Large-scale magnetic fields in texture-seeded cosmological models

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ABSTRACT
After reviewing the problem of large-scale magnetic field generation, it is shown how the Harrison mechanism can generate magnetic fields on galactic scales or larger when protostructures have obtained angular momentum by the recombination epoch. In some models of large-scale structure formation there can be dynamically generated vorticity, or gravitational tidal torquing can cause the dark matter (DM) to acquire rms vorticity before the epoch of recombination. At recombination, the baryonic matter infalling in the DM potential wells acquires a net vorticity arising from dissipative effects in hydrodynamical shocks. For texture models, there is also some initial rms vorticity which contributes to the rms vorticity at recombination. An upper limit to the magnetic field thus generated by the Harrison mechanism in the context of the texture scenario of large-scale structure formation is calculated. Without re-ionization the seed magnetic field upper limit of $10^{-22}$ G falls short of the value required by an exponential dynamo mechanism. If, as is likely in the texture scenario, the Universe was re-ionized at high redshift the resulting seed field is compatible with the dynamo mechanism.

Key words: magnetic fields – cosmology: miscellaneous – large-scale structure of Universe – radio continuum: general – radio lines: general.

1 INTRODUCTION
Measurements of the Faraday rotation of polarized light coming from various astronomical objects have shown that there are large-scale magnetic fields in our Galaxy and in neighbouring galaxies. From the early polarization of starlight measurements (Hall 1949; Hiltner 1949) to the more recent observations of the Faraday rotation of pulsar emissions or other astrophysical sources of polarized emissions, the observational evidence of such fields is well established.

The study of the nucleonic content of cosmic rays hitting the earth requires that the primary protons underwent nuclear interactions consistent with having traversed 30 to 300 times as much matter as there is between us and the Galactic Centre. Alfvén & Fermi (1949) first discussed how a Galactic magnetic field of 10 μG could confine the cosmic rays in the Galactic plane and explain the composition of the cosmic rays. Their prediction is close to the observed 2–3 μG value for the large-scale Galactic magnetic field.

Lastly the presence of a magnetic field in protogalactic gravitationally collapsing gas clouds would explain how such clouds expel, through magnetic braking, the angular momentum which should prevent their collapse.

Such a magnetic field could have been much weaker in our past Universe. Aside from the gravitational collapse which increases the spatial density of field lines, an initial seed field can be amplified by hydrodynamical processes. Current models of Galactic magnetic field creation all require that their initial seed field be amplified by a magnetohydrodynamical process (Ruzmaikin 1990).

Any scenario of large-scale structure formation should predict some seed magnetic field uniform on a Galactic scale, the texture mechanism (Turok 1989) does this in a very natural fashion.

1.1 Outline
Section 2 describes how a magnetic field can be generated in a cosmological scenario where there is vorticity at the recombination epoch. Section 3 briefly describes the texture model of large-scale structure formation and quotes some needed results and equations. In Section 4, after introducing the formalism needed to calculate the rms angular momentum induced by tidal torquing in the texture model, the results of Sections 2 and 3 are used to calculate the magnetic fields induced in the texture model. Conclusions are given in Section 5.
1.2 Notation

In this article the equations are represented in the unit systems most natural to the epoch at which the physics occurs. When discussing physics happening around matter–radiation equality, it is natural to refer scales to that epoch, so the scale factor \( a(t) \) is normalized at matter–radiation equality: \( a(t_{eq}) = 1 \). Occasionally the more conventionally normalized scalefactor \( R \), defined as \( R/R_0 = 1/(1 + z) \), is used.

Comoving spatial distances, denoted by \( x \), are in units of Mpc. Two types of comoving coordinates are used. Spatial distances with the ‘eq’ subscript, such as \( x_{eq} \), denote comoving coordinates with the value they would have at matter–radiation equality, while \( x \) without the ‘eq’ subscript denotes the conventionally used comoving coordinates, where the value of the coordinate (in \( h_{eq}^{-1} \) Mpc) is evaluated at a redshift of zero (present).

Wavenumber variables \( k = 2\pi/x \) are always in comoving units of \( h_{eq}^{-1} \) Mpc\(^{-1} \); here again \( h_{eq} \) is referenced to matter–radiation equality, and \( k \) without a subscript is referenced to the present.

Time \( t \) for a comoving observer is quoted in the computationally convenient units of \( t_{eq} \), defined as \( 1 \) ls = \( 3t_{eq}/(2 - \sqrt{2}) \), where \( t_{eq} \) is the time of matter–radiation equality. For a model with density parameter \( \Omega = 1 \), and a Hubble constant \( H_0 = 50 \) km s\(^{-1} \) Mpc\(^{-1} \), \( 1 \) ls = \( 2.667 \times 10^{12} \) s. Conversion between scalefactor \( a \) and time \( t \) is done using

\[
\frac{t}{2 - \sqrt{2}} \left[ (a - 2)/\sqrt{1 + a} + 2 \right],
\]

where the prefactor inside the brackets has a value of \( 1/3 \) in ls time units. \( \tau \), the comoving conformal time referenced to equality, is defined as \( d\tau = dt/a \). Using \( t(a) \) from equation (1) yields

\[
\tau = \frac{3t_{eq}}{2 - \sqrt{2}} \left( \sqrt{1 + a} - 1 \right).
\]

The prefactor inside brackets being unity defines the ‘ls’ time units.

Finally variables with a subscript ‘i’ (e.g. \( a_i, \tau_i \)) refer to the initial value of that variable at the epoch when a given texture initially collapsed.

