Numerical simulation of thick disc accretion on to a rotating black hole

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Accepted 1996 September 20. Received 1996 September 9; in original form 1996 March 27

ABSTRACT

We study two-dimensional axisymmetric accretion on to a rotating black hole by time-dependent fully relativistic hydrodynamical calculations. We investigate the innermost part of geometrically thick discs, in which most of the gravitationally released energy is advected into the black hole. Relativistic effects are important close to the cusp-like inner edge of the disc, where accretion proceeds with approximately constant angular momentum and the viscous time-scale exceeds the dynamical time-scale of accretion. We use the perfect fluid approximation and calculate isentropic flows with constant angular momentum. We confirm the previous analytical result that the structure of the innermost disc region strongly depends on the black hole spin. We find the mass accretion rate $\dot{M}$ to be in a good agreement with the analytical dependence $\dot{M} \propto (\Delta W)^{1/(\Gamma - 1)}$, where $\Delta W$ is the energy gap and $\Gamma$ is the adiabatic index.

Key words: accretion, accretion discs – black hole physics – hydrodynamics – relativity – methods: numerical.

1 INTRODUCTION

In the standard $z$-model of a thin accretion disc (Shakura & Sunyaev 1973), the binding energy of the accreting gas is radiated away while it is gradually spiralling towards the black hole along nearly equatorial Keplerian orbits. The thin disc model fails when the accretion time-scale becomes comparable to the time-scale of radiative cooling. In this case the released energy remains in the gas being advected to the black hole, the internal pressure grows, and the disc swells. Thick accretion discs have been investigated in various approximations, both analytically and numerically (see review by Rees 1984 and comments by Abramowicz & Lasota 1995). The accretion picture in thick discs is crucially different from the standard thin disc model. Gas flows in an effective (gravitational plus centrifugal) potential forming a cusp-like inner edge of the disc (Kozlowski, Jaroszyński \& Abramowicz 1978). The potential has a barrier at $r_{\text{cusp}} \lesssim 2-3 r_g$, where $r_g = 2GMc^{-2}$ is the gravitational (Schwarzschild) radius of the black hole of mass $M$. The radial pressure gradient drives the gas over the barrier just as it does at the Lagrange $L_1$ point in a close binary. The sonic radius $r_s$ is located inside $r_{\text{cusp}}$, where gas is almost in free fall.

Two-dimensional simulations of geometrically thick accretion flows on to a black hole have been done by several authors. Some of them used Newtonian or pseudo-Newtonian approximations to the black hole gravitational field. Eggum, Coroniti \& Katz (1988) calculated super-Eddington accretion with outflow formation. A stationary transonic adiabatic disc was constructed by Papaloizou \& Szuszkiewicz (1994). Igumenshchev, Chen \& Abramowicz (1996) investigated the role of viscous effects and the convective instability process.

The fully relativistic approach was started by Hawley, Smarr \& Wilson (1984a,b), who developed a relativistic hydrodynamical code in Kerr geometry, and studied the formation of a pressure-supported inviscid torus around a non-rotating black hole. Hawley (1991) investigated the global non-axisymmetric stability of accretion tori orbiting a black hole by two- and three-dimensional simulations. Recently, Yokosawa (1995) has followed the dynamical evolution of both thin and thick $z$-discs using a general relativistic hydrodynamical code.

In the present paper we calculate two-dimensional accretion flows around a rotating black hole in a fully relativistic approach. Like the simulations of Hawley et al. (1984b), in
our models the flow is isentropic and proceeds with constant angular momentum, so viscous processes are neglected. Contrary to the previous investigations, in which the authors constructed the thick accretion disc models in a Schwarzschild geometry (non-rotating black hole), we study influence of the black hole rotation on the disc structure in the innermost region, where this influence is most significant and inviscid approximation is likely to be adequate. We take a non-accreting stationary rotating torus with constant angular momentum (Kozlowski et al. 1978) as our initial condition, and accretion develops due to the instability of such a torus in the case of a positive energy gap (see Section 2.2). This initial condition is different from that of Hawley et al. (1984b), in which gas is initially far from the black hole and starts to fall towards the hole.

