The oscillatory shape of the stationary twisted disc around a Kerr black hole

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ABSTRACT

We consider the twisted accretion disc around a Kerr black hole and derive the stationary twist equation in the leading order on small parameters of relative thickness $\delta$, viscosity $\alpha$ and relativistic corrections $r_e/r$. We take into account post-Newtonian corrections which determine the shape of the stationary twisted accretion disc in the limit of small viscosity, $\alpha < \delta^{4/5}$. We find the phenomenon of radial oscillations of the disc inclination angle $\beta(r)$ due to this correction. The period of the oscillations is about $\Delta r \approx \delta^{-4/5} r_e$ and decreases with increasing distance $r$. We present a qualitative analysis of the oscillatory behaviour and conclude that the oscillations are likely to destroy a disc of low viscosity at sufficiently small $r < r_{\star} \approx \delta^{-3} \beta(\infty) r_e$. We suggest that the twisted disc might be in a highly turbulent state with $\alpha \approx 1$ at $r < r_{\star}$.

Key words: accretion, accretion discs – black hole physics – relativity – celestial mechanics, stellar dynamics.

1 INTRODUCTION

Since the first paper from Bardeen & Petterson (1975) the configuration of the twisted accretion disc around a Kerr black hole has been discussed by a number of authors (Petterson 1977, 1978; Hatchett, Begelman & Sarazin 1981; Papaloizou & Pringle 1983, hereafter PP; Kumar & Pringle 1985; Kumar 1990; Pringle 1992). PP pointed out that the equation describing the geometry of a twisted disc (the twist equation) derived by the previous authors was incorrect because it did not conserve angular momentum. As was shown by PP, the perturbations of the matter density and the velocities should be taken into account for the correct derivation of the twist equation. The terms corresponding to such perturbations appear in the twist equation in the main order due to a resonance in the system, and in conjunction with gravitomagnetic force determine the disc twist. In particular, the characteristic scale of the disc twist becomes smaller as the viscosity parameter $\alpha$ decreases, contrary to previous claims, and therefore in the case of a sufficiently low value of $\alpha$ the post-Newtonian corrections to the equations of motion should be taken into account (PP).

Here we consider the thin (with opening angle $\delta < 1$) twisted accretion disc around a Kerr black hole. We derive and solve the stationary twist equation, and take into account post-Newtonian relativistic correction. Solution of the stationary twist equation shows that the post-Newtonian correction leads to radial oscillations in the disc inclination angle $\beta(r)$ due to this correction. The period of the oscillations is about $\Delta r \approx \delta^{-4/5} r_e$ and decreases with increasing distance $r$. We present a qualitative analysis of the oscillatory behaviour and conclude that the oscillations are likely to destroy a disc of low viscosity at sufficiently small $r < r_{\star} \approx \delta^{-3} \beta(\infty) r_e$. We suggest that the twisted disc might be in a highly turbulent state with $\alpha \approx 1$ at $r < r_{\star}$.

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2 COORDINATE SYSTEM

We use the local cylindrical (twisting) coordinate system (Petterson 1977) in which the disc equations can be written in the simplest form. The position of disc element \( R \) is determined by the cylindrical coordinates \( r, \psi, \zeta \) related to an inclined cylinder of radius \( r \) (see Fig. 1). This cylinder is taken to be oriented perpendicular to the disc ring with the same radius. The cylinder orientation is characterized by two Euler angles: the inclination angle \( \beta \) and the precession angle \( \gamma \). The transformation law from Cartesian coordinate system \( x, y, z \) (where \( z \) coincides with the black hole rotational axis and \( x, y \) lie in the equatorial plane) to the twisting coordinate system can be written as

\[
\begin{pmatrix}
  r \\
  r \sin \psi \\
  \zeta
\end{pmatrix}
= B(\beta) C(\gamma)
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix},
\]

where \( B(\beta) \) and \( C(\gamma) \) are the rotational matrices

\[
B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{pmatrix}, \quad C = \begin{pmatrix} \cos \gamma & 0 & \sin \gamma \\ 0 & 1 & 0 \\ -\sin \gamma & 0 & \cos \gamma \end{pmatrix}.
\]