1.3 CDM model

Cosmological models are conveniently characterized by their power spectrum \( P(k) = \langle \delta_i \delta_{-k} \rangle \propto k^n \), the Fourier transform of the density–density autocorrelation function. In all following discussions standard CDM refers to a cosmology with a scale invariant Peebles–Harrison–Zeldovich \((n = 1) \) power spectrum, and a total density of \( \Omega = 1 \). Throughout this article the dark matter is denoted by DM, while baryonic and leptonic matter are referred to as ‘matter’.

For both CDM and the texture model the density parameter is \( \Omega = 1 \), with a DM fraction \( \Omega_{DM} \approx 0.95 \) and a baryonic fraction of \( \Omega_B = 0.05 \). A Hubble constant of \( 50 \) km s\(^{-1} \) Mpc\(^{-1} \) is used, to be somewhat compatible with data on the age of the Universe. For my purposes, the exact composition of the model makes little difference as long as the density of DM is at least a few times that of baryonic matter.

2 MAGNETIC FIELD GENERATION

There are roughly speaking two frameworks to generate magnetic fields on a galactic scale. In one framework the magnetic field is continually created during the early (and possibly late) life of a galaxy. For example supernovae or active galactic nuclei (AGN) (Daly & Loeb 1990) could generate a weak magnetic field that spreads to the entire galaxy. For a general discussion see the paper by Rees (1994). This framework is not studied in this article.

In the other framework of galactic magnetic field generation, the physics is separated into two eras. First a seed field \( B_{seed} \) is created before the protogalaxy collapses. In the second era the strength of the magnetic field is amplified first by flux conservation during the anisotropic collapse of a protogalaxy and then by some hydrodynamical mechanism such as a dynamo effect.

After the epoch of recombination, any magnetic field present is frozen in the gas filling the universe because this gas is a good conductor. Even long after recombination, the leftover gas has a residual equilibrium ionization \( X_e \sim 10^{-3} - 10^{-4} \) (obtained from the Saha equation) which gives an electron number density \( n_e = \rho_{hydro} (z < z_{rec}) X_e m_n \). The ionized gas conductivity is given by \( \sigma = n_e e^2 \tau_m \), where \( \tau_m = [(4\pi n_e)^2 3/(m_e k_B T_e)^2]/(4\sqrt{2\pi \lambda n_e e^4}) \) is the collisional time-scale, and the Coulomb logarithm \( \lambda \approx 10 \). The resulting conductivity is very high, and the displacement current \( \partial D/\partial t \) can be neglected in the Maxwell equation \( \nabla \times H = - \partial D/\partial t + J \). Taking the curl of that equation, and using \( \nabla \times E = - \partial B/\partial t \) with \( D = -E, B = \mu H \) and \( J = eE \), gives the diffusion equation for the magnetic field \( \nabla^2 B = - (\mu e^2)/(\partial B/\partial t) \). Let \( L_{diff} \) be a diffusion length and \( t_{diff} \) a typical time-scale for diffusion. Dimensional analysis yields that \( t_{diff} \propto \sigma L_{diff}^2 \). Taking a lower limit of \( T_{eq} = 2.75 K (1 + 2)/(1 + z_{rec}) \), \( n_e = 4\pi \times 10^{-7} \) H m\(^{-1} \) and \( L_{diff} \) in units of kpc, yields \( t_{diff} \approx 1.41 \times 10^9 L_{diff}^2 [(1 + 2)(1 + z_{1076})] \) s which is much greater than one Hubble time \( H_0^{-1} = 3.856 \times 10^9 h_{100}^{-1} \) s.

There are two different magnetic field amplification mechanisms that are commonly referred to as dynamo. For an introduction to the dynamo theory see (Moffatt 1978, 1980, 1989; Parker 1979). The first model, often wrongly called linear dynamo, obtains field amplification through differential rotation of the disc. In the second model, often called exponential dynamo, or Parker–Ruzmaikin–Vainsteil dynamo, or simply dynamo, hydrodynamical processes amplify the magnetic field on small scales. Reconnection of this random small-scale field yields a galactic magnetic field with a large-scale coherent component.1

The dynamo amplification mechanism occurs in Galactic objects. The flux is amplified through processes which make flux lines buckle and form loops arcing above the Galactic plane. These loops twist 180°, rejoin, and fall back on the Galactic plane, increasing the flux line density. For example, a cloud of gas can rise as a result of buoyancy forces, it then expands due to the lower density of gas above the Galactic plane. As it expands, the coriolis force makes the cloud rotate, twisting the lines of flux as they are frozen.
is a topic of some controversy as to whether this yields an exponential or a linear increase of the magnetic field (Kulsrud 1989; Kulsrud & Anderson 1992; Parker 1992).

The differential rotation mechanism produces much weaker amplification than the dynamo mechanism but is much less controversial. Exponential dynamo is popular because to obtain the current value of the Galactic magnetic field (3 $\mu$G) requires seed field values of only about $10^{-16}$ to $10^{-15}$ G by the time the galaxy has finished its collapse. On the other hand a linear dynamo requires fields of at least $10^{-12}$ G which are very hard to generate without delicate adjustment of large-scale structure generating models, such as in the inflationary scenario of Ratra (1992).

Generically, seed field creation can happen in three possible epochs. The first one is when the magnetic field has a primordial origin. The last epoch happens once the structure formation is well advanced. It involves magnetic field expelled from powerful beamed radio sources, such as the models of Daly & Loeb (1990). The other intermediate epoch, which this article concentrates on, involves dynamical phenomena around the recombination epoch.

