A non-force-free cylindrical jet embedded in a conical wind envelope

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Abstract

A non-force-free cylindrical equilibrium of the magnetically driven relativistic MHD flows is investigated as a model of astrophysical jets. We present a model in which the magnetosphere has a hybrid asymptotic structure: the polar region and the lower latitude region are filled with the cylindrical jet and the conical wind, respectively. The conical wind was investigated by myself in a previous paper. Here we investigate the cylindrical jet at finite cylindrical radius. In this case, the inertial force is important for the force balance.

In the polar region, the force-balance equation is reduced to a single second-order ordinary differential equation. This reduced equation is numerically solved, and typical solutions having the following properties are obtained.

These solutions indicate self-organized structure, i.e. the solutions fill the entire polar region with finite magnetic flux, and need no external pressure for confinement. The current distribution is completely regular. This cylindrical jet is composed of the cylindrical core with a scale similar to that of the light cylinder, and the ambient cylindrical envelope, which extends to infinite cylindrical radius. The collimation of the core is governed by four parameters (the terminal velocity, the angular momentum, the angular velocity of the field lines and the injection rate).

1 INTRODUCTION

A self-organized stationary and axisymmetric cylindrical magnetohydrodynamic (MHD) equilibrium of relativistic outflows is investigated. Here we present a model for astrophysical jets as follows: the magnetosphere is filled with the hybrid of the highly collimated cylindrical polar jet and the expanded conical wind envelope. In the model, magnetic field lines emanating from the central object (pulsar or black hole + accretion disc is supposed) are rotating, and the frozen plasmas are magnetically accelerated outward and construct a self-organized magnetosphere owing to the Lorentz force and the inertial force.

Astrophysical outflows can be classified into the following three classes at regions far distant from the central object. From now on we define $R$ and $Z$ as the cylindrical radius from the rotational axis of the central object and the height along the rotational axis from the equatorial plane, respectively, and $\Psi$ as the poloidal magnetic flux function ($\Psi = 0$ at the rotational axis). The magnetosphere is supposed to be mirror symmetrical with respect to the equatorial plane. We follow the manner of Chiueh, Li & Begelman (1991) and discuss the asymptotic behaviour along a magnetic field line $\Psi = $ constant (which coincides with a stream line on the poloidal plane in the stationary and axisymmetric case). We can easily show that the ‘asymptotically horizontal’ flow (along a magnetic field line $Z/R \to 0$ as $R \to \infty$) never arises from the consistency with the Bernoulli equation (the poloidal wind equation in Nitta, Takahashi & Tomimatsu 1991).

If $Z/R \to \infty$ as $R \to \infty$ along a magnetic field line, we call it ‘asymptotically paraboloidal’. The paraboloidal class is usually adopted as the jet solution. The paraboloidal equilibrium has been investigated in many papers, for example Blandford & Payne 1982, Eichler 1993, Mestel & Shibata 1994 and Tomimatsu 1994. The cylindrical equili-

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brium also has been investigated in many papers, for example Appl & Camenzind (1993a, b) (one-dimensional equilibrium), Sauty & Tsinganos (1994), Lovelace, Berk & Contopoulos (1991), Li (1993) (two-dimensional equilibrium).

We should remark that the paraboloidal and the cylindrical solutions can fill only the polar region \( Z/R \to \infty \), i.e. \( \theta = 0 \) where \( \theta \) is the polar angle from the rotational axis of the central object) by definition of these classes. Here a question that arises is how is the asymptotic structure of the magnetosphere outside \( 0 < \theta < \theta_0 \) the cylindrical or the paraboloidal flow?

Recent observations revealed that the magnetosphere of the Crab pulsar or young stellar objects consists of the hybrid of the polar jet and the widely expanded wind. It shows that the region outside the collimated jet is not 'empty' but filled with plasma flows, at least in several objects. In order to explain these structures, the hybrid model of a jet + wind system might be plausible.

There are two theoretical reasons adopting this hybrid model. One is that, in this hybrid model, the cylindrical jet fills the polar region \( \theta = 0 \) and the conical wind fills the lower latitude region \( 0 < \theta \leq \pi/2 \), respectively. Hence this hybrid flows can fill the entire far distant region \( 0 \leq \theta \leq \pi/2 \). Another reason is that the asymptotically conical solution cannot stand by itself, because it needs the 'polar current' flowing in a finite cylindrical radius (see Chiueh et al. 1991, Nitta 1995) where the asymptotic approximation \( (R \to \infty) \) which is used to obtain the conical solution is no longer valid. The current carrying cylindrical jet can make up this lack of completeness, and the far distant region is completed by the matching of these two solutions. Hence we can think that this model is one of plausible possibility to explain real hybrid structures. This possibility has been already pointed out in some of the literature, for example Heyvaerts & Norman (1989) for non-relativistic flows and Chiueh et al. (1991) for relativistic flows, but no concrete solution is found in these works.