In our calculations we use the \(-y\) and \(x\) components of unit vector \( \xi \) perpendicular to the ring,

\[
\Psi_1 = \beta \cos \gamma, \quad \Psi_2 = \beta \sin \gamma
\]

instead of \( \beta \) and \( \gamma \), and the angle \( \varphi = \gamma + \psi \) instead of \( \psi \). This enables us to consider the case \( \beta = 0 \) where the twisting coordinate system becomes degenerate. We consider all vectors and tensors to be projected on to the orthonormal basis that is connected with the twisting coordinate system (Petterson 1977):

![Figure 1. The \( r, \psi, \zeta \) coordinate system shown with respect to the \( x, y, z \) system. Locally, it simply consists of a cylindrical coordinate system rotated with respect to the \( x, y, z \) system over angles \( \beta \) and \( \gamma \). \( \gamma \) is an angle between the axis \( OX \) and the line of intersection of the disc plane and the \( OXY \) plane. \( P \) is the projection of the point \( R \) on to the disc ring plane.](https://academic.oup.com/mnras/article-abstract/285/2/394/1151207)

3 BASIC EQUATIONS

The hydrodynamical equations projected on to the basis (4)–(6) reduce to the continuity equation

\[
(\rho \psi)_r = 0, \quad i = 1, 2, 3
\]

and the equations of motion of viscous fluid

\[
\rho (\psi v'_i + A') = \rho \Phi' - P'_i - t''_i + \rho F' - F'_i,
\]

where \( \rho, P, \psi \) denote respectively the density, pressure and velocity components, \( \Phi = M/r \) is the Newtonian potential and \( M \) is the black hole mass, \( \Phi' \) are the components of viscous stress tensor, and the semicolon denotes differentiation with respect to the basis (4)–(6). \( A' \) denotes post-Newtonian corrections to the convective term \( \psi v'_i \). \( F' \) is the gravitomagnetic force, which has only one relevant component:

\[
F'_i = \frac{4av_\text{vis} M^2}{r^3} (\Psi_1 \sin \varphi - \Psi_2 \cos \varphi),
\]

where \( a \) is the black hole rotational parameter.

Following PP we divide the matter density \( \rho \) and velocity components \( \psi, \psi \) into standard parts and perturbations,

\[
\psi = \psi_0 + \psi_1, \quad \psi = \psi_0 + \psi_1, \quad \rho = \rho_0 + \rho_1,
\]

where small perturbations \( \psi_1, \rho_1 \) denote \( \partial \beta/\partial r \) \( \cos (\varphi + \phi_0) \). Substituting equation (11) into the set of basic equations (8) and (9), and separating the terms with different dependences on \( \zeta \) and \( \varphi \), we obtain the sets of equations for the quantities with indices 0 and 1.

The set of equations for the quantities with index 0 is identical to the standard system for flat disc accretion. We use the relation

\[
\frac{\psi_0}{r} + \frac{\partial \Phi}{\partial r} = 0
\]

(12)

to determine the Newtonian \( \psi_0 \):

\[
\psi_0 = r \Omega_r
\]

(13)
where $O = M^2 r^{-3}$. The radial component of velocity $v_r$ can be taken from the equation of angular momentum transfer,

$$v_r = -\frac{3}{2} \frac{v}{r} , \quad (14)$$

where kinematic viscosity (Shakura & Sunyaev 1973)

$$v = \frac{x \zeta^2 \sigma}{r} \frac{v_0}{r} , \quad (15)$$

where $x$ is the viscosity parameter and $\zeta$ is the disc half-thickness. The surface density $\Sigma = \int \rho_0 \, d\zeta$ comes from the equation of mass conservation,

$$\frac{\dot{M}}{2\pi} = -\Sigma v_0 r , \quad (16)$$

where $\dot{M}$ is the mass transfer rate. We use the standard relation

$$\frac{\partial \Phi}{\partial \zeta} + \frac{\xi}{r} \frac{\partial \Phi}{\partial r} = -\frac{\zeta}{r^2} \frac{v_0^2}{r} \xi \sigma , \quad (17)$$

