electromagnetic Structure of the Nucleon

193

a comparison between the theoretical and experimental charge form factors. In order to deduce the proton charge form factor we assume that the iso-vector charge form factor is well approximated by

\[ F^v_1(q^2) = -0.10 + \frac{0.60}{1 + 0.2q^2} \text{ (\(\mu=1\) unit)}, \]  

(3·11)

which has been obtained by Hofstadter et al.\(^3\)

Since the positions of the maxima of \(\text{Im} F^v_1(-\sigma^2)\) are too high, it is obvious that the three-pion contribution with only two-pion correlations is not sufficient to get agreement with experiment in the high momentum transfer region.

3.C. Inclusion of the three-pion resonance

i) \(\text{Im} F^v_1(-\sigma^2)\): In addition to the two-pion resonance the effects of the three-

![Graph](https://example.com/graph.png)

**Fig. 17.** Absorptive parts of the iso-scalar form factors modified by including both the two-pion and three-pion resonance factors. For comparison \(\text{Im} F^v_1,\text{pert}\) are shown by broken lines.
pion resonance are now included through the factor $D(q^2)$. It is shown in Fig.
17 how the absorptive parts are influenced by the inclusion of the three-pion
resonance.

The change of the shapes of the absorptive parts into sharp resonance-like
ones and the low positions of their maxima (a 2 = 42, U 2 ) are remarkable. This
is mainly due to the rapid increase of Im $D(q^2)$.

ii) $\langle r^2 \rangle_{2s}, Q'$ and $\rho'$: The various quantities are now calculated from the
above results and summarized in Table III.

Table III. Inclusion of the three-pion resonance.

<table>
<thead>
<tr>
<th></th>
<th>perturbation + 2π and 3π resonances</th>
<th>perturbation + 3π resonance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle r^2 \rangle_{2s}$</td>
<td>(μ−2)</td>
<td>0.29</td>
</tr>
<tr>
<td>$\mu^2(r^2)_{2s}$</td>
<td>(μ−2e/2m)</td>
<td>0.27</td>
</tr>
<tr>
<td>$Q''$</td>
<td>(ε)</td>
<td>0.85</td>
</tr>
<tr>
<td>$\mu''$</td>
<td>(ε/2m)</td>
<td>1.24</td>
</tr>
</tbody>
</table>

Fig. 18. The iso-scalar charge form factor modified by including both the
two-pion and three-pion resonance factors.

Fig. 19. A comparison of the calculated charge form factor of the proton with
experiments under the assumption that $F_1^P$ is well described by (3.11). Both
the two-pion and three-pion resonance factors are included.
From comparison with experiment it is seen that a considerable improvement has been achieved by including the effects of the three-pion resonance in addition to those of the two-pion resonance. We get a satisfactory value for $\langle r^2 \rangle_1$. The anomalous magnetic moment is still large in magnitude but this is not so serious since the three-pion state might not dominate the anomalous magnetic moment.

The use of the unsubtracted dispersion relation yields the result $\langle r^2 \rangle_1 \simeq 0.17 \mu^{-2}$ which is to be compared with the value $\langle r^2 \rangle_1 \approx 0.07 \mu^{-2}$ obtained by including only two-pion correlations.

iii) $F_1'(q^2)$: Finally the calculated form factor $F_1'(q^2)$ of the charge distribution is shown in Fig. 18. A rather good agreement with experiment is obtained for the low momentum transfer region $q^2 \lesssim 15 \mu^2$ (Fig. 19).

§ 4. Concluding remarks

In the previous sections we have investigated in detail the effects of the two-pion and three-pion resonances on the iso-scalar form factors. The results obtained may be summarized as follows.

(A) The perturbation calculation with the extended effective $\gamma-3\pi$ interaction (3.6) shows that $\text{Im} F_1^{\text{pert}}(-\sigma^2)$ and $\text{Im} F_2^{\text{pert}}(-\sigma^2)$ are both positive definite and the latter is larger than the former. This comes from the fact that the $S$-wave part $\text{Im} F_1^s$ is always positive and dominates over the $D$-wave part and $\text{Im} F_2^s$ is destructive to $\text{Im} F_1^s$ while $\text{Im} F_2^s$ is constructive to $\text{Im} F_2^s$. These properties lead to the relation

$$F_1'(q^2) \text{ (in unit of } \frac{e}{2m}) > F_2'(q^2) \text{ (in unit of } e) > 0 \quad (4.1)$$

In particular we obtain

$$\mu^2 \text{ (in unit of } \frac{e}{2m}) > Q^2 \text{ (in unit of } e) > 0 \quad (4.2)$$

The relations (4.1) and (4.2) are insensitive to the enhancement factors and remain essentially unaltered even after the resonance effects are taken into account. As is seen from the exact expressions (3.1a) \sim (3.1d) these results may be considered as general features of the three-pion state contribution to the iso-scalar form factors.