Of course the entire magnetospheric structure including the region at the finite distance from the central object should be solved as a two-dimensional problem in the poloidal plane. This two-dimensional study is intensely investigated by some groups, and is developing now. As is well known, the structure of the idealized stationary and axisymmetric magnetosphere is denoted by a second-order partial differential equation, the so-called the Grad–Shafranov (G–S) equation. This equation is very difficult to solve not only analytically but also numerically. The core of the difficulty of this problem is that the problem of the G–S equation is a hybrid of the elliptic (boundary-value) problem and the hyperbolic (initial-value) problem with an undetermined boundary for the transmagnetosonic flows (see Sakurai 1990). For the first time, the non-relativistic G–S equation for the hot flows was numerically solved by Sakurai (1985, 1987). These works of Sakurai are epochal because the entire magnetosphere including the starting point of the flow, all the critical surfaces (the slow mode cusp, the slow surface, the Alfvén and the fast surface) and regions farther out are simultaneously solved without artificial assumptions, e.g. the self-similarity or the magnetically dominated flows. However some mathematical ambiguities (for example the boundary condition at the Alfvén surface and the outward integration at the elliptic region) in his procedure are pointed out. The development to relativistic flows was attempted by Camenzind (1987). In this work, flows are assumed to be magnetically dominated and solved for only the elliptic (sub-fast) region, since the entire region should be sub-fast in the magnetically dominated limit. An application to the star-forming region was performed in a series of works by the group at Berkeley (Shu et al. 1994a, b, Najita & Shu 1994 and Ostriker & Shu 1995). In these works, somewhat complicated geometry of the magnetic field modified from the dipole geometry is assumed for a model of young stellar objects (YSOs). In their works, the mathematical ambiguities included in Sakurai's procedure are solved in an elegant way. However, in this paper, our discussion is focused only to the far distant region, and this two-dimensional problem is a too involved subject to be treated further here.

The purpose of this paper is to discuss the asymptotic structure of the non-force-free relativistic cylindrical polar jet which is embedded in the conical wind filling the lower latitude region \( R \to \infty \) with \( 0 < \theta < \pi/2 \). In this paper, our discussion is restricted to the polar region that is far and high from the equatorial plane (with \( Z \to \infty \) and polar angle \( \theta = 0 \)). This restriction enables us to treat the polar jet as a one-dimensional flow. The jet is assumed to be distributed from the rotational axis \( R = 0 \) to the matching boundary \( R = \infty \) with \( \theta = 0 \) with the conical wind. Properties of the conical wind are discussed by Nitta (1994) in detail (briefly summarized in Section 2.4 of this paper). In the region far from the rotational axis \( R \to \infty \), the Lorentz force (the toroidal magnetic force and the electrostatic force) dominates, and the inertial force can be negligible in the force balance. However, in the equilibrium at finite cylindrical radius as in this paper, the inertial force is no longer negligible when the flow has finite angular momentum because the flow is rotating. In addition, observed typical power-law emission from the jet suggests that the flow is supermagnetosonic, at least partially, if this power-law emission is produced by the first Fermi process. This means that the propagation speed of the magnetosonic wave must be finite, i.e. the electromagnetic force does not dominate the inertial force. Hence a non-force-free treatment is needed to discuss the structure at finite cylindrical radius. Added to these, most of the AGN jets have a relativistic speed (the Lorentz factor \( \gamma \approx 2 \to 20 \), and the speed of the wind of the Crab pulsar is very close to the velocity of light \( (\gamma \approx 10^6) \). Hence we should treat the flows in the special relativistic scheme.

We solve the field aligned (the poloidal wind) equation and the trans-field (the G–S) equation self-consistently in order to obtain the equilibrium structure. As discussed above, the calculation is approximated in a one-dimensional approach based on the cylindrical geometry for simplicity. This is justified only at the far distant polar region. In this region, the jet is assumed to be already collimated to the cylindrical shape. We neglect the gravity of the central object and limit the discussion to the non-resistive, inviscid and pressure-free (cold) plasmas in order to discuss selectively the effects of the Lorentz force and the inertial force.

This paper is organized as follows. The basic relativistic equation we use in this work is reviewed in Section 2. We
also clarify the details of the model, and reduce the basic equation to the asymptotic form in this section. We numerically solve this reduced equation. The numerical procedures are discussed in Section 3. The numerical results give us very interesting features of the jet: the cylindrical jet itself has a hybrid structure, i.e. the dense core and the envelope. The degree of the collimation of the core is governed by parameters of the flow. These properties are discussed in Section 4. In Section 5, we summarize the results and compare these with the results of other literature.

2 BASIC EQUATIONS AND MODEL

Let us list the basic equations for relativistic cylindrical equilibrium at the far distant polar region in this section. The notation follows in Camenzind's manner (see Camenzind 1986a,b).

2.1 Relativistic basic equations and constants of the motion

Most of the observed astrophysical outflows seem to be supermagnetosonic, at least partially, because these objects emit typical power-law spectra which can be understood as a result of non-thermal acceleration at the shock surface (the first Fermi process). This results that the inertial terms of the equation cannot be neglected a priori. In this paper, we will study the case that each stream line of magnetized flows converges to corresponding finite cylindrical radius. Since the inertial effect is no longer negligible in this case, we must treat the flows as non-force-free. The flows from pulsars and AGNs have relativistic speed. So we use special relativistic MHD equations as the basic equation which is denoted in Minkowski space-time with usual cylindrical coordinates,

\[ ds^2 = dr^2 - dr^2 - R^2 d\phi^2. \]

For the simplicity, we limit our discussion to non-resistive, pressure-free, inviscid, stationary and axisymmetric flows.