### 2.1 Harrison scenario

If there is some residual primordial vorticity by the epoch of recombination, such as could happen in chaotic cosmology, magnetic fields can be generated using the following mechanism first mentioned by Harrison (1970, 1973).

Before recombination the electron motion is tied to that of the cosmic background radiation (CBR) photons through the large electron Thompson cross-section, $\sigma_T^e$. The Thompson cross-section of the proton, $\sigma_T^p$, is much less important than the hadronic neutron–proton cross-section. As a result, while the electrons move with the photon fluid, the protons coupled with the neutrons act as a separate fluid.

Implicit in the derivation of the Harrison mechanism is the existence of primordial vortices. Let $\rho$ be the physical radius (not in comoving coordinates) of such an eddy. Let the density be denoted by $\rho$ and the angular velocity by $\omega$. Thus $\rho_{\rho}$ and $\omega_{\rho}$ are the density and angular velocity of the radiation. Proton-related quantities bear the ‘p’ subscript, electron-related quantities ‘e’, and matter-related quantities ‘m’.

For an eddy that follows the expansion $\rho_0 r^3$ and $\rho_0 r^{4\gamma}$ remain constant for matter and radiation respectively because the amount of matter and entropy in one comoving volume remains constant. So the angular moments of the matter, $\rho_p \omega_{\rho} r^3$, and the angular momenta of the radiation, $\rho_e \omega_{\rho} r^4$, are separately conserved. Therefore $\omega_p \propto 1/r^2$ and $\omega_e \propto 1/r$. This means that electrons and protons acquire different angular velocities, leading to a circularly rotating electric field and therefore a poloidal magnetic field. The value of this field gets frozen in the gas at recombination and its strength scales as $(r_{rec}/r)^2$, where $r_{rec}$ is the size of the eddy at recombination and $r$ is the size at any later instant of time.

The detailed kinematical analysis can be found in Appendix A. The final result is that the magnetic field just after recombination is

$$B_{rec} = \frac{-2\mu_0 \rho_e}{3} \omega(r_{rec}) = (-7.810^{-17} \text{ G ls}) \omega(r_{rec}) (\text{ls}^{-1}) \Omega h_{70}^2,$$

where $\omega(r_{rec})$ is the angular velocity of a protogalaxy at the redshift of recombination $z_{rec}$. In an expanding universe, the initial vorticity ($\zeta = \text{V} \times \text{v}$) perturbations decay as

$$\zeta(a) \sim 1/a^2.$$

This means initial vorticity is washed away unless the Universe was very chaotic when it started. Limits on the amplitude of primordial density perturbations by the COBE satellite rule out this possibility unless the Universe was re-ionized.

### 2.2 Harrison-like scenario

The above Harrison scenario fails to pass the observational tests because it depends on primordial vorticity. Using instead dynamically generated vorticity, the Harrison mechanism can still be used.

Before decoupling, the Jeans mass is larger than the mass within the horizon, thus baryonic density perturbations cannot grow. While the pressureless non-baryonic DM fluctuations are not pressure supported, the gravitational interaction with the baryons and photons slows down the growth of the DM to yield only a logarithmic growth if the DM fluid was initially expanding with the background. The DM fluid can collapse early if the density perturbation has some initial peculiar velocity or if the spectrum is such that the baryonic density perturbations have mass greater than the Jeans mass. In models where the DM starts collapsing before recombination, rms vorticity can be generated by the tidal torquing mechanism derived in Section 4.

Even in models where the DM only has rms vorticity, the gravity potential of the collapsing spherical DM perturbations has oblate or prolate modes. At decoupling, the acoustic waves in the baryonic matter, which form as the superhorizon density modes enter the horizon, undergo oblique shocks because they are falling in the non-spherical gravitational potential of the DM. Dissipative effects in the shocks create a net vorticity in the matter at the recombination epoch.

Thus in models where the DM acquires a collapsing mode before recombination a magnetic field can be generated by the Harrison mechanism. Additionally for large-scale structure formation models based on topological defects there can be vorticity or rms vorticity dynamically generated at all times. Even after the decay characterized by equation (4), the topological-defect-seeded density perturbations may still have significant vorticity or rms vorticity left at the recombination epoch.
3 LARGE-SCALE STRUCTURE TEXTURE SEEDS

Topological defects as a source of large-scale structure were first discussed by Kibble (1976). Later Turok (1989) showed that global texture could occur naturally in grand unified theories (GUT) and produce interesting density perturbations in the early Universe. See Turok (1988) and Gooding (1991) for reviews.

An SU(2) symmetry seems the most simple particle physics inspired model. A complex Higgs field could have a global SU(2) symmetry. For computational purposes it is easier to use an SO(4) four-component isovector with real fields, so let us consider a GUT with a global SO(4) symmetry.