As for the mean-square radius $\langle r^2 \rangle_1$ of the charge distribution the perturbation theory is quite insufficient to get an agreement with experiment. The magnitude of the anomalous magnetic moment and its wrong sign are not to be taken seriously since the three-pion state will not be a main contributor to the anomalous magnetic moment. The role of more massive states would be important.

(B) The inclusion of the two-pion correlations into the iso-scalar form factors
greatly modifies the results of perturbation calculations. The absorptive parts $\text{Im} F_{1}^{\frac{3}{2}}(\sigma^{2})$ have broad maxima around $\sigma^{2} \approx 80 \mu^{2}$.

The calculated charge radius turns out to be very large, i.e. four times as large as that of experiment. These conclusions are different from those of Blankenbecler and Tarski. Their results have shown that the two-pion resonance at about $20 \mu^{2}$ cannot yield a large charge radius. The difference comes from the different approximations made on the invariant amplitudes.

Instead of calculating the charge radius, the normalization may be chosen so as to fit the observed radius. The iso-scalar charge form factor thus determined is shown to behave like a pole approximation with a mass much larger than $3/2$.

(C) The three-pion resonance, on the other hand, has the effect which reduces the over-enhancement arising from the two-pion correlations to a close agreement with experiment, at least, as to the charge radius. ($\sqrt{\langle r^{2} \rangle_{1}} \approx 0.75 \times 10^{-13} \text{cm}$ is obtained.)

The shapes of the absorptive parts change into resonance-like ones with sharp maxima at about $42 \mu^{2}$. The calculated charge form factor of the proton agrees with experiment in the low momentum transfer region ($q^{2} \leq 15 \mu^{2}$). These features are very satisfactory, though the observed small negative value of the anomalous magnetic moment remains unexplained.

So far we have not attempted to discuss the magnitude of the mean-square radius of the magnetic moment distribution $\langle r^{2} \rangle_{H}$ but only calculated the value of $\mu(\langle r^{2} \rangle_{H})$, since experimental information on $F_{2}^{H}$ are still uncertain at present.

Recently, however, Hofstadter and Herman proposed a set of phenomenological formulas of the charge and magnetic moment form factors which give a good fit to the recent data obtained by Hofstadter et al. From their formulas we find $\langle r^{2} \rangle_{H} \approx -2.0 \mu^{2}$.

On the other hand, the calculated mean-square radius of the magnetic moment distribution $\langle r^{2} \rangle_{H}$, normalized with the observed value of the anomalous magnetic moment $\mu^{2}$, turns out to be $-4.5 \mu^{2}$ (See Table III). In view of a preliminary nature of their formulas, an agreement of both its sign and the order of magnitude with the experimental value may be considered to suggest that the mean-square radius of the magnetic moment distribution $\langle r^{2} \rangle_{H}$ can also be explained satisfactorily by the three-pion state contribution if pion correlations are included.

After these analyses our feeling is that the long-standing expectation that the three-pion state may provide the observed large radius of the charge distribution will be right when the two-pion and three-pion resonances are taken into account, though further investigations must be made before a definite conclusion can be drawn. The small negative value of the anomalous magnetic moment will not be explained simultaneously with the large magnitude of the charge radius by the three-pion state contribution alone.
We have discussed in this paper the contribution from the unphysical region of pair annihilation \( \sigma^2 < 4m^2 \). The contribution from the physical region \( \sigma^2 > 4m^2 \) is in fact very large in the perturbation theory on account of a rapid increase of the phase volume integral which, however, probably violates the unitarity condition. The inclusion of pion correlations must reduce the contribution from the region \( \sigma^2 > 4m^2 \) to a minor one, a point which will be discussed in a separate paper.

