Application of Dispersion Relations to Low Energy Kaon Scattering in Nuclei

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The dispersion relations for the forward scattering amplitudes are applied to estimate the nuclear many-body effect to the amplitudes which describe the kaon-nucleon scattering in nuclei and to obtain an optical model potential. The analysis is made of the assumption that only the effect of the Pauli principle to the intermediate states of the process plays a significant role and that of the multiple scatterings can be neglected. It is shown that, if the $Y^*$ resonant states consist of the $\bar{K}N$ bound systems, in nuclei the $K^-p$ forward scattering amplitude can be positive at low energies, i.e. effectively attractive potential, even if the Dalitz-Tuan ($-$) solutions are adopted for the free $K^-p$ scattering.

§ 1. Introduction

Recent results of the experimental analysis of the $K^-p$ scattering in a nucleus seem to favour the Dalitz-Tuan ($+$) solutions\textsuperscript{3} in connection with an optical model potential,\textsuperscript{2} that is, the attractive $K^-N$ potential. Accordingly we might be on the stage to investigate the characteristic features of the nuclear many-body effect to the low energy kaon-nucleon scattering amplitudes which describe the processes within a nuclear matter.

In a nucleus, the Pauli principle will play an important role to exclude some of the final and intermediate states of the processes, which are included in free space processes. In this paper this effect is assumed to be main correction to the amplitudes for the slow kaon scattering in a nucleus obtained from a simple impulse approximation, on the basis of the dispersion relations.

In § 3, the approximate dispersion relation for the amplitudes in a nucleus will be obtained in the same way as was given by Hamamoto and Miyazawa,\textsuperscript{3} under the assumption that the correction due to the multiple scattering processes is not significant.

Some related arguments to the dispersion relations will be stated in the Appendix.

In § 4, the correction terms included in the dispersion relations, which represent the many-body effect due to the Pauli principle, will be numerically evaluated from the experimental data of the $K^\pm-p$ scattering cross sections.\textsuperscript{4}

It has been known that the absorptive part of the $K^-p$ scattering amplitude has resonance peaks corresponding to the virtual resonant states with the isotopic spin $T=1$ and $0$; i.e. the $Y_{1}^*$ and $Y_{0}^*$ respectively in the unphysical region.\textsuperscript{5}
Our results will essentially depend upon whether those resonant states consist of the \( \pi-Y \) bound systems, or of the \( K-N \) bound systems\(^6\) which must be seriously affected by the Pauli principle in contrast with the former. This situation is closely connected with the interesting problems of the relative parities of the hyperons and the spins of those \( Y^* \) states\(^6,7\) and of the nature of the bound state itself.

In §5, the implication of our results will be discussed. It will be concluded that even if we choose as a free scattering amplitude the one which is given from the Dalitz-Tuan \((-\) ) solutions\(^1\), the low energy \( K^-p \) interaction potential in a nucleus could be effectively attractive, provided that the \( Y^* \) states are assumed to be the \( K-N \) bound systems and forbidden largely in a nucleus.

\section{2. Kaon-nucleon dispersion relations}

Between the dispersive parts \( D_{\pi \pm} (\omega) \) \( (D_{n \pm} (\omega)) \) and the absorptive parts \( A_{\pi \pm} (\omega) \) \( (A_{n \pm} (\omega)) \) of the forward scattering amplitudes, \( T_{\pi \pm} (\omega) = D_{\pi \pm} (\omega) + i \varepsilon (\omega) A_{\pi \pm} (\omega) \) \( (T_{n \pm} (\omega) = D_{n \pm} (\omega) + i \varepsilon (\omega) A_{n \pm} (\omega)) \), which describe the \( K^\pm p \) \((K^\pm n)\) scattering processes, respectively, the following dispersion relations have been derived:\(^8\)

\[
D_{\pi \pm} (\omega) = \frac{P}{\pi} \int_0^\infty d\omega' \frac{A_{\pi \pm} (\omega')}{\omega' - \omega} + \frac{P}{\pi} \int_0^\infty d\omega' \frac{A_{n \pm} (\omega')}{\omega' + \omega},
\]

and

\[
D_{n \pm} (\omega) = \frac{P}{\pi} \int_0^\infty d\omega' \frac{A_{n \pm} (\omega')}{\omega' - \omega} + \frac{P}{\pi} \int_0^\infty d\omega' \frac{A_{n \pm} (\omega')}{\omega' + \omega},
\]

where \( \omega \) represents the total energy of an incident kaon in laboratory system.