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \Phi^\dagger) (\partial^\mu \Phi) - \frac{\lambda}{4} (\Phi^2 - \Phi^0)^2 \]  

(5)

with

\[ \Phi = \frac{1}{\sqrt{2}} (\Phi_1, \Phi_2, \Phi_3, \Phi_4) \]  

(6)

where the Lagrangian is invariant under the global symmetry transformation \( \Phi \rightarrow \exp(iz) \Phi \) with \( z \) a constant. In the ground state \( \Phi \) develops an expectation value, \( \langle 0 | \Phi | 0 \rangle = (\Phi_1, 0, 0, 0) \), which breaks the symmetry of the Lagrangian. Since the one-direction is arbitrary, fields in regions not in causal contact point in different directions. As various regions come into causal contact they may form a field configuration known as texture. The four components of a spherically symmetric solution are

\[ \Phi = \eta (\sin \chi \sin \theta \cos \phi, \sin \chi \sin \theta \sin \phi, \sin \chi \cos \theta, \cos \chi) \]  

(7)

with \( \chi (r=0) = 0, \chi (r = + \infty) = \pi, \) and \( \eta = |\Phi_0| \). To visualize this hedgehog solution start by visualizing a set of outward pointing arrows laid out on the surface of a sphere. Now add shells, but at each successive cell the vector in \( R^3 \) appears smaller as it rotates in a fourth dimension. At some point one shell has zero length vectors, the next shell has a vector pointing inward. The layering of shells continues until the vector reaches its maximum length again, that is a \( \pi \) rotation of \( \chi \) in \( R^3 \).

This large size field configuration is unstable by Derrick's theorem (Derrick 1964), which states that no static soliton solution exists in three or more dimensions.\(^3\) Thus the energy stored in the gradients of the field configuration collapses to the centre. At that point the energy density is such as to allow the texture field configuration to topologically unwind over the potential. This leaves the field aligned within the current Hubble volume. As the Hubble radius grows, other texture field configurations may appear and collapse. Each time a texture collapses, the energy of the field configuration concentrates at the centre of the texture producing a rapid gravitational attraction toward the centre of the texture. The resulting radial velocity kick can be approximated by a ramp velocity profile

\[ v = \epsilon c \left( 1 - \frac{x}{c \tau} \right) \]  

(8)

where \( \epsilon = \frac{8nG\Phi^0}{c^2} \) is normalized to COBE, and \( c \tau \) is the comoving horizon size. The velocity kick is calculated for an ideal spherically symmetric texture collapse. Numerical textures simulations show that most textures are not perfectly symmetric and that a small azimuthal velocity is also imparted to the matter surrounding the texture. This small rms azimuthal velocity from the initial collapse produces no net vorticity because of the Kelvin circulation theorem for pure gravity.

The radial velocity kick allows an unfolding texture to produce a collapsing density perturbation even though the matter contained within the horizon at that epoch is lower than the Jeans mass before decoupling. Of course the baryonic component of that collapsing mode is pressure supported and is damped after a few acoustic wave oscillations, but the pressureless DM component can keep on collapsing.

The probability that a texture collapses is

\[ \frac{dn}{d\tau} = \frac{\dot{\epsilon}}{\tau^2} \]  

(9)

where \( n \) is the number of texture per comoving volume, \( \tau \) is the conformal time, and \( \dot{\epsilon} = 0.04 \), found by analytical triangulation argument (Turok 1989) and later confirmed by numerical simulations.

3.1 Power spectrum

There are two computations of the power spectra. Gooding (1991) found the linear theory solution for the growth of density perturbations with equation (8) as initial conditions. Using that solution combined with equation (9), they computed the linear theory power spectrum induced by textures at any era. This expression is particularly good at early times, but is expected to be inaccurate at late times as a result of non-linear effects at small scales.

Pen, Spergel & Turok (1993) calculated from numerical simulations the power spectrum \( P_{\text{in}} \) caused by texture as it would look today. They give a fitting function which I rewrite as

\[ P_{\text{in}}(k) = \frac{890(\text{Mpc}^2 h_{\text{eq}})^2 k}{V} \left[ 1 + 0.74 k / k_p + 0.13232 (k / k_p)^{1.5} \right] \]  

(10)

To find the comoving power spectrum at other epochs I assume that the spectrum can be factorized as \( P(k, a) \propto T(k, a) k^2 D^2 (a) \), with \( n = 1 \) for textures. The \( k^2 \) factor is the scale invariant primordial spectrum. The \( D(a) \) factors are the linear theory growth factors of density perturbations valid in both the matter and radiation era from Peebles (1980). There are two factors of \( D \), because \( P(k, a) \propto \langle \delta^2 \rangle \). \( T(k, a) \) is a transfer function that dampens modes that have become non-linear. The value of this function today is the denominator of equation (10).
with \( k_p = 0.525 \ h_z^2 \) being the wavenumber of the peak of \( T(k) \). I assume that this transfer function is self-similar, and that the location of peak of that function is constant in comoving coordinates.

Thus the numerical power spectra can be extrapolated using

\[
P(k_{\text{comov}}, a) = \frac{P_{\text{in}}(k)}{V} \frac{D(a)^2}{D(a_{\text{today}})^2}.
\]

Additionally the spectrum is truncated for wavenumbers with \( k < 2\pi/\text{horizon} \).

The two spectra agree reasonably well, within a factor of a few, and can be taken as the two limiting forms of the true power spectrum.

4 MAGNETIC FIELD BY TEXTURES

For textures to produce a magnetic field, they need to produce perturbations with non-zero angular momentum by the epoch of recombination. While some residual asphericity is caused by a texture unwinding event, let us study the asphericity induced by tidal torquing.

4.1 Angular momentum

The acquisition of angular momentum has been discussed by many authors, starting from the early suggestion of Hoyle and then later in the work of Peebles (1980) up to the currently accepted picture (Barnes & Efstathiou 1987; Ryden 1987). What follows is inspired from Ryden (1987) who computed the tidal torquing for a standard CDM model with a Gaussian spectrum.