Finally we make two remarks about the pion correlation factors. It is an interesting point of the solution (2.24) that because of its product form the \( P \)-wave resonance in a particular two-pion channel is reflected in another two-pion channel as the enhancement of other waves than \( P \)-state. Such correlations are not unusual in the ordinary dispersion relation approaches but in the present case it must be further justified because all pairs of pions are now possible simultaneously to exist in the physical regions of their own channels. In this paper it is assumed that the analytic continuation in the squared virtual photon mass \(-q^2\) is allowed.

As stated in §2, we have normalized them at the symmetry point \( s_i = \mu^2 \) \((i = 1, 2, 3)\) and \( q^2 = 0 \). An alternative condition for fixing the normalization is

\[
\lim_{s \to \infty} \exp U(s) \to 1.
\]

This means that the enhancement by the final state interaction should vanish as the energy goes to infinity. This condition, though physically natural, is not compatible with the approximate solution (2.26), which suggests us that the solution (2.26) is adequate only in the low energy region

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**Appendix A**

**Simple models of the three-pion scattering resonance and final state interaction**

Let us first discuss the case where the \( T = 0 \) three-pion resonance is induced by an unstable vector meson \( \omega \). Since the \( \omega \)-meson has the same transformation property as the iso-scalar photon, we can take its decay interaction as

\[
\langle 0| J_\mu^w | k^+, k^-, k^0 \text{ in} \rangle = -i \left( \frac{1}{g_{\omega \omega \omega \omega}} \right)^{1/2} \varepsilon_{\mu \nu \lambda \sigma} k^\nu k^\lambda k^\sigma k^0.
\]
where we have extracted the final state pion-pion correlation factors. $g_\omega$ is the effective decay coupling constant.

We calculate the three-pion scattering matrix element from diagrams shown in Fig. A.1.

\[
\langle k', k^-, k^0 \text{ out} | J^0 | k^+, k^- \text{ in} \rangle = -\frac{1}{8\omega_+\omega_-\omega_0} \varepsilon_{\mu\nu\lambda\sigma} k_{\mu'}' k_{\nu'}' k_{\lambda'}' k_{\sigma'}' \times \exp \{ U(s_1') + U(s_2') + U(s_3') \} g_\omega^2 D_{\pi^\nu} (q^2) \\
\times \exp \{ U(s_1) + U(s_2) + U(s_3) \} \varepsilon_{\mu\nu\lambda\sigma} k^+ k^- k^0 \left( \frac{1}{4\omega_+\omega_-} \right)^{1/2}, \tag{A.2}
\]

where $q^2 = (k^+ + k^- + k^0)^2$ and the $\omega$-meson propagator $D_{\pi^\nu} (q^2)$ is defined by

\[
D_{\pi^\nu} (q^2) = \frac{1}{q^2 + m_\omega^2 - \frac{1}{\pi} \int_{(2\pi)^3} d^m \varepsilon^\mu \varepsilon^\nu \varepsilon^\lambda \varepsilon^\sigma \sum \frac{(-m_\omega^2 g_\omega^2)}{m^2 + q^2 - i\epsilon}, \tag{A.3}
\]

\[
\text{Im} \sum (q^2, g_\omega^2) = \frac{\pi}{3} \int_{(2\pi)^3} d^3 k^+ d^3 k^- d^3 k^0 \frac{|\langle 0 | j_{\pi^\nu}^\omega | k^+, k^-, k^0 \rangle|^2}{(2\pi)^3} \times \delta(k^+ + k^- + k^0 - q), \\
= \frac{\pi}{3(2\pi)^6} (-q^2) \int_{(2\pi)^3} d^3 k^+ d^3 k^- d^3 k^0 \left( \frac{k^+ \times k^-}{8\omega_+\omega_-\omega_0} \right) g_\omega^2 \times |\exp \{ U(s_1) + U(s_2) + U(s_3) \} |^2 \delta(k^+ + k^- + k^0 - q). \tag{A.4}
\]