In such a simplified case, we obtain two basic equations and four constants of the motion (see Takahashi et al. 1990 and Nitta et al. 1991). The two basic equations are the so-called field aligned (Bernoulli) equation and the trans-field (Grad–Shafranov) equation. The four constants of motion are the energy of a particle, the angular momentum of a particle, the angular velocity of plasma flows, and the ratio of the particle number flux to the magnetic flux (the injection rate) defined as below:

\[ E = \mu u_\phi - \frac{\Omega_\phi B_\phi}{4\pi \eta}, \]

\[ L = - \mu u_\phi + \frac{B_\phi}{4\pi \eta}, \]

where \( \mu \) is the specific enthalpy that in the cold limit reduces to the rest-mass energy of a plasma particle, and \( B_\phi \) is the \( \phi \) component of the magnetic field defined by \( B_\phi = \frac{1}{2} (\sqrt{-g}) \partial_{\mu} F^{\mu \nu} k^\nu \) (\( k^\mu \) is the time-like Killing vector), \( u_\mu \) is the covariant component of the four-velocity,

\[ \Omega_\phi = - \frac{F_{\phi \rho}}{F_{\phi \phi}}, \]

\[ \eta = - \frac{R m u^\phi}{F_{\phi \phi}}, \]

\[ = - \frac{R m u^z}{F_{\phi \phi}}, \]

\[ = - \frac{R m u^z (\Omega - \Omega_\phi)}{F_{\phi \phi}}, \]

where

\[ \Omega = \frac{u^\phi}{u^t}, \]

is the angular velocity of plasma flows, \( n \) is the proper number density observed in the plasma-comoving frame and \( F_{\phi \phi} \) is the electromagnetic tensor defined as \( F_{\phi \phi} = \frac{\partial_\phi A_\rho - \partial_\rho A_\phi}{\partial_\rho A_\phi}. \)

The first basic equation is the poloidal wind equation,

\[ (1 + u_\phi^2)(k_0 - M^2)^2 = \frac{E^2}{\mu}, \]

\[ (k_0 k_2 - 2k_2 M^2 - k_4 M^4), \]

where \( k_0, k_2 \) and \( k_4 \) are

\[ k_0 = 1 - R^2 \Omega_\phi^2, \]

\[ k_2 = \frac{E}{\mu}, \]

\[ k_4 = \frac{-R^2 + (L/E)^2}{R^2}, \]

\[ \epsilon = E - L \Omega_\phi \] and \( M^2 \) is the square of the Alfvén Mach number defined as

\[ \frac{4\pi \mu n^2}{n} = 4\pi \mu \sqrt{\frac{u_\phi^2}{B_\phi}}, \]

with

\[ u_\phi^2 = - u^\mu u_\mu \quad (A = R, Z) \]

\[ B_\phi^2 = - B^\mu B_\mu \]

The equation (9) denotes the evolution of the quantities along a stream line, hence is often called the field aligned equation or the Bernoulli equation.

The second basic equation is the Grad–Shafranov (G–S) or the trans-field (force-balance) equation,

\[ R(k_0 - M^2)^2 \left\{ \frac{\partial_\phi}{\partial_\rho} \left( k_0 - M^2 \right) \partial_\phi \Psi \right. \]

\[ \left. + \partial_2 \left( \frac{k_0 - M^2}{R} \right) \partial_2 \Psi + \frac{k_0 - M^2}{R} \left[ \partial_\phi^2 \Psi + \partial_2^2 \Psi \right] \right\}. \]

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where the prime \( ' \) denotes the derivative with respect to \( \Psi \), because \( E, L, \eta, \) and \( \Omega_r \) are functions of \( \Psi \) only. This equation denotes the force balance in the trans-field direction. These two equations (9) and (16) completely determine the global structure of the magnetosphere filled with magnetized flows.

### 2.2 The polar equation

We will study the structure of the polar region \((\mathcal{Z} \to \infty)\) with the polar angle \( \theta_0 = 0 \) from the rotational axis. In this region we can approximate \( \hat{a}_z \to 0 \) (the so-called 'polar limit' used in Appl & Camenzind 1993a,b).

In the region the poloidal velocity \( u_\phi \) settles to the terminal value \( U_{\infty} \). We now convert the independent parameters from \((E, L, \Omega_r, \eta)\) to \((U_{\infty}, L, \Omega_r, \eta)\) using the poloidal wind equation.

From the equation (9), we can find a relation

\[
(k_0-M^2)^2E^2-2L\Omega_x(k_0-2M^2)E
+[(L^2\Omega_x^2(k_0-2M^2)-M^2L^2R^2)]
-\mu^2(1+u_\phi^2)(k_0-2M^2)^2 = 0.
\] (17)

We obtain two roots of \( E = E(\Psi, R) \) by solving this equation. One root will be unphysical because this is inconsistent with the value from the definition (2) as discussed in the following sections. So we must adopt the alternative consistent root as the physical solution.

We convert the independent variables \((R, \mathcal{Z})\) to the new variables \((x, y)\) defined as

\[
R = R_0 \tan(x/2),
\]

\[
\mathcal{Z} = R_0 \tan(y/2),
\] (18) (19)

where \( R_0 \) is a typical length, e.g. the light cylinder radius of a typical magnetic field line. It will be set as \( R_0 = 10^{14} \) m in the following discussion for the typical value of the light cylinder radius of AGNs. These new variables are favourable to denote the infinity \( R, \mathcal{Z} \to \infty \).

For the 1D cylindrical flows at the far distant polar region, we can reduce the above two basic equations (the poloidal wind equation and the G–S equation) to a single second-order ordinary differential equation of \( \Psi \),

\[
\frac{d^2\Psi}{dx^2} + S(x, \Psi, d\Psi/dx) = 0
\] (20)

where

\[
S(x, \Psi, d\Psi/dx) = N/D
\] (21)

with

\[
N = f_1 \frac{2\Psi \cos^3(\pi x/2) \sin(\pi x/2)}{nR_0^2/2} + f_2
\]

\[
D = f_1 \frac{\cos^2(\pi x/2)}{(nR_0/2)^2}
\]

\[
f_1 = \frac{k_0(k_0-M^2)}{8\pi^2} + 2R^2\eta \frac{\partial E}{\partial \Psi_R} \frac{1}{\Psi_R} \left( \frac{E-L\Omega_x}{M^2} - E \right)
\]