Consider the evolution of shells centred on a spherical perturbation. A given shell contains a fixed amount of matter and shells do not cross at least until they first collapse to the centre of the perturbation. Consider a coordinate system centred on a particular texture. Label shells of constant mass by their initial comoving position \( x \). The gravitational comoving torque induced on a shell by all the surrounding matter of density \( \rho \) is

\[
\tau(x) = -G \left[ \int_{\text{shell}} \rho(s) s \times \nabla \Phi(s) \, ds \right],
\]

where

\[
\rho(s) = \rho_0 (1 + \delta(s)) [1 + \epsilon(s)].
\]

All the non-sphericity of \( \rho(s) \) is in \( \epsilon(s) \) for \( s \) much smaller than the size of the texture (i.e. the other earlier textures that collapsed within the volume of the currently collapsing texture provide the asphericity). Note that in comoving coordinates the comoving matter density of the background is constant.

\( \Phi \) is the gravitational potential caused by other distant textures. It can be expanded in spherical harmonics \( Y_{lm}(\theta, \phi) \) as

\[
\Phi(s) = \sum_{l=2}^{\infty} \frac{4\pi}{2l+1} \sum_{m=-l}^{l} a_{lm}(s) Y_{lm}(\theta, \phi) s^l,
\]

where

\[
a_{lm}(s) = \rho_0 \int_{s_{\text{outside shell}}} Y_{lm}(\theta_1, \phi_1) [1 + \delta(y)] \epsilon(y) y^{l-1} \, dy.
\]

Substituting equation (14) into equation (12) and using the fact that

\[
s \times \nabla [s^l Y_{lm}(\theta, \phi)] = is^l L Y_{lm}(\theta, \phi),
\]

where

\[
L = i(\mathbf{x} \times \nabla) = \hat{x}(L_+ + L_-) + \hat{y}(L_- - L_+) + zL_z
\]

is the angular momentum operator familiar from quantum mechanics. The \( z \) component of the torque is then

\[
\tau_z = -iGM_{\text{shell}} \rho_0 \sum_{l=2}^{\infty} \frac{x^l}{2l+1} \sum_{m=-l}^{l} m a_{lm}
\times (x) \int_{s_{\text{shell}}} \epsilon(s) Y_{lm}^*(\theta, \phi) \, d\Omega,
\]

where \( M_{\text{shell}} \) is the mass of the shell.

Let us introduce the multipole moments of the shell

\[
q_{lm}(x) = \int_{s_{\text{shell}}} Y_{lm}^*(\theta, \phi) s^l \rho(s) \, ds
\]

\[
= \frac{x^l}{4\pi} M_{\text{shell}} \int_{s_{\text{shell}}} Y_{lm}^*(\theta, \phi) \epsilon(s) \, d\Omega,
\]

where the right-hand side form is reached after substituting equation (13) and keeping only the non-symmetric part. \( \tau_z \) can now be rewritten as

\[
\tau_z = -i4\pi G \sum_{l=0}^{\infty} \frac{1}{2l+1} \sum_{m=-l}^{l} m a_{lm}(x) q_{lm}^*(x).
\]

The mean torque on a shell should be zero by Kelvin’s theorem, but the rms torque on a shell is non-zero. Keeping only the first contribution to rms torque, the quadrupole term \((l=2)\), yields

\[
\langle |\tau|^2 \rangle = 3 \langle |\tau_z|^2 \rangle = 3 \left( \frac{4\pi}{5} \right) \frac{G}{2} \sum_{m=-2}^{2} \sum_{n=-2}^{2} \sum_{m-n} \langle a_{2m}(x) a_{2n}^*(x) q_{2m}^*(x) q_{2n}(x) \rangle.
\]

with

\[
\langle a_{2m}(x) a_{2n}^*(x) q_{2m}^*(x) q_{2n}(x) \rangle = \left( \frac{M_{\text{shell}}^2}{4\pi} \right)^2 \int dy_1 \int dy_2 \int_{x_1} dy_3 \int_{x_2} dy_4
\times \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \langle \delta_1 \delta_2 \delta_3 \delta_4 \rangle Y_{2m}(1) Y_{2n}(2)
\times Y_{2m}^*(3) Y_{2n}^*(4)
\]

The final form is reached under the assumption that the four-point correlation function can be written as

\[
\langle \delta_1 \delta_2 \delta_3 \delta_4 \rangle = \langle \delta_1 \delta_2 \rangle \langle \delta_3 \delta_4 \rangle + \langle \delta_1 \delta_3 \rangle \langle \delta_2 \delta_4 \rangle + \langle \delta_1 \delta_4 \rangle \langle \delta_2 \delta_3 \rangle.
\]
I am thus neglecting the reduced four-point correlation function, which is zero for a random Gaussian process (Peebles 1980). The texture model is non-Gaussian, but the contribution from the reduced four-point correlation function is smaller than the contribution from the two-point functions.