All our calculations are performed with an adiabatic index \( \Gamma = \frac{4}{3} \), since we are mainly interested in radiation-supported discs. Such a regime of accretion can take place when the accretion rate exceeds \( L_{\rm E} c^{-2} \), \( L_{\rm E} \) being the Eddington limit. The temperature of the flow is low (non-relativistic), and the released energy is converted to the radiation which is trapped by the optically thick flow and advected to the black hole (e.g. Rees 1984); results may also be relevant to ion-supported discs (Rees et al. 1982), since the accretion pattern weakly depends on the adiabatic index in our models.

Basic equations of the problem and the stationary torus solution are given in Section 2. In Section 3 we briefly describe the numerical method. The results are presented in Section 4 and discussed in Section 5.

2 BASIC EQUATIONS

2.1 Hydrodynamical equations in Kerr geometry

We assume that the gas self-gravity is negligible. Then the space–time metric is determined by two parameters only: the black hole mass \( M \) and its specific angular momentum, \(-M \leq a \leq M\) (we hereafter use units \( c=G=1 \)). The flow parameters are comoving mass density \( \rho \), pressure \( P \) and four-velocity \( u^\mu \). Then the specific internal energy, \( e \), can be expressed in terms of \( P \) and \( \rho \) from the perfect gas equation of state, \( P = (\Gamma - 1) \rho e \).

The dynamical equations of the accreting matter can be derived from the conservation laws for the rest mass and the stress energy (Misner, Thorne & Wheeler 1973),

\[
\nabla_u (\rho u^\mu) = 0 ,
\]

\[
\nabla_u T^\mu_\nu = 0 ,
\]

where the stress-energy tensor of a perfect fluid is given by

\[
T^\mu_\nu = (\rho + p) u^\mu u_\nu + P \delta^\mu_\nu ,
\]

and \( \nabla_u (\ldots) \) signifies a covariant derivative. Multiplying equation (2.2) by \( u^\nu \) and combining the result and equation (2.1) we obtain the following equation for internal energy:

\[
\nabla_u (\rho u^\mu) + \nabla \cdot (P u^\mu) = 0 .
\]

To study numerically the equations (2.1), (2.2), and (2.4), we use the Boyer–Lindquist coordinate system \((t, r, \theta, \phi)\) and the Kerr solution for a metric tensor \( g_{\mu \nu} \) given e.g. in Misner et al. (1973),

\[
ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 .
\]

We assume the signature \((- + + +)\), so that the ‘lapse function’ of the zero angular momentum observer (ZAMO) is given by \( \alpha = (1 - g^{\theta \theta})^{-1} (\text{Bardeen, Press & Teukolsky 1973}) \).

Then one arrives at the continuity equation

\[
\frac{\partial}{\partial t} (\rho \alpha) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{g} \gamma \rho \alpha u^\mu \right) = 0 ,
\]

the equations of motion

\[
\frac{\partial}{\partial t} S^\mu + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{g} S_{\nu} u^\nu \right) + \frac{\partial P}{\partial t} + \frac{1}{2 S} \frac{\partial}{\partial x^\mu} \left( \sqrt{g} \gamma u^\mu \right) = 0 ,
\]

and the internal energy equation,

\[
\frac{\partial}{\partial t} (\rho e) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{g} \gamma \rho e u^\mu \right) + \frac{\partial P}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{g} \gamma u^\mu \right) = 0 .
\]

Here \( \gamma \) denotes the spatial coordinates, \( v^\mu = u^\mu / u^t \) is the coordinate three-velocity, \( \gamma = 2 u^t \) is the gas Lorentz factor measured by ZAMO, \( \gamma = \sqrt{g} \) is the determinant of the three-metric tensor \( g_{\mu \nu} \), and \( S^\mu = (\rho + p + \rho) \gamma v^\mu \). \( S^\mu \) and \( S^\nu \) coincide with \( r \)- and \( \theta \)-components of the momentum density vector in ZAMO frame. In a stationary flow, \( \varepsilon = - u_j / u^t \) is a constant of motion (does not change along a flow line). It has the same meaning as the specific angular momentum (Kozlowski et al. 1978).