take the disc to be isothermal with density profile

$$\rho_0 = \rho_* \exp \left[ -\left( \frac{\zeta}{2\zeta_*} \right) \right] \quad (18)$$

and assume also that the gas pressure dominates over the radiation pressure. For this case we have approximately linear dependence of the disc thickness $\zeta_*$ on large $r$ (Shakura & Sunyaev 1973),

$$\zeta_* = \delta r , \quad (19)$$

where $\delta$ is the disc opening angle.

The quantities with index 1 come from the perturbed continuity equation (8) and the equations of motion (9). From the perturbed continuity equation (8) we have

$$\frac{\rho_0}{r} \frac{\partial v_{p1}}{\partial \varphi} + \Omega \frac{\partial \rho_1}{\partial \varphi} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho_0 v_{p1}) = 0 . \quad (20)$$

The perturbed radial component of equation (9) can be written as

$$\rho_0 \Omega \left[ \frac{\partial v_{p1}}{\partial \varphi} - 2 v_{p1} + A_{\varphi} \right] = -\nabla_P \rho + \frac{\partial t_{\xi}}{\partial \xi} , \quad (21)$$

and from the equation for the $\varphi$-component we have

$$\rho_0 \Omega \left[ \frac{\partial v_{p1}}{\partial \varphi} - 2 v_{p1} + 2 \frac{\partial}{\partial \varphi} A_{\varphi} \right] = -2 \frac{\partial}{\partial \varphi} \frac{\partial t_{\xi}}{\partial \xi} . \quad (22)$$

The $r$ component of the gas pressure gradient $\nabla_P \rho$ in (21) has to be taken into account due to the disc twist:

$$\nabla_P \rho = \zeta \tau W \Omega^2 \rho_0 . \quad (23)$$

The perturbed components of the viscous tensor $t_{\xi}$ and $t_{\varphi}$ in equations (21) and (22) are

$$t_{\xi} = -\eta \frac{\partial}{\partial \xi} v_{p1} , \quad t_{\varphi} = -\eta \frac{\partial}{\partial \varphi} v_{p1} , \quad (24)$$

where $\eta = \rho_0 v$.

The relativistic corrections can be written in the explicit form (see Appendix)

$$A_{\varphi} = \frac{M}{r} \left( \frac{3}{2} \frac{\partial v_{p1}}{\partial \varphi} + v_{p1} \right) , \quad (25)$$

$$A_{\varphi} = \frac{M}{4 \alpha} \left( \frac{3}{2} \frac{\partial v_{p1}}{\partial \varphi} - \frac{5}{4} v_{p1} \right) . \quad (26)$$

Note, that the system (21), (22) is degenerate in the leading order; therefore, the small viscous terms and the relativistic corrections should be retained in equations (21) and (22) to solve them.

Finally, we consider the perturbed $\xi$-component of equation (9). Integrating this component with respect to the vertical coordinate $\zeta$ we can see that the gravitomagnetic force $F_\xi$ is balanced by the $\xi$-component of Newtonian gravity force and $\xi$-component of perturbed pressure gradient $\partial P_\xi/\partial \xi$,

$$\int \frac{d\zeta}{r} \frac{\partial \Phi}{\partial \zeta} + \Sigma F_\xi = 0 , \quad (27)$$

where the perturbed pressure gradient term is omitted since this term vanishes after integration over $\zeta$. Equation (27) has the simple form of two forces balancing in the leading order on small parameters $M/r$ and $\alpha$ only. The corresponding equation in general case of a large value of $\alpha$ and time dependence was derived by PP and in the twisting coordinate system (Demiansky & Ivanov 1997).

4 TWIST EQUATION

To obtain the twist equation we proceed as follows. With the help of equations (7), (12)–(19) and (21)–(26) we find and solve the equations for the velocity perturbations. Using equation (20) we express the density perturbations in terms of perturbed velocities. Substituting the result into equation (27) we obtain the twist equation.