\[
f_2 = \frac{R^2}{8\pi^2} (k_0-M^2)
\]

\[
\times \left\{ \left[ \frac{1}{R^2} + \Omega_x^2 + 4\pi\mu(\eta u_\phi)^2 \right] \Psi_R - R\Psi_R(\eta^2)^{1/2} \right\}
\]

\[
+ \left\{ \frac{\partial E}{\partial \Psi_R} + \frac{\partial E}{\partial \Psi} \right\} 2R^2\eta^2 \left( \frac{E-L\Omega_x}{M^2} - E \right)
\]

\[
+ \left\{ - (E-L\Omega_x)^2 - L^2 + R^2E^2 + \frac{\mu^2}{M^2} \right\} (\eta^2)^{1/2}
\]

\[
- \frac{\mu^2}{M^2} \eta \left( \frac{E-L\Omega_x}{\mu} \right) - (k_0-M^2) \Omega_x^2 \right\}.
\] (25)

Here \( \Psi_0 \) and \( \Psi_R \) are the derivative with respect to \( x \) and \( R \), respectively. These are related as \( \Psi_R = \Psi \cos^3(\pi x/2) \) \( (R_0/\pi)^2 \). From now on we call equation (20) the 'polar equation'.

### 2.3 Boundary conditions

We are interested in the structure of the self-organized magnetosphere in this paper. The following is a necessary condition that the entire magnetosphere is filled with a finite amount of the poloidal magnetic flux \( \Psi_0 \) which can be produced by the central object. So the entire polar region must be filled with a finite amount of magnetic flux \( \Psi_0 < \Psi_0 \). We set the boundary conditions as

\[
\Psi = 0 \text{ at } x = 0 \text{ (} R = 0 \text{)}
\]

and

\[
\Psi = \Psi_0 \text{ at } x = 1 \text{ (} R = \infty \text{)}.
\] (26) (27)

The first condition is trivial for non-singular magnetic distribution, and the second condition comes from the above discussion for self-organized magnetosphere. Thus the problem is well defined with these boundary conditions, and we will try to solve the polar equation (20) numerically because the source term \( S(x, \Psi, d\Psi/dx) \) of the polar equation is very complicated.

### 2.4 Connection to the conical wind envelope

We adopt a model in this paper that the cylindrical jet is embedded in a conical wind envelope. The jet and the wind fill the polar region \((\theta = 0)\) and the lower latitude region \((0 < \theta)\), respectively. Detail of the conical wind is discussed in Nitta (1994).
Let us briefly summarize the result of Nitta (1994). This is the general solution of the asymptotically conical class far outside the light cylinder. At the asymptotic region \((R \gg R_1\), where \(R_1\) is a typical scale of the light cylinder), the Lorentz force (the toroidal magnetic pinching force and the relativistic electrostatic force) dominates other forces. In this region, all of the field lines must be denoted as \(Z = f(\Psi)R\), because of the definition of the conical class \((Z/R \to \text{finite as } R \to \infty)\). The \(G-S\) equation for the conical class can be reduced to an ordinary differential equation for \(f(\Psi)\). We can easily obtain the general solution of this reduced equation. The inclination \(f(\Psi)\) is given as a function of the angular velocity of magnetic field lines \(\Omega_\Psi(\Psi)\), the terminal velocity of flow \(u_t(\Psi)\) and a parameter \(\lambda\) denoting the amount of the polar current, and is denoted as

\[
f(\Psi) = \sinh \left[ \frac{1}{\lambda} \int_\Psi^{\Psi_0} \frac{\Omega_\Psi(\Psi')}{u_t(\Psi')} d\Psi' \right],
\]

where \(\Psi_0\) is the total poloidal magnetic flux of the magnetosphere. This solution expresses a force-free equilibrium between the electrostatic force and the toroidal magnetic pinching force. Generally the word 'force-free' is confused with 'magnetically dominated' in much of the literature, but we should notice that 'force-free' does not mean 'magnetically dominated', so this solution does not lose the generality. This solution is very favourable because of the following. This solution can fill the entire asymptotic region with a finite amount of poloidal magnetic flux, and can explain both the thin jets and the equatorial winds according to the parameters.

We should note that this conical solution cannot stand by itself because it needs the polar current \(\lambda\) flowing inside the innermost conical flux surface (see Chiueh et al. 1991 or Nitta 1994). This region is the inner exterior of the conical solution. In order to clarify the origin of the polar current, we must study the region of finite cylindrical radius from the rotational axis. The flow in this region is 'cylindrical' by definition \((R \to \text{finite as } Z \to \infty)\).

The cylindrical jet can be connectable with the conical wind envelope only at the matching boundary \(x = y = 1\) with \(\theta = 0\) as shown in Fig. 1. We should note that the 'shape' of the innermost surface of the conical wind \((\Psi \to \Psi_c + 0)\) and the outermost surface of the cylindrical jet \((\Psi \to \Psi_c - 0)\) are the same at the matching boundary, i.e. these surfaces are vertical. We shall return to this point in the final section.

We should note an additional matching condition (see Chiueh et al. 1991) as follows. Since a finite magnetic flux \(\Psi_c\) must fill the entire polar region \(0 \leq R \leq X\) for the self-organized magnetosphere, we should demand that

\[
\Psi_c = \int_0^X B_\phi R \, dR = \text{finite},
\]

where \(B_\phi\) is the poloidal magnetic field. This requires that \(B_\phi R^2 \to 0\) as \(R \to \infty\). If the injection rate \(\eta\) is finite everywhere (this is reasonable for MHD flows), i.e.

\[
\eta = \frac{n u_\phi}{B_\phi} = \frac{(nR^2) u_\phi}{B_\phi R^2} = \text{finite},
\]

where \(L\) is assumed to be finite. Here we will consider the cylindrical jet as the polar part of the hybrid magnetosphere. Note that the result of Nitta (1994) shows that the polar region of the conical wind should not be current free \((B_\phi \neq 0\) as \(R \to \infty\) except the case of the non-magnetized flows. This is the property of the conical wind (see Chiueh et al. 1991). Hence this case must be rejected for the hybrid solutions.