2.2 Stationary thick disc solution

To set initial and boundary conditions in our time-dependent calculations, we use a stationary rotating torus with \( u^\mu = u^\mu = 0 \). Non-zero four-velocity components \( u^t \) and \( u^\theta \) in such a torus meet the equation (Abramowicz, Jaroszynski & Sikora 1978)

\[
\nabla P \rho \rho + P = \nabla \cdot \left( \nabla \rho + \frac{1}{e} \rho \varepsilon \right) ,
\]

where \( W_0 \) is the potential at the boundary of the torus (where \( P = 0 \)). We will consider tori with constant \( \varepsilon \). Then the potential can be written simply as \( W = W_0 + W_0 \). In the case \( \varepsilon > \varepsilon_{\text{ms}} \), where \( \varepsilon_{\text{ms}} \) is the angular momentum of the marginally stable Keplerian orbit, the dependence \( W(r, \theta) \) describes a surface with saddle point \( W_{\text{cusp}} \) at \( r = r_{\text{cusp}} \), \( \theta = \pi/2 \). If \( \Delta W = W_0 - W_{\text{cusp}} < 0 \), the rotating torus is separated from the black hole by a potential barrier of height \( \Delta W \). If \( \Delta W > 0 \), the solution (2.9) is unstable, and any perturbations lead to accretion. Note that the isentropic non-accreting
torus with $\ell=\text{constant}$ is marginally stable with respect to local axisymmetric adiabatic perturbations (Seguin 1975).

The tori were found to be unstable to a variety of non-axisymmetric global modes (Papaloizou–Pringle instability, Papaloizou & Pringle 1984). However, according to two- and three-dimensional time-dependent numerical simulations, this instability does not disrupt the thick disc, and the saturation of unstable modes occurs at low amplitude (see Narayan & Goodman 1989 for a review). Moreover, even a modest mass accretion rate suppresses the instability (Hawley 1991).

3 NUMERICAL TECHNIQUE

3.1 The code

We use a time-explicit Eulerian method to solve the time-dependent equations (2.5)–(2.7) in axial symmetry. The details of the method can be found in the papers of Hawley et al. (1984b) and Stone & Norman (1992). The numerical procedure uses operator splitting, which divides the solution into several steps, and the contribution of a single term in the equations is calculated on each step. The source terms in the equations operate in the usual manner. The advection terms are evaluated with the help of the upwind algorithm, where the piecewise parabolic interpolation procedure originally described by Colella & Woodward (1984) is used to compute the fluxes. No artificial viscosity is used in the present calculations.

The code has been tested on two problems which have analytical solutions: spherical Bondi accretion in a Schwarzschild metric and a stationary torus in a Kerr metric (cf. Hawley et al. 1984a). In addition, we tested the code on the problem of relativistic Bondi–Hoyle accretion with shocks. The results obtained were similar to those of Petrich et al. (1989).

The sonic surface is defined as

$$r_{\text{sonic}} = \frac{\rho c^2}{\rho c^2 + \rho_{\text{in}} v^2} = h_{\text{Lorentz}}.$$ 

We use reflection boundary conditions at $\theta=0$ and $\theta=\pi/2$. At the inner boundary the gas is supposed to be in free fall into the black hole. As the outer boundary condition at $r_{\text{out}}$, we take the stationary thick disc solution (see Section 2.2). This condition allows both outflow and inflow through the boundary. A more complete discussion of such boundary conditions is given by Igumenshchev et al. (1996). We assumed $r_{\text{out}} = 20r_g$; test calculations showed that the accretion flow in the inner ($r \lesssim 10r_g$) part of the disc weakly depends on the choice of $r_{\text{out}} > 20r_g$. The number of grid points varies from $n_r \times n_\theta = 50 \times 128$ to $50 \times 150$, depending on the black hole spin parameter $a$ in each model.

4 NUMERICAL RESULTS

We calculated three groups of models with adiabatic index $\Gamma = 4/3$, black hole spin $a/M=0, 0.9, -0.9$, and three different values of gas angular momentum $\ell$. Each model started from an initially non-accreting stationary torus with energy gap $\Delta W = W_0 - W_{\text{inflow}}$ at $r_{\text{inflow}}$ (see Section 2.2). In each group we took four values of $\Delta W$. Parameters of the models are listed in Table 1, where $a$ and $\ell$ are given in $GMc^{-1}$ units, and $W_{\text{inflow}}$ and $\Delta W$ in $c^2$ units. After the beginning of accretion, the models were followed until a quasi-stationary flow was established, which usually required time of order $10^2 r_g^{-1} c^{-1}$, and several models were calculated up to $\sim 10^3 r_g^{-1} c^{-1}$.