In detail, substituting (7), (12)–(19) and (23)–(26) into equations (21) and (22), and subtracting (22) from (21), we find

$$\frac{\xi}{r} \left( \frac{4 \alpha}{4 \alpha} \frac{\partial}{\partial \varphi} \frac{\partial v_{p1}}{\partial \varphi} - W v_{p1} \right) = \frac{12 M}{r^2} , \quad (28)$$

where in the leading order we have

$$\frac{\partial v_{p1}}{\partial \varphi} = 2 v_{p1} . \quad (29)$$

Solution of (28) gives

$$v_{p1} = -\frac{\xi v_{p0}}{4 \alpha} \left[ \frac{\partial W}{\partial \varphi} + kW \right] . \quad (30)$$

where $k = 3 M/\alpha r$ determines the relative role of relativistic corrections.

Now we express the perturbed density $\rho_1$ in terms of perturbed velocities with the help of continuity equation (20) and equation (29):
The twisted disc around a Kerr black hole

Figure 2. The dependence of the inclination angle $\beta/\beta(\infty)$ on radius $r/M$ for $\delta=0.1$, $a=1$ and for the different values of $\alpha$. Curves 1, 2 and 3 correspond to $\alpha=0.01$, 0.1 and 1 (for illustration). We have $R_{1}/M=20$ and $R_{2}/M=4.2$, 25, 90 for cases 1, 2 and 3 respectively. When $\alpha$ becomes comparable or less than $\delta$ and $R_{2} \gg R_{1}$, we obtain oscillations.

Substituting (30) into equation (31), and (31) into equation (27), and explicitly performing the integration over $\xi$ in (27) with help of (18), we obtain the twist equation,

$$\frac{1}{2\pi r^{2}} \frac{\partial}{\partial r} \left\{ 2\delta \Omega r^{2} \left( \frac{\partial W}{\partial \varphi} + kW \right) - 1 + k^{2} \right\} + \Sigma F_{2} = 0. \quad (32)$$

Transforming the functions in the brackets in equation (32) with the help of the equation (14)–(16), we find the twist equation in the final form,

$$\frac{\partial}{\partial x} \left\{ \frac{1 + i k \delta w}{1 + k^{2}} \right\} + \frac{32a}{\delta^{2}} w x = 0, \quad (33)$$

where we introduce new variables $w = \Psi_{1} + i \Psi_{2} = \beta \exp i \gamma$ instead of $W$ and $x = (r/M)^{-1/2}$. Note, that the analogous equation obtained in the previous works (PP; Kumar & Pringle 1985) does not take into account the relativistic correction term $[(1 + i k)/(1 + k^{2})]$.

5 ANALYTIC SOLUTIONS

The shape of twist equation (33) is determined by the relation between two small parameters $\alpha$ and $\delta$ (see equation 40 below). In general, the twist equation can be solved numerically (see Figs 2 and 3), but for the limiting cases of $\alpha \gg \delta$ and $\alpha \ll \delta$ the twist equation has analytic solutions.

First we consider the case of the large values of $\alpha$. In this case we can neglect the relativistic correction term $k$ in equation (33), and the analytical solution of (33) with boundary conditions $w(0) = 0$ and $w(r \to \infty) = e^{-i\gamma} w(\infty)$ is expressed in terms of Bessel functions:

$$w = (y_{1}/2)^{3} w(\infty) \Gamma(2/3) \left\{ J_{-1/3}(y_{1}) - e^{-i\gamma} J_{1/3}(y_{1}) \right\}, \quad (34)$$

where $J_{n}$ are the Bessel functions, $\Gamma$ is the gamma function, $y_{1} = (2/3) \exp (i\gamma/4)$, $C_{n^{2}} = 32a/\delta^{3}$, and $C = 32a/\delta^{3}$. This solution in the limit of $x \to \infty$ ($r \to 0$) is