The second case is favourable in the viewpoint of continuity because the terminal speed of the conical wind on the innermost vertical \((\theta = 0)\) flux surface must vanish. This is trivial because \(f(\Psi)\) of the equation (28) must diverge. Here we have obtained the matching condition of the cylindrical flows with the conical wind envelope:

\[
u_t(\Psi_c) = 0.
\]

Figure 1. The hybrid of the polar jet and the wind. The top boundary and the right wall denote the polar region and the asymptotic region, respectively. The matching boundary locates at \(x = 1, y = 1\) \((\theta = 0)\).

This requirement results in \(nR^2 \to 0\) or \(u_\phi \to 0\) as \(R \to \infty\) \((\Psi \to \Psi_c)\).

The first case follows that the polar region is 'current-free' \((B_\phi \to 0)\) because

\[
B_\phi = \lim_{R \to \infty} \left[ - \frac{4\pi \eta}{\lambda^2 (nR^2) + 4\pi \mu n^2} (\mu - M^2) \right] = 0
\]

We numerically solve the polar equation (20) by Newton–Raphson method in a difference scheme called the 'first-order explicit method'. The entire polar region \((0 \leq x \leq 1\) with \(y = 1\) is divided into small segments that are evenly spaced in \(x\). In the following results, we use 100 mesh points. The solution is obtained as \(\Psi = \Psi(x)\). Of course this problem must have a unique solution.

Calculation is started from a trial function and iteratively performed until well converged. We can verify the convergence of the approximated numerical solution to the real solution by performing the following check. Initially we set

3 NUMERICAL PROCEDURE

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We adopt a trial function $\Psi(x) = \Psi_0 x$. The solution converges to an approximated solution (see Fig. 2 for the reference case defined later) by iterative calculation. In order to check whether this solution converges well to the real solution, we try to start from another initial trial function $\Psi(x) = \varepsilon \Psi_0 x$ for $0 \leq x < 1$ and $\Psi(1) = \Psi_0$, where $\varepsilon$ is a very small number. These iterations from two different trial functions converge to the similar solution. So we can believe that the obtained solution is well converged to the real solution.

3.1 Model for calculation and normalization

The parameters of this problem are the terminal velocity $u_0(\Psi)$, the angular momentum $L(\Psi)$, the angular velocity $\Omega_0(\Psi)$ of the field line and the injection rate $\eta(\Psi)$. We numerically obtain typical solutions of the polar equation (20) under a simple model of these parameters. In order to connect to the conical envelope at $x = 1$ ($R = \infty$) with $y = 1$, the cylindrical flow must extend to $R = \infty$. In this case, we must set $u_0 \to 0$ when $\Psi \to \Psi_0$ as (32). So we adopt a model of the functional form of $u_0$ as

$$u_0(\Psi) = u_0(1 - \Psi/\Psi_0)$$

where $u_0$ is a constant. For the regularity at the rotational axis $R \to 0$, we must set $L \to 0$ as $\Psi \to 0$. We adopt a simple model as

$$L(\Psi) = L_0 \Psi/\Psi_0$$

where $L_0$ is a constant. In the general case, the angular velocity $\Omega_0$ of field lines and the injection rate $\eta$ depend on $\Psi$. However, for simplicity, we set these constants of the motion as constants independent of $\Psi$,

$$\Omega_0(\Psi) = \Omega_0$$

and

$$\eta(\Psi) = \eta_0$$

where $\Omega_0$ and $\eta_0$ are constants. These mean rigidly rotating magnetosphere and uniformly magnetized flows, respectively. In the following results, we treat these four constants $u_0, L_0, \Omega_0$ and $\eta_0$ as parameters.

4 NUMERICAL RESULTS

We perform the calculation and obtain the new features of the cylindrical jet as follows.

4.1 The poloidal current distribution

Recent works emphasize that the regularity of the poloidal current distribution is an index whether a solution is plausible or not (Heyvaerts & Norman 1989; Chiueh et al. 1991). The regularity is also important for the hybrid model discussed here because this poloidal current is certainly the polar current, and is essential for the conical envelope. Let us study the poloidal current distribution of this cylindrical jet.

Fig. 3 shows the poloidal current density as a function of $x$ for the reference case. We can clearly find that the current density is finite everywhere. On the contrary, in the previous paper (Nitta 1995), the current density behaves as $j_p \propto R^{-1}$ when $R \to 0$ and diverge on the rotational axis. Let us discuss this difference more carefully. In the previous paper we investigated a cylindrical equilibrium as an inner solution of the polar region of the conical wind under a severely restricted condition that $u_0 = L = 0$ everywhere and $\Psi = 0$. The solution shows that the centrifugal force and the Lorentz force due to the polar current are in balance. While this

We adopt the units of each dimension as follows. The units of length, mass, time and the magnetic field are $R_0, \mu_0 c^2, R_0/\mu_0 c$ and $\Psi_0/R_0^2$, respectively.