In the same approximation as for the three-pion scattering resonance (See Fig. A.2), the final state interaction for electro-production of three pions is written as follows:

\[
\langle 0 | j_{\pi^\nu}^\omega | k^+, k^-, k^0 \text{ in} \rangle = -i \left( \frac{1}{8\omega_+\omega_-\omega_0} \right)^{1/2} \varepsilon_{\mu\nu\lambda\sigma} k^+ k^- k^0 \times g_\omega (q^2 + m_\omega^2) D_{\pi^\nu} (q^2) \exp \{ U(s_1) + U(s_2) + U(s_3) \}, \tag{A.5}
\]
where \( g_\gamma \) is the effective coupling constant of the \( \gamma - 3\pi \) interaction. Thus the function \( \tilde{H} \) has the structure given by

\[
\tilde{H} = g_\gamma \frac{q^2 + m_\omega^2}{q^2 + m_\omega^2 - \frac{1}{\pi} \int dm^2 \text{Im} \frac{\sum (-m^2, g_\omega^2)}{m^2 + q^2 - i\varepsilon} .
\]  

(A·6)

\( \tilde{H} \) has a vanishing point at \( q^2 = -m_\omega^2 \). This is characteristic of the unstable meson model for the final state interaction.

We next consider a model in which the three-pion resonance occurs through a chain mechanism. Feynman diagrams for the three-pion scattering process and electro-production of three pions are shown in Figs. A.3 and A.4 respectively.

Fig. A. 3. Chain model for the three-pion scattering process.

Fig. A. 4. Chain model for the final state three-pion interaction.

The matrix elements corresponding to Figs. A.3 and A.4 are obtained at once if only we take the limit of infinite \( \omega \)-meson mass in (A·2) and (A·5) and replace \( g_\omega^2/m_\omega^2 \) by a new coupling constant \( \lambda \). We then find

\[
\langle k^+, k^-, k^0 \text{ out} | J^0 | k^+ , k^- \text{ in} \rangle = - \left( \frac{1}{32 \omega \omega' \omega_0 \omega_0' \omega - \omega_-} \right)^{1/2} \lambda
\]

\[
\times \epsilon_{\mu\nu\lambda\sigma} k_0^{\mu+} k_0^{\nu+} k_0^{\lambda+} k_0^{\sigma+} \epsilon_{\mu\nu\lambda\sigma} \ k_0^{\mu-} k_0^{\nu-} k_0^{\lambda-} k_0^{\sigma-}
\]

\[
\times \exp \{ U(s'_1) + U(s'_2) + U(s'_3) \} \cdot \exp \{ U(s_1) + U(s_2) + U(s_3) \} D(q^2) \]

\[
A\cdot7\]

where

\[
D(q^2) = 1 - \frac{1}{\pi} \int dm^2 \text{Im} \frac{\sum (-m^2, \lambda)}{m^2 + q^2 - i\varepsilon} .
\]  

(A·9)

If we assume that the function \( Re D(q^2) \) develops a zero at \( q^2 = -m_\omega^2 \), the formula (A·7) shows a resonance behaviour around \( q^2 = -m_\omega^2 \). In this case we can rewrite (A·9) as

\[
D(q^2) = (q^2 + m_\omega^2) \int dm^2 \text{Im} \frac{\sum (-m^2, \lambda)}{(m^2 + q^2)(m^2 - m_\omega^2)} - i\theta (-q^2 - 9\mu^2) \text{Im} \sum (q^2, \lambda)
\]

\[
A\cdot8\]

\[
\approx (q^2 + m_\omega^2) \int dm^2 \text{Im} \frac{\sum (-m^2, \lambda)}{(m^2 - m_\omega^2)^2} - i\theta (-q^2 - 9\mu^2) \text{Im} \sum (q^2, \lambda) .
\]  

(A·10)
(A·10) obviously shows that for the scattering process the chain approximation model is substantially equivalent to the unstable particle model (A·2)

\[
\left( \lambda \int_{(m_{\omega})^2}^{\infty} dm^2 \frac{\text{Im} \sum (-m^2, \lambda)}{(m^2 - m_{\omega}^2)^2} \right)^{-1}\text{corresponds to } g_{\omega}^2.
\]

On the other hand it is very interesting to note that the function \( \widetilde{H} \) given by (A·8) does not vanish at \( q^2 = -m_{\omega}^2 \). This is an example of the remarkable difference for final state interactions between the dynamical resonance model and the unstable particle model.\(^{27}\)