$$w \approx K(r) \exp \left\{ (r/R_{1})^{-3/4} \right\} \left[ \gamma(\infty) - \frac{5\pi}{24} + \left( \frac{r}{R_{1}} \right)^{-3/4} \right], \quad (35)$$

where the coefficient $K(r) \approx \text{constant}$. The characteristic scale

$$R_{1} = \left( \frac{2C_{n^{2}}}{9} \right)^{3/2} M \approx 4.2a^{5/2} \delta^{5/2} \delta^{-4/3} M \quad (36)$$

describes two effects: the monotonic exponential decrease of the inclination angle $\beta$ with decreasing $r$ and the precession $\gamma(r)$ of the disc rings in the direction of black hole rotation (Bardeen & Petterson 1975).

When we consider the case of small $\alpha$ we neglect the unity in the equation (33) in comparison with $k$. The approximate solution of equation (33) drastically changes in this case: instead of the well-known effect of a smooth decrease of inclination angle $\beta$ we have the radial oscillations $\beta(r)$. The oscillating solution has the form

$$w = y_{2}^{1/2} \left\{ 2^{-2/3} \Gamma(2/3) w(\infty) \right\} \left. J_{-5/6}(y_{2}) + 2^{2/3} A_{5} \right\}, \quad (37)$$

where $y_{2} = (6/5) C_{s}^{5/2}$, $C_{s} = 32a/\delta^{3}$ and $A$ is some constant determined by the inner boundary condition. In the ordinary case when the factors affecting the disc inclination are determined by the large scales we can set $A = 0$. When $r \to 0$ we have

We consider the case $a > 0$ hereafter.

Figure 3. The result of the integration of the twist equation presented in a parametric form \( \Psi_1(r) = \text{Re} \psi, \Psi_2(r) = \text{Im} \psi \). The points A, B, C, and D correspond to the radii \( r/M = 2, 10, 100 \) and 1000 respectively. We take \( \delta = 0.1 \) below. (a) The case of \( \alpha = 0.01 \), corresponding to the oscillating curve 1 in Fig. 2. At small \( r \) the 'spiral' is strongly elongated (max \( |\Psi_1| \gg \max |\Psi_2| \)). The number of peaks (5) on curve 1 in Fig. 2 corresponds to the number of max \( |\Psi_1| \) and to the growth of \( \gamma = \arctan (\Psi_2/\Psi_1) \) on \( 5\pi \). (b) The same as (a) but \( \alpha = 0.1 \). The amplitude of oscillations is suppressed with respect to the previous case due to the growth of the viscosity. (c) The case of large viscosity \( \alpha = 1 \). The oscillations are damped.
\[ w = \left( \frac{2}{\pi} \right)^{1/2} \left( \frac{r}{R_s} \right)^{-1.8} 2^{-3/5} \Gamma (2/5) w(\infty) \cos \left[ \left( \frac{r}{R_s} \right)^{-5/4} + \frac{\pi}{20} \right], \]  
\[ \text{(38)} \]

where
\[ R_s = \left[ \frac{6^{4/5}}{5} \right] \left( \frac{2^5}{3} \right)^{5/5} a^{2.5} \delta^{-0.5} M \approx 3.1 a^{2.5} \delta^{-0.5} M \]  
\[ \text{(39)} \]

is the characteristic scale of relativistic oscillations.

From the condition \( R_s > R_1 \), we find that the twist disc oscillations take place when
\[ \alpha < a^{-5/2} \delta^{4/5}, \]  
\[ \text{(40)} \]

and solution (37) approximately describes the disc twist on any scales \( r < R_s \).

6 OSCILLATORY REGIME

Now we consider the origin of perturbations in the disc and clarify the physical meaning of the oscillations in the inclination angle. First we need to consider the trajectories of gas particles \( r(t) \) in the stationary disc (see Fig. 3). We rewrite the equations of motion (21) and (22) in the dynamical form using the standard leading order relation, \( \delta \partial t = \Omega (\partial \delta \phi) \).