The final flow pattern is variable in all the models. There are time-dependent circulations of gas in the outer part of the accretion disc. These vortices have various lengths ($\lesssim 0.1 r_g$) and signs, and their strength can be characterized by Mach number $M = \sqrt{|g_m(v^2) + g_m(v^2)\ell^2/c^2|}$, $c \lesssim 0.1$. Accretion behaviour is not seen in these non-regular motions unless they are averaged. An irrotational in $(r, \theta)$-plane inflow takes place only in the innermost part of the disc, inside some radius which depends on the black hole spin. This inflow contains the transonic region.

Example snapshots of flow pattern in the inner part of the accretion disc are shown in Figs 1–3 for Models Ie, Ia, and IIb, respectively. In all the figures, the inner circle around the coordinate origin represents the black hole horizon $r_h$, and the outer one shows the grid inner radius $r_{in}$, Figs 1(a), 2(a) and 3(a) show the contours of density $\rho$ and, and the relativistic momentum vector field $(S, \sqrt{g_{\theta\theta}} S_\theta, S_\phi/\sqrt{g_{\theta\theta}})$. The sonic surface is defined as $M = 1$, and the inflow is supersonic closer to the black hole. In models with $a=0$ (group I) and $a/M = -0.9$ (group III) the density distribution on $(r, \theta)$-plane forms a surface with a saddle point located in the equatorial plane between $r_h$ and the sonic point $r_s$ (see Figs 1a and 3a).

A special feature of Models IIa–d is the instability of the transonic inflow. In this group of models the transonic

<table>
<thead>
<tr>
<th>Model</th>
<th>$a/M$</th>
<th>$\ell/M$</th>
<th>$W_{\text{inflow}}$</th>
<th>$r_{\text{inflow}}/r_g$</th>
<th>$\Delta W$</th>
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<tr>
<td>Ia</td>
<td>0.0</td>
<td>3.9136</td>
<td>2.1</td>
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</tr>
<tr>
<td>Ib</td>
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<td>2.6088</td>
<td>0.88</td>
<td>0.04</td>
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<tr>
<td>Ic</td>
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<td>4.4751</td>
<td>3.34</td>
<td>0.03</td>
<td></td>
</tr>
<tr>
<td>Id</td>
<td>0.9</td>
<td>4.4751</td>
<td>0.12</td>
<td>0.24</td>
<td></td>
</tr>
</tbody>
</table>

*The parameter $a$ is chosen to be positive when the black hole spin is codirected with the angular momentum vector of the disc.
Figure 1. Flow pattern in Model Ic ($a=0$, Schwarzschild black hole). The two circles around the coordinate centre show the black hole horizon (inner circle) and the numerical grid inner boundary (outer circle). (a) Contours of density $\gamma p$ measured by the zero angular momentum observer (ZAMO). The contour lines are spaced with $\Delta \log \gamma p = 0.1$. The density distribution in the equatorial plane has a maximum at $r=4.5r_g$ and a minimum at $r=1.6r_g$, and increases towards the black hole horizon. (b) The arrows (in relative units) show the momentum vector with the components $(S_\perp \sqrt{g_\perp}, S_\parallel \sqrt{g_\parallel})$ measured by ZAMO. The sonic surface (Mach number $M=1$) is represented by the heavy line.

Figure 2. Model IIa of an accretion disc (rotating black hole, $a/M=0.9$; the signs of rotation of the disc and the black hole coincide). See caption of Fig. 1 for details.

inflow is located very close to the black hole horizon. The instability results in sonic waves of significant amplitude (especially in Models Ila and IId), which propagate outwards. The nature of the instability (numerical or hydrodynamical) is not yet clear.

In agreement with the previous analytical study (Kozlowski et al. 1978), the structure of the accretion flow depends strongly on the cusp position, which depends on the spin of the black hole. The location of sonic radius $r_s$ in the equatorial plane is always within $r_{cusp}$ and it turns out to be determined mainly by the spin parameter $a$ rather than by the disc angular momentum $\ell$. 

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Figure 3. Model IIIb of accretion disc (rotating black hole, \(a/M = -0.9\); the rotations of the disc and the black hole have opposite signs). See caption of Fig. 1 for details.