**Appendix B**

**Perturbation calculations of \( \widetilde{H} \), \( \bar{\alpha} \) and \( \bar{\beta}^8 \)**

i) \( \widetilde{H} \): Explicit calculations are necessary only for one diagram of Fig. 8. Contributions from other diagrams are inspected by the consideration of symmetry. With \( j^a_\mu = ie \varphi \gamma^a_\mu /2 \), we find for Fig. 8.a

\[
\langle 0 | j^a_\mu | k^+, k^-, k^0 \text{ out} \rangle (8a) = -8eg^3 m \varepsilon_{\mu \nu \lambda \sigma} k^\nu k^\lambda k^\sigma \text{Tr}[\bar{\tau}_+ \tau_- \tau_0] (8\omega_+ \omega_- \omega_0)^{-1/4} (2\pi)^{-4}
\]

\[
\times \int d^4p \text{Tr} \left[ \gamma^a_\mu \gamma^a_\nu \gamma^a_\lambda \gamma^a_\sigma \frac{1}{(p+q/2)+m} \frac{1}{(p+q/2-k^+) + m} \right] (B·1)
\]

\[
= -8eg^3 m \varepsilon_{\mu \nu \lambda \sigma} k^\nu k^\lambda k^\sigma \text{Tr}[\bar{\tau}_+ \tau_- \tau_0] (8\omega_+ \omega_- \omega_0)^{-1/4} (2\pi)^{-4}
\]

\[
\times \int d^4p \frac{1}{(p+q/2)^2 + m^2} \frac{1}{(p+q/2-k^+)^2 + m^2} \frac{1}{(p-q/2+k^-)^2 + m^2}
\]

\[
\times \frac{1}{(p-q/2)^2 + m^2}. \quad (B·2)
\]

The integration in (B·2) is easily performed by Feynman parametrization and we have

\[
\langle 0 | j^a_\mu | k^+, k^-, k^0 \text{ out} \rangle (8a) = -8eg^3 m \varepsilon_{\mu \nu \lambda \sigma} k^\nu k^\lambda k^\sigma \text{Tr}[\bar{\tau}_+ \tau_- \tau_0]
\]

\[
\times (8\omega_+ \omega_- \omega_0)^{-1/4} (2\pi)^{-4} \frac{i\pi^3}{6m^8} I(q^2; k^+, k^-), \quad (B·3)
\]

where the function \( I(q^2; k^+, k^-) \), which is symmetric for \( k^+ \) and \( k^- \), is written as

\[
I(q^2; k^+, k^-) = 6 \int_0^1 dx \int_0^1 dy \int_0^1 dz (1-z) dz
\]

\[
\times \left[ 1 - \left\{ xz - x^2 z^2 + y(1-z) - y^2 (1-z)^2 \right\} \frac{\mu^2}{m^2} \right]
\]

\(^{27}\) The same calculations were performed by M. Kato.\(^{11}\)
Electromagnetic Structure of the Nucleon

Noticing that Tr[τ⁺τ⁻τ⁻]εₐ⁺⁺⁺k⁺k⁻k₀ is invariant for the exchange between any two of +, −, and 0, the matrix elements for Figs. 8b and 8c are immediately obtained from (B·3) by the following replacements:

\[ I(q⁺; k⁺, k⁻) \rightarrow I(q⁺; k⁺₀, k⁻) \text{ for Fig. 8b,} \]
\[ \rightarrow I(q⁺; k⁺, k₀) \text{ for Fig. 8c.} \]

Comparison of (B·3) with (2·4) yields the amplitude \( \vec{H} \) as

\[ \vec{H} = \frac{eg²}{12π²m³} \{ I(q⁺; k⁺, k⁻) + I(q⁺; k⁺₀, k⁻) + I(q⁺; k⁺, k₀) \}. \]  

\[ \text{(B·5)} \]

ii) \( \vec{α} \) and \( \vec{β} \): We first calculate the matrix element for Fig. 9.a. With \( f = -iγ⁺σ⁺ϕ⁺µ⁺ \) we find

\[ \bar{v}(\vec{p}) \langle k⁺, k⁻, k₀ \text{ out} | f | \vec{p} \rangle (9a) = ig³ \left( \frac{m}{8ω⁺ω⁻ω₀E} \right)^{1/2} \]
\[ \times \bar{v}(\vec{p}) τ⁺γ⁺ \frac{1}{-iγ⁺(\vec{p} − k⁺) + m} \tau⁺γ⁺ \cdot (p − k⁻) + m \tau⁻γ⁻(p). \]  

\[ \text{(B·6)} \]