Neglecting the viscous corrections in (21) and (22), we obtain
\[ \dot{\Delta} = -2 \omega_0 \Omega \left[ 2 - \frac{2M}{r_0} \right] = -\rho_0^{-1} \nabla P, \]  
\[ \text{(41)} \]

where
\[ \Delta = r - r_0, \]  
\[ \frac{\delta}{\partial t} = \Omega (\partial \delta \phi) \]  
\[ \text{(42)} \]

where \( \Delta = r - r_0 \) is the elliptical deviation of the trajectory from a circle \( \Delta = \nu \), and dot denotes the time derivative. All background quantities are taken at \( r_0 \) (we omit this index below), we use the explicit form for relativistic corrections (25) and (26) and equation (23) for the pressure gradient. Integrating (42) over \( t \) and substituting the result into (41), we obtain
\[ \dot{\Delta} = \left( 1 - \frac{6M}{r} \right) \Omega^2 \Delta = -\rho_0^{-1} \nabla P. \]  
\[ \text{(43)} \]

If one neglects the relativistic correction term \( 6M/r \) and the pressure gradient in (43), the trajectories are the stationary ellipses with small arbitrary eccentricities \( \Delta = e \sin (\Omega \Delta + \phi_0) \). In the absence of the pressure gradient term the relativistic correction leads to Einstein precession of such ellipses and to the absence of stationary orbits.

In the isothermal case the pressure gradient \( \nabla P \) is perpendicular to the lines of equal unperturbed density \( \rho_0 \) in the leading order. In the twisted discs these lines have some inclination to the particle orbit plane, therefore we should take into account the non-zero projection of the pressure gradient on to the orbit plane. In Fig. 4 we show the parallelogram-like cross-section of the disc element at radius \( r \) and angle \( \phi = \pi/2 \) by the ZOY plane. The disc surface (solid lines) is inclined with respect to the \( r \)-axis due to non-zero

\[ \Psi_1. \]  
\[ \text{The case shown, } \Psi_1 > 0, \Psi_1' = \beta' < 0, \text{ corresponds to the position of the pericentre of the trajectories at point } P \]  
\[ \text{(44)} \]

Thus the twisted disc consists of the ellipses with small eccentricities \( e \), which are the odd functions of height \( \xi \) and depend on \( r \) due to the disc twist \( r \Psi_1' \). The dependence of eccentricities on \( r \) leads to the density perturbation \( \rho_1 \). When the eccentricities increase with decreasing \( r \) the particle trajectories concentrate near the apocentres of the ellipses producing the density excess \( \rho_1 > 0 \) there, and deviate near the pericentre of the ellipses producing the lack of density \( \rho_1 < 0 \) (see Fig. 5a). In the opposite case of decreasing eccentricity with decreasing \( r \) we have \( \rho_1 > 0 \) near pericentre and \( \rho_1 < 0 \) near apocentre (see Fig. 5b).

Now we consider the force balance equation (27). Both terms in (27) depend on \( \phi \) identically. We consider force balance for the angle \( \phi = \pi/2 \), where we may have either the apocentre (at \( \xi \Psi_1' > 0 \)) or the pericentre (at \( \xi \Psi_1' < 0 \)). At this point \( F_x > 0 \) (when \( \Psi_1 > 0, \alpha > 0 \)), and we need to have the negative \( \xi \) projection of Newtonian force, \( F_{\xi} = \int d\xi P_{\xi} (\partial \Phi / \partial \xi) = -F_x \). Since the Newtonian acceleration \( \partial \Phi / \partial \xi \) is always directed to the central plane, \( \xi = 0 \), we
Finally, we estimate the $r$-dependence of $F_{\mathrm{grav}}$ and of $F_t$. From equations (17), (30) and (31) we have $F_{\mathrm{grav}} \propto C_1 r \Psi_1^0$ and from equation (10) $F_t \propto C_2 r^{-3.5} \Psi_1$, where $C_1$, $C_2$ are constant. Therefore we write the approximate expression for the balance equation in the form

$$\Psi_1^0 + C_2 r^{-4.5} \Psi_1 \approx 0,$$

where the ratio $C_2/C_1$ should be positive for the validity of the inequalities (21). Obviously we have the oscillations of the inclination angle with frequency $\propto r^{-4.5}$ sharply increasing with decreasing $r$ (as described by equation 58).