3.2 Reference case

We treat the case $u_0 = L_0 = \Omega_0 = \eta_0 = 1$, in the non-dimensional value as the reference case. These value correspond to $u_0 = 1$, $L_0 = \mu R_0/c \sim 10^{-9}$ kg m$^2$ s$^{-1}$, $\Omega_0 = c/R_0 \sim 10^{-5}$ s$^{-1}$ and $\eta_0 = \Psi_0/(\mu R_0^2) \sim 10^{13}$ S kg$^{-1/2}$ m$^{-3}$ in dimensional expression if $\mu c^2 = m_s \sim 10^{-30}$ kg (e$^+$ + e$^-$ pair plasma), $R_0 = 10^{12}$ m and $\Psi_0 = 10^{25}$ T m$^2$ (these values are the same with Appl & Camenzind 1993a, b). In the following sections, we try to show some typical quantities for this reference case, and compare with different cases of parameters.
solution does not involve the line current flowing just on the axis $R=0$, but the current density diverges at $R=0$. This strange behaviour of the current density is responsible to the cold (pressure-free) assumption and the restriction $u_i=0$. In this situation, the toroidal velocity cannot vanish on the rotational axis $R=0$ from the following discussion. Since a finite amount of the polar current, say $-\lambda$, flows in a region of finite radius, then $B_\phi= -\lambda \neq 0$ at the asymptotic region $R=\infty$, and the speed of the flow vanishes ($u_i=1$) on the innermost conical surface $\Psi=0$. Hence the energy of the flow has the value $E_{\text{asympt}} = \mu + \frac{\lambda \Omega_i}{(4\pi\eta)} > \mu$ on $\Psi=0$ from equation (2). On the rotational axis $R=0, B_\phi \to 0$ when the line current does not exist as in this case, hence $E_{\text{asympt}} = \mu u_i$ on the axis. The value of the flux function does not change from the axis to the asymptotic region because $\Psi=0$ everywhere in this polar region. We should remember that the energy $E$ is a flux function, and $E_{\text{asympt}}=E_{\text{asympt}}=E(0)$. As a result, we obtain that $u_i > 1$ on the axis. This means the presence of the toroidal flow on the axis, because there is no poloidal flow. In order to confine this toroidal flow by the Lorentz force, the poloidal current density must diverge. However, also in the present work, the gas pressure is ignored, but the flow can have non-vanishing poloidal velocity, so the toroidal velocity can vanish on the axis, and divergence of the current density can be avoided.

Fig. 4 shows the poloidal current distribution. We can find that the total current inside the jet is finite ($\sim 10^{19}$ A for the reference case). This is favourable for the following two reasons. First, the total current generated by the central object seems to be finite. Secondly, the conical wind envelope must be current carrying, so the cylindrical jet must involve the finite current (see Nitta 1994, Chiueh et al. 1991). From these discussions, we can understand that the current distribution is completely regular in this solution.

4.2 Energy and angular momentum flux

We should note the finiteness of the energy flux and the angular momentum flux. From equation (2) and the finiteness of the numerical result of $B_\phi$, we can easily find that $E$ is finite everywhere. The total energy flux is estimated as

$$P_{\text{tot}} = \int_0^\infty E \eta \ d\Psi.$$  (37)
This is obviously finite because $E$ and $\eta$ are finite. The total angular momentum flux is similarly estimated as

$$N_{\text{tot}} = \int n R d\Psi.$$  

(38)

This is also finite when $L$ is finite everywhere as supposed in this work. We have checked the finiteness of the magnetic flux, the current flux, the energy flux and the angular momentum flux. These finitenesses are necessary for a plausible self-organized solution.

### 4.3 Core–envelope structure

We can obtain typical numerical solutions for a cylindrical jet. These structures are results of the equilibrium between the Lorentz force and the centrifugal force. Fig. 5 clearly shows the core–envelope structure of the jet itself. Fig. 5 shows logarithmic plots of the proper number density $n$, the poloidal magnetic field $B_p$ and the poloidal current density $j_p$ versus the cylindrical radius $R$. Each of these curves is obviously composed of two different power laws. For example, in the vicinity of the rotational axis ($0 < R < R_0$), $n$ is proportional to $R^0$. On the contrary, in the outer region ($R_0 < R < \infty$), $n$ is proportional to $R^{-2}$ (note that, in the reference case, $R_0$ is the same as the light cylinder radius $R_L$). We call them the 'jet core' and the 'jet envelope', respectively. We also find that the core radius is nearly equal to the radius of the light cylinder $R_L$.

One might wonder is it plausible to call a 'collimated flow' if the cylindrical flow is infinitely expanded with respect to the cylindrical radius $R$. However, in our solution we can clearly distinguish the core and the envelope as in the above discussion. Our answer is that the collimation of the flow means core formation is evident.

### 4.4 Dependence of the collimation on the parameters

Figs 6(a)–(d) show the dependence of the core radius on the parameters $u_0$, $L_0$, $\Omega_0$ and $\eta_0$, respectively. If a parameter varies under the condition that other parameters are fixed, $u_0$ suppresses the collimation due to the outward electrostatic force. This is a relativistic effect. $L_0$ and $\eta_0$ also suppress the collimation due to the inertial force (increment of the centrifugal force). However, $\Omega_0$ amplifies the collimation due to the magnetic pinching force.

### 5 SUMMARY AND DISCUSSION

We have numerically solved a non-force-free and special relativistic cylindrical MHD equilibrium across the light cylinder as a one-dimensional problem under a model that this cylindrical jet is embedded in the conical wind envelope. The matching boundary is located at the infinity $R = \infty$. In order to simplify and clarify the effect of the Lorentz force and the inertial force, we neglect the resistivity, the pressure and the viscosity. We impose the ideal symmetry, i.e. the flows are stationary and axisymmetric.