In the \( K^+ p \) and \( K^+ n \) processes, from the strangeness conservation rule, only the elastic and charge exchange scatterings, i.e. \( K^+ p \to K^+ p \) and \( K^+ n \to K^+ n \), \( K^0 + p \) are allowed, respectively, at low energies above the threshold energy \( \mu \), kaon mass. For the unphysical region, \( 0 < \omega < \mu \), \( A_{\pi \pm} (\omega) = A_{n \pm} (\omega) = 0 \).

On the other hand, in the \( K^- p \) and \( K^- n \) processes there occur several absorption processes, i.e. \( K^- p \to \Sigma^\pm, n + \pi^\pm, A + \pi^\pm \) and \( K^- n \to \Sigma^0, K^- \pi^- \), \( A + \pi^- \), respectively, for \( \omega > \mu \), as well as the elastic and charge exchange scattering processes, i.e. \( K^- p \to K^- p \), \( \Omega^0 n \) and \( K^- n \to K^- n \), respectively.

The unphysical part of the \( A_{\pi \pm} (\omega) \) contains spectra corresponding to the virtual resonant states \( Y_1^* \) and \( Y_0^* \) above the energy \( \omega_{\pi \pi} = [(m_A + m_\pi)^2 - m^2 - m^2]/2m \), where \( m_A \), \( m_\pi \), \( m_\pi \) and \( m \) represent the masses of the \( A \) particle, pion, kaon and nucleon, respectively.

The masses and the half-widths of these \( Y_1^* \) and \( Y_0^* \) states have been observed to be \( M_{Y_1^*} = 1385 \text{ Mev} \) and \( M_{Y_0^*} = 1405 \text{ Mev} \), and \( \Gamma_{Y_1^*}/2 = 15 \text{ Mev} \) and
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In the physical region, \( \omega \gg \mu \), using the optical theorem

\[ A_{\pm \pm}(\omega) = -\frac{k}{4\pi} \sigma_{\pm \pm}(\omega) \quad \text{and} \quad A_{\pm \mp}(\omega) = -\frac{k}{4\pi} \sigma_{\pm \mp}(\omega), \tag{2} \]

where \( k = (\omega^2 - \mu^2)^{1/2} \), we can express \( A_{\pm \pm}(\omega) \) and \( A_{\pm \mp}(\omega) \) with the total cross sections, \( \sigma_{\pm \pm}(\omega) \) and \( \sigma_{\pm \mp}(\omega) \), respectively.

For later convenience, we rewrite the dispersion relations of Eq. (1) in the other forms, using Eq. (2),

\[ D_{\pm \pm}(\omega) = \frac{P}{4\pi^2} \int_0^\infty d\omega' k' \left[ \frac{\sigma_{\mp \pm}^{\,\prime}(\omega')}{\omega' \pm \omega} + \frac{\sigma_{\mp \mp}^{\,\prime}(\omega')}{\omega' \mp \omega} \right] \]

\[ + \frac{P}{\pi} \int_0^\infty d\omega' \frac{A_{\mp \mp}^{\,\prime}(\omega')}{\omega' \mp \omega} \tag{3} \]

and

\[ D_{\pm \mp}(\omega) = \frac{P}{4\pi^2} \int_0^\infty d\omega' k' \left[ \frac{\sigma_{\mp \mp}^{\,\prime}(\omega')}{\omega' \pm \omega} + \frac{\sigma_{\pm \pm}^{\,\prime}(\omega')}{\omega' \mp \omega} \right] \]

\[ + \frac{P}{\pi} \int_0^\infty d\omega' \frac{A_{\mp \pm}^{\,\prime}(\omega')}{\omega' \pm \omega}. \tag{4} \]