Figure 4. Time-average mass accretion rate \(\dot{m} = M c^2/\mathcal{L}_e\) as a function of the energy gap \(\Delta W\) for models with \(a = 0\) (group I, circles), \(a/M = 0.9\) (group II, squares), and \(a/M = -0.9\) (group III, triangles). The bars show the variability of \(\dot{m}\). The lines represent the dependences \(\dot{m} \propto (\Delta W)^{\Gamma - 1}\) where \(\Gamma = 4/3\) is the adiabatic index. The plot is given for the case of the black hole mass \(M = 1 M_\odot\) and the polytropic constant \(K = 1.5 \times 10^{20}\) in cgs units. Rescaling of \(\dot{m}\) for different \(M\) and \(K\) is given by

\[
\dot{m}_{\text{new}} = \dot{m} (K_{\text{new}}/K)^{-3} M_\odot/M_\odot.
\]

We calculated the accretion rate at the inner grid boundary \(r_a\) as \(M = 4\pi \int_0^{2\pi} \sqrt{g} r^2 v_\theta^2 d\theta\). Time-average values of \(\dot{m}\) are shown as a function of the energy gap \(\Delta W = W_0 - W_{\text{cusp}}\) in Fig. 4 (points) for all calculated models. The bars display the mean squared deviation of \(M(t)\) from its mean value. This deviation shows the time variability of \(M(t)\) connected with the vortices in outer part of the disc. In the case of Models IIa–d, the significant variability of the accretion rate can be explained by the instability (discussed earlier) in the transonic part of the inflow. In Fig. 4 we also show dependences \(M \propto (\Delta W)^{\Gamma - 1}\) (lines) for each type of model. This analytical dependence was obtained by Koslowski et al. (1978). One can see good agreement between the analytical and numerical results in all models, with a modest variability of \(M\). Only two models, IIa and IIb, show large deviations from the law \(M \propto (\Delta W)^{\Gamma - 1}\), and both have highly variable \(M(t)\) (see Fig. 4, square points).

5 DISCUSSION AND CONCLUSIONS

We developed a two-dimensional fully relativistic hydrodynamical code and calculated several inviscid models of transonic disc accretion in a Kerr gravitational field of a rotating black hole. We took an isentropic stationary rotating torus of constant angular momentum as our initial condition and followed its time evolution until a quasi-stationary accretion flow was established. Accretion occurred because the initial configuration was unstable, and proceeded near the equatorial plane in the cusp-like inner edge of the disc. We assumed that the radiative cooling was negligible and used an adiabatic approximation with index \(\Gamma = 4/3\), which corresponds to a radiation-supported disc with superEddington accretion rate, \(\dot{M} \gg L_e\ c^{-2}\).

The calculated structure of the accretion flow strongly depends on the cusp position, in agreement with the previous analytical study (Kozłowski et al. 1978). The location of the sonic radius and the flow geometry are sensitive to the black hole spin. We investigated three cases: \(a/M = 0, 0.9, -0.9\), where \(a\) was the specific angular momentum of the black hole of mass \(M\) in geometrical units \((c = G = 1)\). We found the flow to be irrotational in the \((r, \theta)\)-plane only close to the cusp. Outer parts of the disc showed complex
quasi-periodical vortex motions of gas. These motions are likely to be a consequence of our choice of initial conditions. They develop from initial perturbations in the cusp region and propagate outwards. The vortices produced exist without damping in an inviscid constant angular momentum medium (see note in Section 2.2).

Our numerical models confirm the analytical dependence \( M \propto (\Delta W)^{3/4} \) (Koslowski et al. 1978) for all three values of the black hole spin \( a \) (see Fig. 4). Significant deviations from this dependence (seen in two models, IIa and IIId) are likely to be connected with the high amplitude variability of the flow arising from an instability of unclear nature (numerical or hydrodynamical) in the innermost part of the disc.

In our previous paper (Igumenshchev et al. 1996), isentropic accretion disc models were calculated in pseudo-Newtonian gravitational potential. Comparing the results we note that, outside of the sonic radius \( r_s \), there are no qualitative differences between pseudo-Newtonian models and fully relativistic models in the case of a Schwarzschild black hole. The quantitative differences in distributions of density and pressure are less than 10 per cent.

ACKNOWLEDGMENTS

We are grateful to Marek Abramowicz, Xingming Chen and Alexander Polnarev for discussions, and Birgitta Högmans for improving the grammar of the text. We thank the referee, Jean-Pierre Lasota, for his careful reading and useful remarks. This research was supported by the Swedish Institute and the Royal Swedish Academy of Sciences. We thank the Department of Astronomy and Astrophysics, Göteborg University and Chalmers University of Technology for hospitality. AMB acknowledges partial support by the Russian Foundation for Fundamental Research (project code 95-02-06063 and 93-02-17059).

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