7 RESTRICTIONS

Solutions (34) and (37) are applicable only for sufficiently small values of initial inclination angle $\beta(0)$. The shear velocities (29) and (30) are proportional to $\beta^2$, increase with increasing of $\beta(\infty)$ and lead to a shear instability (Kumar 1990; Coleman & Kumar 1993) when

$$v_1 > \frac{\xi}{\xi^*} v_3,$$

where $v_1 = \sqrt{v_{\phi 1}^2 + v_z^2}$ and $v_3 = (\xi_*/r_{\mathrm{av}}$ is the sound velocity. In the case of large $x > \delta^2$ (see equation 40), we find that solution (34) is stable when

$$\beta(\infty) < 2 \sigma.$$

In the opposite case of small viscosity the shear velocities increase with decreasing $r$ due to oscillations (see equations 29, 30 and 37, 38), and always lead to instability at small radii:

$$r < r_1 \approx \left(\frac{R_2 \beta(\infty)^{1/3}}{6M}\right)^{1/3} < R_2 \approx \delta^{-2} \beta(\infty)^{1/3} M.$$

The relativistic oscillations begin when $r < R_2$, and we find from the condition $r_t < R_2$ that the oscillating disc is stable when the initial angle is small:

$$\beta(\infty) < 2 \delta^{4/5}.$$

The other restrictions come from stability analysis of the twisted discs in the case of $x < \delta$. Papaloizou & Lin (1995) showed that twist waves propagate over a Newtonian disc. In our case of a relativistic twisted accretion disc this effect also takes place (Demianski & Ivanov 1997). The effect of the persistent sources of these waves might prevent the disc relaxation to our stationary oscillatory solution (37) if the amplitude of the waves is greater than the amplitude of the oscillations.

8 DISCUSSION

We have derived the stationary twist equation for the twisted accretion disc around a Kerr black hole in the twisting coordinate system. We take into account the post-
Newtonian relativistic corrections to the equations of motion.

We have found the radial relativistic oscillations of the inclination angle when \( x < x_0 \) for the stationary twisted accretion disc with low viscosity.

In general, let us consider a low viscous twisted accretion disc in a slightly modified Newtonian potential and some force \( F \), acting perpendicular to the disc rings. The modification of the potential leads to the precession of the free particle orbit along an axis which is characterized by precession vector \( \omega_1 \), while the presence of \( F \) leads to the precession of the orbit plane which is characterized by precession vector \( \omega_2 \). We suppose that if the precessions corotate \( [(\omega_1 \cdot \omega_2) > 0] \) we have oscillations of the inclination angle. In the opposite case (for example the oppositely rotating black hole \( a < 0 \)) we have the smooth non-oscillating dependence on \( \beta \) on \( r \).

The shear velocities lead to the shear instability at any distances \( r \) when \( \beta(\infty) \cdot x \) for the large values of \( x \) and when \( r < r_0 \) for the low values of \( x \). This instability may generate the turbulence in the disc and effectively increases the \( x \) parameter. When \( v_s \approx v_t \) at the height of disc atmosphere \( \xi_s \), the increment of instability growth on scale \( \xi_s \) is comparable with the dynamical frequency \( \Omega \) (Coleman & Kumar 1993). The calculations (Canuto, Goldman & Hubickyj 1984) show that in this case the turbulent motion develops and the resulting value of the \( x \) parameter is about 1. Therefore the twisted accretion disc might be in a highly turbulent state with \( x \approx 1 \) at radii smaller than \( r_0 \). In this case the oscillatory shape of the disc is damped.

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APPENDIX

For the derivation of the post-Newtonian corrections (25) and (26) we should calculate \( r \) and \( \varphi \) components of 4-acceleration \( F = \nabla U \) in the corresponding approximation. We write the post-Newtonian metric in the form

\[
ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 + \frac{2M}{r}\right) dr^2 - d\xi^2 - r^2 d\varphi^2, \tag{A1}
\]

where we neglect the twisting of the coordinate system and the black hole rotation which are unimportant for our purpose.