In the above Eqs. (3) and (4), we employed the following notations:

- \( \sigma_{\mp \pm}^{\,\prime}(\omega) \): total \( K^-p \) scattering cross section (i.e. the sum of the total elastic and charge exchange scattering cross sections),
- \( \sigma_{\mp \mp}^{\,\prime}(\omega) \): total \( K^-p \) absorption cross section,
- \( \sigma_{\pm \pm}^{\,\prime}(\omega) \): total \( K^n-n \) elastic scattering cross section,
- \( \sigma_{\pm \mp}^{\,\prime}(\omega) \): total \( K^-n \) absorption cross section,

and \( \sigma_{\pm \mp}^{\,\prime}(\omega) \): total \( K^+p \) scattering cross section.

In obtaining these Eqs. (3) and (4), we have dropped the integrals of \( A_{\mp \pm}(\omega) = -A_{\pm \mp}(\omega) \) and \( A_{\mp \mp}(\omega) = -A_{\pm \pm}(\omega) \) for the unphysical region, \( 0 < \omega < \mu \).

Here we notice that, in order to establish well the convergence of the integrals in the dispersion relations, it will actually be necessary to carry out a subtraction procedure to each expressions in Eqs. (3) and (4). But, when we obtain an expression of the dispersion relation for the difference between the amplitude which describes a process in a nucleus and the corresponding amplitude in free space, our procedure will be equivalent to the subtraction, provided that the high energy limit behaviours of both amplitudes are assumed to be the same.
§ 3. Many-body effect to the scattering amplitudes

The dispersion relation for the \( K \)-nucleus scattering amplitude \( T^{(N)} \) is very complicated due to the composite nature of the target. We shall simplify the problem by considering only a two-body scattering, a kaon and a nucleon, and taking into account the effect of other nucleons as the Pauli exclusion principle. This type of dispersion relation was discussed by Hamamoto and Miyazawa\(^3\) and we follow the same approach.

When a \( K \)-\( N \) scattering takes place in a nuclear matter, some of the intermediate states will be excluded, if the states are not allowed in the nucleus. Thus for instance in Eq. (3), the term containing \( \sigma^{\text{st}}_{\rho} (\omega) \) will be zero in the nucleus, if the incident energy \( \omega' \) has not enough energy to excite the recoil nucleon above the Fermi surface. In this way we arrive at a simple approximate result for the scattering amplitude \( t^{(N)} (\omega) \) which describes the kaon scattering by one of the nucleons at rest in the nucleus;

\[
t^{(N)} (\omega) = \frac{1}{\pi} \int_{-\infty}^{\omega} \text{Im} t^{(N)} (\omega') \frac{d\omega'}{\omega' - \omega},
\]

where \( \text{Im} t^{(N)} (\omega') \) differs from \( \text{Im} t^{(N)} (\omega') \) in free space if the final and intermediate states are not allowed in the nucleus. Some discussions on the dispersion relation is given in the Appendix.

On the basis of Eqs. (3), (4) and (5), we can express the differences of the \( K^\pm-p \) or \( K^\pm-n \) forward scattering amplitudes: \( \Delta D_{p \pm} (\omega) \) or \( \Delta D_{n \pm} (\omega) \), respectively, between the amplitudes which describe kaon scatterings in a nucleus and those in free space.

\[
D_{p \pm}^{(N)} (\omega) = D_{p \pm} (\omega) + \Delta D_{p \pm} (\omega)
\]

and

\[
D_{n \pm}^{(N)} (\omega) = D_{n \pm} (\omega) + \Delta D_{n \pm} (\omega),
\]