The factor \( τ⁺τ⁻τ⁻ \) can be replaced with \( \text{Tr}[τ⁺τ⁻τ⁻]/2 \) since we are interested in the \( T=0 \) state. We rewrite (B·6) in terms of 2-component Pauli spinors and retain only those terms which contribute to the \( ^3S_1 \) and \( ^3D_1 \) nucleon pair states. Then we find

\[ \bar{v}(\vec{p}) \langle k⁺, k⁻, k₀ \text{ out} | f | \vec{p} \rangle (9a) = g³ \left( \frac{1}{8ω⁺ω⁻ω₀E} \right)^{1/2} \frac{1}{2} \text{Tr}[τ⁺τ⁻τ⁻] \]
\[ \times \chi⁺* \left[ \sigma \cdot k⁺ \times k⁻ + \frac{E − m}{m} \sigma \cdot \hat{p} \hat{p} \cdot k⁺ \times k⁻ \right] \chi⁺ \]
\[ \times \frac{1}{2\vec{p} \cdot k⁻ + μ²} \frac{1}{2\vec{p} \cdot k⁺ + μ²}. \]  

\[ \text{(B·7)} \]

We compare (B·7) with (2·7) and project out the \( ^3S_1 \) and \( ^3D_1 \) amplitudes as follows:

\[ \bar{α}(9a) = -\frac{g³}{4π} \int dΩₚ \left( 1 + \frac{E − m}{m} \hat{p}₂² \right) F(p; k⁺, k⁻), \]
\[ \bar{β}(9a) = -\frac{g³}{4π} \int dΩₚ \left( \frac{3*p₂²}{2} + \frac{E − m}{m} \hat{p}₂² \right) F(p; k⁺, k⁻), \]

\[ \text{(B·8a)} \]

where \( \hat{p} = p/|p| \) and the \( z \)-axis is taken parallel to \( k⁺ \times k⁻ \). \( F(p; k⁺, k⁻) \) is defined by

\[ F(p; k⁺, k⁻) = \frac{1}{-2p \cdot k⁻ − 2Eω⁻ + μ²} \frac{1}{2p \cdot k⁺ − 2Eω⁺ + μ²}. \]  

\[ \text{(B·9)} \]
Note that the expressions (B·8a) and (B·8b) are symmetric for the replacement $k^+ \rightarrow k^-$. Contributions to $\tilde{\alpha}$ and $\tilde{\beta}$ from other diagrams are obtained simply by replacing arguments of the function $F(p; k^+, k^-)$. Thus the full expressions for $\tilde{\alpha}$ and $\tilde{\beta}$ are

\[
\tilde{\alpha} = \frac{-2g^2}{4\pi} \int d\Omega_p \left( 1 + \frac{\varepsilon}{m} \hat{\mathbf{p}}_p \right) \{ F(p; k^+, k^-) - F(p; k^0, k^-) + F(p; k^+, k^0) \},
\]

\[
\tilde{\beta} = \frac{-1}{\sqrt{2}} \frac{g^2}{4\pi} \int d\Omega_p \left( \frac{3\hat{s}_p^2 - 1}{2} + \frac{\varepsilon}{m} \hat{\mathbf{p}}_p \right) \{ F(p; k^+, k^-) - F(p; k^0, k^-) + F(p; k^+, k^0) \}.
\]

A part of angular integrations is carried out explicitly, and we find

\[
\tilde{\alpha} = -g^2 \int d\mathbf{z} \left( 1 + \frac{\varepsilon}{m} \mathbf{z} \right) \{ \hat{F}(p; k^+, k^-) - \hat{F}(p; k^0, k^-) + \hat{F}(p; k^+, k^0) \},
\]

\[
\tilde{\beta} = -\frac{g^2}{\sqrt{2}} \int d\mathbf{z} \left( \frac{3\mathbf{z}^2 - 1}{2} + \frac{\varepsilon}{m} \mathbf{z} \right) \{ \hat{F}(p; k^+, k^-) - \hat{F}(p; k^0, k^-) + \hat{F}(p; k^+, k^0) \},
\]