Using (A1) we calculate the \( \varphi \) and \( r \) components of \( F \):

\[
F_\varphi^r = U \varphi \frac{\partial U^\varphi}{\partial \varphi} + \frac{U^r}{r^2} \frac{1}{\partial r}, \tag{A2}
\]

\[
F_r^\varphi = U^r \left(1 - \frac{3M}{r}\right) + \frac{U^\varphi}{r^2} \frac{\partial U^r}{\partial \varphi}. \tag{A3}
\]

From the normalization condition \( U^\varphi U^\varphi = 1 \) \( (\varphi = 0, 1, 2, 3) \) we have

\[
U^\varphi = 1 + \frac{M}{r} + \frac{r(U^\varphi)^2}{2}. \tag{A4}
\]

Substituting (A4) into (A3) we obtain

\[
F_r^\varphi = \frac{M}{r^2} \left(1 - \frac{3M}{r}\right) + \frac{U^\varphi}{r^2} \frac{\partial U^r}{\partial \varphi}. \tag{A5}
\]

As was done in the main text we divide \( U, F \) into a main part with index 0 (which describes the circular motion in the field of post-Newtonian source) and a perturbed part with index 1. From the equation \( F_r^\varphi = 0 \) we determine the motion of free particle on the circular orbit:

\[
U^\varphi_0 = \Omega \left(1 + \frac{3M}{r}\right). \tag{A6}
\]

For the perturbed part we have, in the linear approximation,

\[
F_r^\varphi_1 = U_0^\varphi \frac{\partial}{\partial \varphi} U_0^r + \frac{U_0^r}{r^2} \frac{1}{\partial r} r^2 U_0^\varphi, \tag{A7}
\]

\[
F_r^\varphi_1 = U_0^\varphi \frac{\partial}{\partial \varphi} U_0^r - 2U_0^\varphi U_0^r \left(1 - \frac{3M}{r}\right). \tag{A8}
\]

Projecting (A7) and (A8) on to an orthonormal basis, we obtain

\[
\omega^r = \left(1 - \frac{M}{r}\right) dr, \quad \omega^\varphi = \left(1 + \frac{M}{r}\right) dr, \quad \omega^\varphi = d\xi, \quad \omega^r = r d\varphi, \tag{A9}
\]

and substituting (A6) we obtain

\[
\tilde{F}_r^\varphi = \Omega \left[ \frac{\partial}{\partial \varphi} \tilde{U}_r - 2\tilde{U}_r^\varphi + A \right]. \tag{A10}
\]


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\( \dot{\Phi}_i - \Omega \left\{ \frac{\ddot{\Phi}_i}{2} + \frac{\partial}{\partial \phi} \dot{\Phi}_i + A^e \right\}, \) \hspace{1cm} (A11)

where

\[ A' = \frac{M}{r} \left( \frac{3 \partial \ddot{U}_i}{2 \partial \phi} + \ddot{U}_i \right), \] \hspace{1cm} (A12)

\[ A^e = \frac{M}{r} \left( \frac{3 \partial \dot{U}_i}{2 \partial \phi} - \frac{5 \dot{U}_i}{4} \right), \] \hspace{1cm} (A13)

and \( \dot{\Phi}_i, \ddot{U}_i \) correspond to the basis (A9). The equations (A10) and (A11) correspond to the equations (21) and (22) respectively.

In our calculations the post-Newtonian corrections appear only in the combination (see 21, 22 and 28)

\[ B = \dot{\Phi}_i - 2 \frac{\partial \dot{\Phi}_i}{\partial \phi} \Omega \left( A' - 2 \frac{\partial A^e}{\partial \phi} \right), \] \hspace{1cm} (A14)

therefore for our purpose it is sufficient to replace the components of the 4-velocity in (A12) and (A13) by their classical limit,

\[ \dot{U}_i \rightarrow v_i, \] \hspace{1cm} (A15)

and to obtain (25) and (26).