where

\[
\Delta D_{p -} (\omega) = -\frac{P}{4\pi^2} \int_{\rho}^{\omega} d\omega' k' \left[ \frac{\{\sigma^{\pm}_{p -} (\omega') - \sigma^{(N)}_{p -} (\omega')\} + \{\sigma^{a}_{p -} (\omega') - \sigma^{(N)a}_{p -} (\omega')\}}{\omega' - \omega} \\
+ \frac{\sigma^{\text{st}}_{\rho} (\omega') - \sigma^{(N)\text{st}}_{\rho} (\omega')}{\omega' + \omega} \right] - \frac{P}{\pi} \int_{\rho}^{\omega} d\omega' A_{p -} (\omega') - A^{(N)}_{p -} (\omega'),
\]

\[
\Delta D_{p +} (\omega) = -\frac{P}{4\pi^2} \int_{\rho}^{\omega} d\omega' k' \left[ \frac{\{\sigma^{\pm}_{p +} (\omega') - \sigma^{(N)\pm}_{p +} (\omega')\} + \{\sigma^{a}_{p +} (\omega') - \sigma^{(N)a}_{p +} (\omega')\}}{\omega' + \omega} \\
+ \frac{\sigma^{\text{st}}_{\rho} (\omega') - \sigma^{(N)\text{st}}_{\rho} (\omega')}{\omega' - \omega} \right] - \frac{P}{\pi} \int_{\rho}^{\omega} d\omega' A_{p +} (\omega') - A^{(N)}_{p +} (\omega'),
\]
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\[ D_{n-} (\omega) = - \frac{P}{4\pi^2} \int_0^{\infty} d\omega' k' \left[ \frac{\sigma_{n-}^{(a)} (\omega') - \sigma_{n-}^{(s)} (\omega')}{\omega' - \omega} \right] + \frac{\sigma_{n-}^{(s)} (\omega') - \sigma_{n-}^{(a)} (\omega')}{\omega' + \omega} \right] - \frac{P}{\pi} \int_0^{\infty} d\omega' A_{n-} (\omega') - A_{n-}^{(a)} (\omega') \right] \left[ \frac{\omega' + \omega}{\omega' - \omega} \right], \] (10)

and

\[ D_{n+} (\omega) = - \frac{P}{4\pi^2} \int_0^{\infty} d\omega' k' \left[ \frac{\sigma_{n+}^{(a)} (\omega') - \sigma_{n+}^{(s)} (\omega')}{\omega' - \omega} \right] + \frac{\sigma_{n+}^{(s)} (\omega') - \sigma_{n+}^{(a)} (\omega')}{\omega' + \omega} \right] - \frac{P}{\pi} \int_0^{\infty} d\omega' A_{n+} (\omega') - A_{n+}^{(a)} (\omega') \right] \left[ \frac{\omega' + \omega}{\omega' - \omega} \right]. \] (11)

In the above equations the index \((N)\) distinguishes the quantities with respect to the scattering processes in a nucleus from those in free space.

When we know the forward scattering amplitudes, the optical model potentials\(^9\) for the scatterings of \(K^\pm\) on nuclei are given by, in our approximation,

\[ V_\pm + iW_\pm = - \frac{2\pi}{(4\pi/3) r_0^3} \frac{1}{\omega} \left[ A \left\{ ZD_{p\pm} (\omega) + N D_{n\pm} (\omega) + i ZA_{p\pm} (\omega) + N A_{n\pm} (\omega) \right\} \right] + \left[ ZD_{p\pm} (\omega) + N D_{n\pm} (\omega) + i ZA_{p\pm} (\omega) + N A_{n\pm} (\omega) \right], \] (12)

where \(r_0\) is related to nuclear radius \(R\) by \(R = r_0 A_1^{1/3}\), and

\[ \Delta A_{p-} (\omega) = - \frac{k}{4\pi} \sigma_{p-}^{(s)} (\omega) \quad \text{for} \quad \mu < \omega < \omega_F \]

\[ = - \frac{k}{4\pi} \frac{\omega_F^2 - \mu^2}{\omega^2} \sigma_{p-}^{(s)} (\omega) \quad \text{for} \quad \omega_F < \omega, \]

\[ \Delta A_{p+} (\omega) = - \frac{k}{4\pi} \sigma_{p+}^{(a)} (\omega) \quad \text{for} \quad \mu < \omega < \omega_F \]