After using the definition of the comoving power spectrum in terms of the two-point function,

\[
\int d\Omega_1 d\Omega_2 Y_{2m}(1)^* Y_{2n}(2) \int d^3k P(k) \times \int d^3k \frac{1}{V_{\text{com}}} e^{-ik \cdot x} Y_{2m}(1)^* Y_{2n}(2) d\Omega_1 d\Omega_2,
\]

the expansion of a plane wave in spherical harmonics

\[e^{-i k \cdot x} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} j_l(kr) Y_{lm}(\theta, \phi) Y_{lm}(\theta, \phi),\]

where \(\theta, \phi\) and \(\theta', \phi\) are the orientations of the vectors \(x\) and \(k\) respectively, and the orthogonality relations of spherical harmonics, one gets

\[
\tau(x) \equiv \tau(\mathbf{x}) = \sqrt{30} \frac{4\pi G}{k^2 \int dk P(k, \tau) [j_1(kr)]^2},
\]

where

\[
\langle a_{2m}(x) \rangle = \frac{\pi}{(4\pi)^2 V_{\text{com}}} \int k^2 dk P(k, \tau) [j_1(kr)]^2,
\]

\[
\langle a_{2m}(\mathbf{x}) \rangle = \frac{2n(n+1)}{nV} \rho_0 \int dk P(k, \tau) [j_1(kr)]^2,
\]

\[
\langle a_{2m}(x) a_{2m}(\mathbf{x}) \rangle = \frac{\rho_0 \tau M_{\text{shell}}}{2\pi^2 V_{\text{com}}} \int k^2 dk P(k, \tau) [j_1(kr)]^2 j_1(k|x|^2),
\]

and \(j_n(kr)\) is the spherical Bessel function of \(n\) th order. The lower limit of the \(k\) integral is limited by the horizon, and the upper limit by the smallest scale where dissipative effects can be neglected (typically a fraction of the texture horizon scale). The volume factor in equation (26) exactly cancels out the volume factor of the power spectrum. For the sake of conciseness the \(\tau, \tau\) dependence was omitted.

The comoving torque is related to the physical space torque by \(\tau_{\text{phys}} = \tau_{\text{com}} / R^2\) for comoving coordinates referred to today, and by \(\tau_{\text{phys}} = \tau_{\text{com}} / a^2\) for comoving coordinates referenced to equality. The rms angular momentum of a shell initially at comoving position \(x_0\) is given by the time integral of the torque

\[
J(x_{0}, a_{i}) = \int_{x_{0}}^{x_{i}} \frac{\tau_{\text{com}}}{R^2} dt,
\]

the rms torque being evaluated at the time-evolving position of the shell \(x(x_{0}, a)\) for a shell initially at comoving position \(x_0(a_i)\), at scalefactor \(a\) from a texture seeded at \(a_i\). The position of the shell is followed using the spherical collapse model as in Gooding (1991). Changing the integration variable from \(t\) to \(a\) in equation (30) and dividing \(J\) by \(M_{\text{shel}} a^2\) yields the rms angular velocity acquired

\[
\omega(x, a, a_i) \equiv \frac{J(x, a, a_i)}{M_{\text{shel}} a^2} = \frac{3\epsilon_{\text{eq}}}{2(2 - \sqrt{2}) x(a, a_i)^2 M_{\text{shel}}}
\]

\[
\times \int_{a_{\text{eq}}}^{a} \frac{da'(a', a_i) a'(a', a_i)}{\sqrt{1 + a^2 a'^2}},
\]

where the \(M_{\text{shel}}\) factor cancels exactly the mass factor in equation (26), making equation (31) independent of the mass of the shell.

Thus the texture power spectrum equation (11) can be used to figure out the rms torque, and then the rms angular momentum produced by gravitational torquing. In the texture model, DM shells can collapse before recombination, even in the radiation era. While the spherical collapse model can be used in the matter era, collapse is modelled in the radiation era by assuming a purely linear behaviour

\[x(x_{0}, a, a_i) = [1 + \delta(a, a_i)_{\text{DM}}]^{-1/3}.
\]

For the evolution of DM density perturbation \(\delta(a, a_i)_{\text{DM}}\), I use the analytic solution in Gooding (1991). Once the shell enters the matter era, the position of the shell is tracked using the spherical collapse model (Peebles 1980) after matching the peculiar velocity of the collapsed shell, the time derivative of equation (32). The integral for the torque runs only until the turn-around time of a shell, or until recombination, whichever happens first, because very little acquisition of angular momentum occurs after this point.

To convert the rms vorticity into vorticity, the baryonic fluid should undergo dissipative processes or shocks. While the DM cannot undergo dissipative processes, it is still possible to convert the angular momentum from the DM shell into vorticity for the matter.

As new modes of the density fluctuations enter the horizon, the matter mode grows until it is pressure supported, and undergoes a few cycles of dissipative acoustic oscillations (Peebles 1980). It is these shock waves, oscillating in the rotating aspherical gravitational potential, that transfer the angular momentum to the matter. Obtaining an accurate calculation of the efficiency of vorticity transfer from this highly non-linear phenomenon will have to await future developments in non-linear hydrodynamics codes.

4.2 Numerical results

To compute the magnetic field requires knowing how efficiently the DM transfers its angular momentum to the matter component. Assuming 100 per cent conversion, the matter acquires an angular velocity of \(\omega\).

For reference our Galaxy has a rotational velocity of 250 km s\(^{-1}\) at a radius of 10 kpc (mid-radius), which is \(\omega \sim 2.16 \times 10^{-4}\) in my units. For a galaxy-sized texture perturbation I calculate \(w \sim 1.2 \times 10^{-5}\) at recombination for the fitted and linear power spectrum respectively, with a nearly solid-body constancy over a range of \(x\), with a time evolution of \(\omega \sim a^{-0.8}\). Using equation (3) yields a magnetic field of \(B \sim 1.2 \times 10^{-20}\) G. At that early epoch, the density
of the protogalaxy is actually bigger than that of the final object. Upon complete collapse of the spheroid to the density and shape of a galaxy, flux conservation dictates that the field value to be multiplied by the windup factor \((P_\text{initial}/P_\text{galaxy})^{2/3} \sim 10^{-2}\), yielding a seed field of \(1 \times 10^{-22} \text{ G}\) for galaxy-sized perturbations. In fact the calculated magnetic field is nearly constant for structures up to \(M = 10^{14} \text{ M}_\odot\).