#### 5.1 Properties of the solution

This equilibrium is characterized by the following noteworthy properties. (1) This cylindrical jet is a self-organized structure. (2) The current distribution is completely regular. (3) The jet itself has hybrid core–envelope structure. Let us look at these properties more carefully.

(1) The jet fills the entire polar region ($0 \leq R \leq \infty$ at $Z = \infty; \theta = 0$) with a finite amount of the poloidal magnetic flux ($0 \leq \Psi \leq \Psi_0$). This means that such magnetosphere can be filled only with the magnetic field lines that are emanated from the central object, e.g. a pulsar or a black hole + accretion disc system. Another remarkable feature is that the jet needs no external pressure of ambient medium for confinement.

(2) The electric current distribution is important to check whether the solution is physically plausible (see Chiueh et al. 1991) or not. Fig. 3 shows that the current density does not diverge not only at the rotational axis $R = 0$ but also everywhere. Fig. 4 shows that the total current of
Figure 6. The dependence of the collimation on the parameters. Figs 6(a)–(d) denote the logarithmic plot of the number density distributions versus the logarithm of $R$. The collimation of the jet core is governed by parameters $u_0$ (a), $L_0$ (b), $\Omega_0$ (c) and $\eta_0$ (d). The units of the number density and the cylindrical radius are $\Psi/(\mu R_0^3) \sim 10^{14} \text{ m}^{-3}$ and $R_0 \sim 10^{14} \text{ m}$, respectively. The solid line of each figure is the reference case with $u_0 = L_0 = \Omega_0 = \eta_0 = 1$. The smaller $u_0$, $L_0$, $\Omega_0$ and the larger $\eta_0$ lead to a higher degree of collimation comparing with the reference case, and vice versa.
the magnetosphere is finite. This is plausible because the total current generated by the central object seems to be finite. So we can conclude that the current distribution is completely regular in this solution.

(3) As shown in Fig. 5, the cylindrical flow is composed of two clearly distinguishable components, i.e. the jet core and the jet envelope. These two components can be distinguished by two different power laws of the density distribution as the function of the cylindrical radius $R$. The jet core is confined inside a scale similar to the radius $R_i$ of the light cylinder. The jet envelope fills outside the core region and smoothly connects to the conical wind envelope. The collimation of the core is governed by the parameters, i.e. the terminal speed of the flow $u_t$, the angular momentum $L$, the angular velocity of the field line $\Omega_e$ and the injection rate $\eta$. $u_t$ suppresses the collimation due to the outward electrostatic force. $L$ and $\eta$ also suppress the collimation due to the centrifugal force. On the contrary $\Omega_e$ enhances the collimation due to the magnetic pinching force (the screw pinch).

When the angular momentum is small, the region near the rotational axis rotates in opposite direction against the rotation of the magnetic field line. For example, when $L_i = 0.2$, the region $x < 0.4 \ (R < 0.7 R_l)$ has negative angular velocity. This curious behaviour is discussed in Nitta (1995) where the discussion is restricted in the case $u_t = L = \Psi_\infty = 0$. From the definition of the angular momentum (3), we can find that the region $L < -B_\phi/(4\pi\eta)$ rotates in opposite direction. In the previous paper, the equilibrium of the inner region is understood as that the centrifugal force due to this counter rotation is in balance with the magnetic pinching force. In the present work, we have generalized it.

There are series of interesting works of Appl & Camenzind (1993a, b) worthy to compare with our result. They studied the relativistic cylindrical equilibrium at the polar region in very similar way, but they assume that the Lorentz force dominates than the inertial force, so their discussion is limited in the force-free case. They also pointed out the hybrid core–envelope structure of the cylindrical equilibrium due to the force balance between the magnetic pinching force and the relativistic electrostatic force. They showed that the core radius varies as $R_{core} \sim u_t R_l$ where $u_t$ is the terminal speed of the flow. This is plausible because, in the force-free situation, the core radius is determined only by equilibrium of the Lorentz force, i.e. the magnetic pinching force and the electrostatic force. The magnetic pinching force depends on the angular velocity $\Omega_e$ of the field line which also determines the light cylinder radius $R_{lv}$, and the electrostatic force depends on the terminal velocity $u_t$. Hence the core radius depends on $u_t$ and $R_{lv}$. On the contrary in our case, the inertial force (the centrifugal force) arising from the rotation of the flow around the rotational axis should be added to the force balance. The core radius depends also on the angular momentum $L$ which denotes the degree of the rotation and the injection rate $\eta$ which denotes the relative ‘weight’ of the flow to the Lorentz force. We should note that there is a remarkable difference between their works and this work. In their solution the jet extends to a finite cylindrical radius, and needs external pressure to confine the jet. So they impose a matching condition with the ambient medium on the outermost flux surface. However our solution is a self-organized structure as discussed above. This difference comes from whether imposing the matching condition (32) with the conical wind envelope or not.

5.2 Matching with the conical wind

We discuss the matching with the conical wind further here. In the hybrid model taken up here, we must slightly change the condition at the innermost flux surface of the conical solution. In Nitta (1994), we set the inclination of the innermost flux surface $\Psi = 0$ as $f(0) = \infty$ [see equation (28)]. In order to realize it, we should choose the terminal velocity on the innermost flux surface as $u_t(0) = 0$. However in the present model, the innermost surface is not $\Psi = 0$ but $\Psi = \Psi_\infty$. Hence the condition should be altered to $u_t(\Psi_\infty) = 0$.