\[ = - \frac{k}{4\pi} \frac{\omega_F^2 - \mu^2}{\omega^2} \sigma_{p+}^{(a)} (\omega) \quad \text{for} \quad \omega_F < \omega, \] (13)

\[ \Delta A_{n-} (\omega) = - \frac{k}{4\pi} \sigma_{n-}^{(a)} (\omega) \quad \text{for} \quad \mu < \omega < \omega_F \]

\[ = - \frac{k}{4\pi} \frac{\omega_F^2 - \mu^2}{\omega^2} \sigma_{n-}^{(a)} (\omega) \quad \text{for} \quad \omega_F < \omega, \]

and

\[ \Delta A_{n+} (\omega) = - \frac{k}{4\pi} \sigma_{n+}^{(s)} (\omega) \quad \text{for} \quad \mu < \omega < \omega_F \]
In the right-hand side of Eq. (12), the terms in the second square bracket represent the corrections to the optical model potentials obtained in the impulse approximation. The meaning of the approximation in Eq. (13) will be understood in the next paragraph.

In order to evaluate \( \Delta D_{p\pm}(\omega) \), we make approximations as follows:

1) In the reaction processes, we may neglect the small effect of the exclusion principle to the intermediate states of them, under the assumption that the multiple scattering effect does not decisively affect the result. Thus, we put

\[
\sigma_{p-}^{a}(\omega) - \sigma_{p-}^{(N)a}(\omega) = 0.
\]  

(14)

2) For the scattering processes, we only take into account the fact that the final states with a nucleon scattered below the Fermi surface of a nucleus are forbidden, therefore these real processes do not occur. We neglect the effect of exclusion of the intermediate states which include two or more nucleons.

In this approximation,

\[
\sigma_{p-}^{(N)a}(\omega) = 0 \quad \text{and} \quad \sigma_{p-}^{(N)a}(\omega) = 0, \quad \text{for} \ \mu < \omega < \omega_{r},
\]  

(15)

where the conventional parameter \( \omega_{r} \) may be identified with the incident kaon energy to be just enough to excite a ground state nucleon into the Fermi level, viz.,

\[
\omega_{r} = \left[ \mu^{2} + \left( 1 + \frac{\mu}{m} \right)^{2} p_{r}^{2}/4 \right]^{1/2} = 539 \text{ Mev.} \]  

(16a)

Instead, the \( \omega_{r} \) may be identified with the average of the necessary kaon energy with respect to the distribution of momenta of nucleons in random directions. Assuming a constant K-N scattering cross section, we get a measure of the \( \omega_{r} \), below which no real scattering process occurs,

\[
\omega_{r} = \left[ \mu^{2} + \left( 3 \left( 1 + \frac{\mu}{m} \right) \rho_{p} \right)^{2} \right]^{1/2} = 519 \text{ Mev.} \]  

(16b)

For the sake of comparison, two values of the \( \omega_{r} \), (16a) and (16b), are used to calculate the amplitudes. The values listed in column (a) or (b) in Tables I, II and III correspond to case (16a) or (16b), respectively. For \( \omega > \omega_{r} \), assuming that s-wave scattering dominates the low energy kaon-nucleon interactions and the angular distribution of the scattering is isotropic, we get

\[
\sigma_{p-}^{a}(\omega) - \sigma_{p-}^{(N)a}(\omega) = \frac{\omega_{r}^{2} - \mu^{2}}{\omega^{2} - \mu^{2}} \sigma_{p-}^{a}(\omega)
\]  

and

\[
\sigma_{p+}^{a}(\omega) - \sigma_{p+}^{(N)a}(\omega) = \frac{\omega_{r}^{2} - \mu^{2}}{\omega^{2} - \mu^{2}} \sigma_{p+}^{a}(\omega).
\]  

(17)
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3) As we shall discuss the behaviour of the scattering amplitudes in the physical region, we may neglect the small terms with the factor $1/(\omega' + \omega)$ in the integrands in Eqs. (8) and (9) except for the last term in Eq. (9).