In the texture model there is also some rms angular velocity resulting from the non-radial rms velocity induced by the non-central field gradients during texture collapse. This rms angular velocity is directly induced in the matter, the DM, and the photons, so that the standard Harrison mechanism can be used. The generated orthonal velocity is typically a small fraction of the radial velocity. I will approximate it by

\[ v_\perp = \frac{\eta(x)}{x} \frac{c}{\tau}, \]  

(33)

where \(|\eta(x)| \ll 1\) can be obtained from numerical simulations and is typically smaller than 0.02. This creates an initial angular momentum per unit DM mass \(J/M_\text{bary} = r v_\perp\) at scale factor \(a\). At recombination, this will create an angular velocity of \(\omega = J/M_\text{rec} r^2 = a v_\perp/a x\), since \(r = ax\). Thus at recombination there will be a rms angular velocity of

\[ w_\text{rms} = \frac{v_\parallel(x)}{x} \frac{a^2}{a_\text{rec}} \]  

(34)

from the texture collapse. At mid-comoving radius \((x = c\tau/2)\), a texture seeded at \(a = 0.032\) (galaxy sized) has a leftover rms velocity at recombination \((a_\text{rec} = 5)\) of \(\omega \sim 0.032 \mu 25 (\sqrt{1 + 5 - 1} = 10^{-2} \mu \text{ s}^{-1}\). Assuming 100 per cent conversion of this rms angular velocity into angular velocity and \(\mu = 0.01\), this yields an upper limit magnetic field of \(7.8 \times 10^{-20} \text{ G}\) with residual rms vorticity after using Equation (3) and multiplying by the windup factor.

In fact, during the field re-ordering there are many field gradients sloshing about not leading to full texture collapse. These field gradients cause rms bulk flows which will lead to magnetic fields outside the site of texture seeds. The stronger of these fields are generated near recombination and are horizon sized. A similar calculation to that above yields a field of \(10^{-22} \text{ G}\) at recombination. In this case this field is not necessarily connected with structures, thus it is not multiplied by the windup factor, but instead by \((1 + z_\text{rec})^{-2}\). This predicts an upper limit field of \(10^{-22} \text{ G}\) in some zones in intergalactic space today.

5 CONCLUSIONS

The creation of a seed field even as small as \(10^{-18} \text{ G}\), required for the exponential dynamo, is a non-trivial requirement for a model of large-scale structure formation. The upper limit I compute for the magnetic field, \(2 \times 10^{-20} \text{ G}\), falls short of the modern estimates of the required seed field. Although the texture model fares better in this problem than the standard CDM model which does not produce any rms angular momentum by recombination, it does not pass this test. On the other hand the texture model could still produce magnetic fields through early AGN. This would be more effective in the texture model than in CDM because the structures form earlier, and it is possible for early AGN to have generated the seed field in protogalaxies. In addition, if the Universe re-ionizes at high redshift, highly probable in the texture scenario, the Harrison mechanism I use would produce a much higher magnetic field since by then much more angular momentum is acquired by the texture shells.

The scenario I have developed shows us that any model that yields non-baryonic collapsing density perturbations by the epoch of recombination can produce magnetic fields. Models with isocurvature fluctuations with a lot of power on small scales, such as the PBI model, could create magnetic field using the mechanism in this paper because they too lead to the formation of early structures which can acquire a significant vorticity before recombination. The PBI model has so much more small-scale power that the rms angular momentum would be much more important than in the texture model. That model would also lead to early re-ionization which would lead to an even larger magnetic seed field.

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APPENDIX A: DERIVATION OF THE EQUATION FOR THE HARRISON MECHANISM

Here I derive the equation for the Harrison mechanism. The expanding early Universe is treated as four different interpenetrating fluids interacting with each other. The four fluid components are the electrons, protons, photons (γ) and neutrals (neutrons).

The standard convective derivative of fluid dynamics for the jth fluid element in a fluid is

$$\frac{D_j}{D\tau} = \frac{d}{dt} + v_j \cdot \nabla = \frac{\partial}{\partial t} + v_j \cdot \nabla,$$

(A1)

where \(d/dt\) follows the fluid. In an expanding universe \(V_j = \dot{R} x + \ddot{R} x\), where \(\dot{R} x\) is the Hubble flow term, the other term being the peculiar velocity. The convective derivative must thus be replaced by

$$\frac{1}{R} \frac{D}{D\tau} (R v) = \frac{\partial v}{\partial t} + v + \frac{1}{R} (V \cdot \nabla v)$$

(A2)

where the definition of the Hubble expansion rate \(H = \dot{R}/R\) is used.