As discussed before, the shape of the innermost flux surface of the conical solution and the outermost flux surface of the cylindrical solution are matched at the boundary $x = y = 1$ with $\theta = 0$. Both of these are vertical here. We now try to check the continuity of physical quantities. The magnetic flux function $\Psi$ and the velocity $u_t$ of the flow are continuous at the boundary [$\Psi = \Psi_\infty$ and $u_t(\Psi_\infty) = 0$]. We can choose the covariant component $B_\phi$ of the toroidal magnetic field to be continuous, because we can arbitrarily set $B_\phi$ at the innermost flux surface of the conical solution as $B_\phi = -\lambda$ (this is a free parameter of the conical solution). The continuity of $\Psi$ and $B_\phi$ guarantees the continuity of the poloidal and the toroidal magnetic field, and guarantees the continuity of the poloidal and the toroidal electric current via Ampere’s law. Similarly, the continuity of $B_\phi$ and $u_t$ denotes the continuity of the electric field and the continuity of the electric charge. Hence all of the electromagnetic quantities are continuous at the boundary. Next we check the continuity of the fluid quantities. The velocity $u_t = 0$ is continuous. The number density $n$ can be continuous if we choose the injection rate $\eta$ to be continuous from the definition of the injection rate $\eta = u_t/\beta$, because we have already checked the continuity of the velocity $u_t = (u_t)$ and the poloidal magnetic field $B_\psi$. We should note that the injection rate is a free parameter in the conical solution because this is a force-free solution (see Nitta 1994), hence we can choose $\eta$ continuously.

We can conclude that the cylindrical jet can be matched with the conical wind at this matching boundary.

5.3 Alfvén singularity and Mach number

Fig. 7 shows square of the Alfvén Mach number and the fast Mach number as functions of $x$ for the reference case. We can find that the flow is superAlfvénic everywhere at $Z = \infty$. As is well known, the G–S equation has a singularity at the Alfvén point. So the Alfvén singularity does not appear in this problem. We also notice that the superfast region is bounded at a finite cylindrical radius $x < 0.9 (R < 7 R_l)$ in Fig. 7. Outside this region, the flow is subfast as Chieu et al. (1991) predicted. The reason is that, as discussed in Section 2.4, the terminal speed of infinitely extended and current-carrying jet must behave as $u_t \rightarrow 0$ when $x \rightarrow 1$, while the fast mode propagation speed is finite because $B_x \rightarrow \text{finite}$. So the flow is subfast at the outer region.
Fact that the intermediate region of the observed jet fills the entire polar region discussed in Nitta (1994) fills the entire polar region (0 = 0) with \(Z = \infty\), i.e. \(\theta = 0\) where \(\theta\) is the polar angle from the rotational axis, and the conical wind envelope discussed in Nitta (1994) fills the entire asymptotic region 0 \(\leq y \leq 1\) (0 \(\leq Z \leq \infty\)) with \(x = 1\) (\(R = \infty\)), i.e. 0 \(< \theta < \pi/2\). It is very important how these structures at far distant region connect to the inner region of finite distance from the central object. In order to discuss the connection with the inner region, very complicated a two-dimensional calculation is intrinsically needed. The interest of the author as a future work is solving the entire magnetosphere as complete two-dimensional equilibrium with these results as an outer boundary condition.

### 5.4 Connection to the inner region

The model of cylindrical jet with conical wind envelope is attractive because this hybrid flow can occupy the entire region of infinitely distant region (see Fig. 1). The cylindrical jet discussed in this paper fills the entire polar region 0 \(\leq x \leq 1\) (0 \(\leq R \leq \infty\)) with \(y = 1\) (\(Z = \infty\)), i.e. \(\theta = 0\) where \(\theta\) is the polar angle from the rotational axis, and the conical wind envelope discussed in Nitta (1994) fills the entire asymptotic region 0 \(\leq y \leq 1\) (0 \(\leq Z \leq \infty\)) with \(x = 1\) (\(R = \infty\)), i.e. 0 \(< \theta < \pi/2\). It is very important how these structures at far distant region connect to the inner region of finite distance from the central object. In order to discuss the connection with the inner region, very complicated a twodimensional calculation is intrinsically needed. The interest of the author as a future work is solving the entire magnetosphere as complete two-dimensional equilibrium with these results as an outer boundary condition.

### 5.5 Discussion

We now discuss the properties of the structure of the entire magnetosphere. Our solution is designed to be an inner solution of the polar region (0 \(\leq R \leq \infty\) with \(Z \to \infty\)) which cannot be covered by the asymptotically conical solution (\(R \to \infty\)) discussed in Nitta (1994). As Chiueh et al. (1991) pointed out, this conical solution must involve a finite amount of poloidal current inside the innermost flux surface (the polar current). The distribution of this poloidal current is important to decide whether this conical solution is physical or not. However any asymptotic analysis (based on the limit \(R \to \infty\)) cannot clarify the structure of such region at finite cylindrical radius. Nitta (1995) shows the distribution of the poloidal current at finite radius in a restricted case (\(\Psi_c = 0\), \(L = u_c = 0\)). The present work is performed as more generalization to this.

Most of the parameters are still free in this work. If the connection to the inner region is clarified, these parameters should be restricted from the critical conditions to pass the critical points (the slow, the Alfven and the fast point). This restriction will give us important information of the magnetosphere. For example, from the view point of the force balance at the far distant region, the effects from the pressure seem to be negligible. This follows that the effects of the critical condition at the slow point seem to be negligible. However, as Kudoh & Shibata (1995) pointed out, the critical condition at the slow point in the hot wind will give us the information about the mass-loss rate.

Recent observation of the Crab nebula shows hybrid structure of the polar jet and the equatorial wind. Flows of YSOs also show hybrid of the thin optical jet and the high speed wind. In order to explain these hybrid structures, the model that the cylindrical jet embedded in the conical wind envelope will be one of plausible possibility.

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