\[
\hat{F}(p; k^+, k^-) = \left[ \frac{2E_{w_+} - \mu^a}{(2E_{w_+} - \mu^a)^2 + (1 - \mathbf{z}^2) 4\mathbf{p}^2 (k^-)^2} \right]^{1/2} \left[ \frac{2E_{w_-} - \mu^a}{(2E_{w_-} - \mu^a)^2 + (1 - \mathbf{z}^2) 4\mathbf{p}^2 (k^+)^2} \right]^{1/2} \left[ \frac{(2E_{w_+} - \mu^a)(2E_{w_-} - \mu^a)}{(2E_{w_+} - \mu^a)^2 + (1 - \mathbf{z}^2) 4\mathbf{p}^2 (k^+)^2} \right]^{1/2} \left[ \frac{(2E_{w_-} - \mu^a)}{(2E_{w_-} - \mu^a)^2 + (1 - \mathbf{z}^2) 4\mathbf{p}^2 (k^-)^2} \right]^{1/2} \left[ \frac{2E_{w_+} - \mu^a}{(2E_{w_+} - \mu^a)^2 + (1 - \mathbf{z}^2) 4\mathbf{p}^2 (k^-)^2} \right]^{-1/2} - \cos \phi(1 - \mathbf{z}^2) 4\mathbf{p}^2 k^+ k^- \right],
\]

Since we are considering in the unphysical region ($E < m$) of nucleon pair annihilation, we have defined (B·12) as the analytic continuation from the physical region ($E > m$).

iii) Approximate expressions for $\tilde{H}$, $\tilde{\alpha}$ and $\tilde{\beta}$: In the numerical discussions in § 3 we have introduced the following approximations for the invariant amplitudes:

\[
\tilde{H}(q^2; s_1, s_2, s_3) \approx \tilde{H}(q^2; \bar{s}, \bar{s}, \bar{s}),
\]

\[
\tilde{\alpha}(q^2; s_1, s_2, s_3) \approx \tilde{\alpha}(q^2; \bar{s}, \bar{s}, \bar{s}),
\]

\[
\tilde{\beta}(q^2; s_1, s_2, s_3) \approx \tilde{\beta}(q^2; \bar{s}, \bar{s}, \bar{s}),
\]
where \( \bar{s} = \mu^2 - q^2/3 \), i.e. \( \omega_+ = \omega_- = \omega_0 = -q^2/9 \). These approximate functions are written explicitly as

\[
\bar{H}(q^2; \bar{s}) = \frac{g^2}{4\pi^2 m^2} I(q^2), \quad (B\cdot14)
\]

\[
I(q^2) = \frac{1}{0} dx \frac{1}{0} dy \frac{1}{0} dz [1 + \{xyz(1-z) - xz
+ x^2 z^2 - y(1-y) + y^3(1-z)^3 \} \frac{\mu^2}{m^2} + \frac{1}{3} z(1-z)
\times \{(2-x)(2-y) - 1\} \frac{q^2}{m^2}]^2, \quad (B\cdot15)
\]

\[
\bar{\alpha}(q^2; \bar{s}) = -\frac{g^2}{4\pi} \frac{16\pi (2E\omega - \mu^2)}{(m^2 - E^2) (\omega^2 - \mu^2)} \left( \mathcal{J}_4 + \frac{E-m}{m} \mathcal{J}_5 \right), \quad (B\cdot16a)
\]

\[
\frac{1}{\sqrt{2}} \bar{\beta}(q^2; \bar{s}) = -\frac{g^2}{4\pi} \frac{8\pi (2E\omega - \mu^2)}{(m^2 - E^2) (\omega^2 - \mu^2)} \left( \mathcal{J}_4 - \frac{2E+m}{m} \mathcal{J}_5 \right), \quad (B\cdot16b)
\]

\[
\mathcal{J}_4 = \frac{1}{3\sqrt{1-4t^2}} \frac{1}{\sqrt{r-t}} \left[ \sin^{-1} \left( \frac{s}{r} + \frac{t}{r(1+s)} \right) \right.
\left. - \sin^{-1} \left( \frac{s}{r} - \frac{t}{r(1-s)} \right) \right], \quad (B\cdot17)
\]

\[
\mathcal{J}_5 = -\frac{2}{3} \sin^{-1} r + s^2 \mathcal{J}_4, \quad (B\cdot18)
\]

\[
t = -\frac{(2E\omega - \mu^2)^2}{3(m^2 - E^2)(\omega^2 - \mu^2)}, \quad s = \sqrt{1-4t}, \quad r = \sqrt{1-3t}, \quad (B\cdot19)
\]

where \( \omega = \frac{2}{3} E \).

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