In an expanding universe with scalefactor \(R\), with the density and number density of the ‘i’ species being \(\rho_i\) and \(n_i\), respectively, the total fluid pressure \(p_i\) and the Newtonian peculiar gravitational potential \(\phi\), the equation for the electron motion is

$$\frac{\rho_e}{R} \frac{D}{D\tau} (R v_e) = -\frac{en_e}{c} \left( E + \frac{1}{c} v_e \times B - \frac{j}{\sigma} \right)$$

$$+ \frac{4}{3} \rho_i n_i c \sigma_{\gamma i} (v_e - v_i) - \frac{\nabla p_e}{R} - \frac{\rho_e \nabla \phi}{R},$$

(A3)

The four terms represent the electromagnetic forces, Compton drag, pressure gradient and gravitational force, \(\sigma\) is the electrical conductivity of the whole fluid, and \(\sigma_{\gamma i}, \sigma_{\gamma e}\) are the Thompson cross-section of the electron and proton respectively. The positive ion equation is

$$\frac{\rho_i}{R} \frac{D}{D\tau} (R v_i) = -\frac{en_i}{c} \left( E + \frac{1}{c} v_i \times B - \frac{j}{\sigma} \right)$$

$$+ \frac{4}{3} \rho_i n_i c \sigma_{\gamma i} (v_i - v_e) - \frac{\nabla p_o}{R} - \frac{\rho_o \nabla \phi}{R},$$

(A4)

and the photon equation is

$$\left( \rho_i + \frac{1}{c^2} p_i \right) \frac{1}{R} \frac{D}{D\tau} (R v_p) = -\frac{1}{c} \frac{d}{dt} p_i - \left( \rho_i + \frac{1}{c^2} p_i \right) \nabla \phi$$

$$- \frac{4}{3} \rho_i n_i c \sigma_{\gamma i} (v_i - v_e) - \frac{4}{3} \rho_i n_i c \sigma_{\gamma i} (v_i - v)$$

$$+ \frac{4}{3} \frac{\rho_o c}{5n_\sigma} \nabla^2 \phi,$$

(A5)

The last term can be dropped because the conductivity is very large at that epoch. Using the continuity equation

$$\nabla \cdot \mathbf{v} = -\frac{1}{(\rho_i + (1/c^2) p_i) \frac{dt}{dt}},$$

(A6)

\(\rho_i = \frac{1}{3} \rho, c^2, \) and \(\nabla \cdot \mathbf{v} = 3H + \nabla v, \) yields

$$\rho_i \frac{D}{D\tau} v_i = -\frac{1}{3} \rho_i v_i (V \cdot v) - \rho_i \nabla \phi$$

$$- c \sigma_{\gamma i} \rho_i n_i (v_i - v_e) - c \sigma_{\gamma i} \rho_i n_i (v_i - v),$$

(A7)

Taking the curl of equations (A4), (A3), and (A7) gets rid of the pressure and gravitational term. A short calculation of the curl of equation (A4) yields

$$\frac{1}{R^2} \frac{D}{D\tau} (R^2 V) + \nabla (V \cdot v) - (V \cdot \nabla) v = -\frac{n_e e}{\rho} \frac{1}{R^2} \frac{D}{D\tau} (R^2 B)$$

$$+ \frac{c}{4 \pi \sigma} \left[ \frac{1}{\mu} \nabla^2 B - \frac{e}{c^2} \right] - \frac{4}{3} \frac{n_e c \sigma_{\gamma e} (\zeta_e - \zeta_e)}{m_e},$$

(A8)

where to find the RHS I assumed that \(B \cdot VV = HB\) which is only exactly true at \(\theta = \pi/2\). The conductivity \(\sigma\) being very high, the wave equation term in \(B\) can be dropped. With

$$\sigma_{\gamma} = \frac{n_e e}{\rho} \frac{e}{m_e},$$

(A9)

the curl of equation (A4) can be written as

$$\frac{1}{R} \frac{D}{D\tau} [R^2 (V + \sigma_\gamma B)] = -\frac{4}{3} \frac{\rho_i}{m_e} c \sigma_{\gamma i} (\zeta_e - \zeta_e),$$

(A10)

Similarly the curl of equation (A3) yields

$$\frac{1}{R} \frac{D}{D\tau} [R^2 (V + \sigma_\gamma B)] = -\frac{4}{3} \frac{n_e c \sigma_{\gamma e}}{m_e} (\zeta_e - \zeta_e).$$

(A11)

Notice that the electron dynamics are much more affected by the photon drag due to the smaller electron mass in the denominator of the right-hand side of equation (A11). Also notice that in equations (A10) and (A11) \(\sigma_{\gamma i}/m_{\gamma} = \sigma_{\gamma e}/m_{\gamma} \).

Taking the curl of equation (A7) yields

$$\rho_i \frac{1}{R} \frac{D}{D\tau} (R \zeta_i) = -n_e c \sigma_{\gamma i} \rho_i (2 \zeta_e - \zeta_e - \zeta_e)$$

(A12)

where \(\nabla \cdot v\) was assumed to be very small.

Adding equations (A11) and (A10) yields

$$\frac{1}{R} \frac{D}{D\tau} [R^2 (\zeta_x - \zeta_x)] = \frac{1}{R} \frac{D}{D\tau} [R^2 (\alpha \times B)],$$

(A13)

where \(\alpha_x \gg \zeta_x\) was used. Since the proton spins down faster than the electron, the proton vorticity is small. Integrating equation (A13) yields the desired result of equation (3).

Equation (3) was derived under the assumption that the matter vorticity is spinning down faster than the radiation vorticity after having the same initial angular velocity. In the
original Harrison mechanism, the vortices were primordial and the radiation had the same initial velocity as the matter.

In the case of angular momentum acquired by tidal torquing the situation is a little different. No perturbations of the photon fluid can grow before recombination, making the moment of the photon shell zero. Thus the photon fluid, as well as the electron fluid, remains with zero angular momentum.

In the same manner the baryons are pressure supported, thus before recombination the moments of the baryonic component of the shell will be null, and they will not acquire angular momentum. Only the pressureless DM shells can acquire rms angular momentum.