On the above assumptions, Eqs. (8) and (9) are rewritten in simpler approximate forms;

$$\Delta D_{p-}(\omega) = I_{p-}^1 + I_{p-}^2,$$

$$I_{p-}^1 = -\frac{P}{4\pi^2} \int_0^\infty d\omega' k' \frac{\sigma_{p-}(\omega')}{\omega' - \omega} - \frac{\omega p^2 - \mu^2}{4\pi^2} \int_0^\infty d\omega' \frac{k' \sigma_{p-}(\omega')}{(\omega^2 - \mu^2)(\omega' - \omega)},$$

$$I_{p-}^2 = -\frac{P}{\pi} \int_0^\infty d\omega' \frac{A_{p-}(\omega') - A_{p-}^{(N)}(\omega')}{\omega' - \omega},$$

and

$$\Delta D_{p+}(\omega) = I_{p+}^1 + I_{p+}^2,$$

$$I_{p+}^1 = -\frac{P}{4\pi^2} \int_0^\infty d\omega' k' \frac{\sigma_{p+}(\omega')}{\omega' - \omega} - \frac{\omega p^2 - \mu^2}{4\pi^2} \int_0^\infty d\omega' \frac{k' \sigma_{p+}(\omega')}{(\omega^2 - \mu^2)(\omega' - \omega)},$$

$$I_{p+}^2 = -\frac{P}{\pi} \int_0^\infty d\omega' \frac{A_{p+}(\omega') - A_{p+}^{(N)}(\omega')}{\omega' - \omega}.$$

With Eqs. (18) and (19), we shall numerically evaluate $\Delta D_{p \pm}(\omega)$ in the next section.

§ 4. Numerical results

From the experimental data of the total $K^\pm p$ scattering cross sections, we get the values of $I_{p \pm}$ in Eq. (18), which are tabulated in Table I. If the spectrum in the unphysical region is not affected by the exclusion principle, $A_{p-}(\omega) = A_{p-}^{(N)}(\omega)$, or $I_{p-}^2 = 0$. Therefore,

$$\Delta D_{p \pm}(\omega) = I_{p \pm}^1.$$

The values of $D_{p \pm}^{(N)}(\omega)$ for this case are shown also in Table I. In this evaluation we use the free scattering amplitude $D_{p-}(\omega)$ which is given by the Dalitz-Tuan $(a -)$ solution which gives a peak to the absorptive part $A_{p-}(\omega)$ at the suitable position 30 Mev below the threshold energy.

The $D_{p \pm}^{(N)}(\omega)$ in Table I will give rather good approximate values, provided

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* For instance, the contribution from the term, $-\frac{(1/4\pi^2)\int_{\mu}^\infty d\omega' \frac{\sigma_{p+}(\omega') - \sigma_{p+}^{(N)}(\omega')}{\omega' + \omega} \frac{\kappa' d\omega'}{k'}$, in the right-hand side of Eq. (8) is smaller than the order of $10^{-4}$ in the range of energy in which we are interested, so that the neglect of it causes no trouble.
Table I. The contribution to the $\Delta D_{p\pm}(\omega)$ from the scattering processes: $D_{p\pm(N)}(\omega) = D_{p\pm}(\omega) + \tilde{F}_{p\pm}$.

<table>
<thead>
<tr>
<th>K.E. (Mev)</th>
<th>$I_{p\pm}^1$ (a)</th>
<th>$I_{p\pm}^1$ (b)</th>
<th>$D_{p\pm}(\omega)$ (a)</th>
<th>$D_{p\pm}^{(N)}(\omega)$ (a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>-0.60 $f$</td>
<td>-0.69 $f$</td>
<td>-8.8 x 10$^{-2}f$</td>
<td>-6.9 x 10$^{-2}f$</td>
</tr>
<tr>
<td></td>
<td>-0.53</td>
<td>-0.39</td>
<td>-8.3</td>
<td>-5.4</td>
</tr>
<tr>
<td>15</td>
<td>-0.31</td>
<td>0.04</td>
<td>-7.3</td>
<td>-3.2</td>
</tr>
<tr>
<td>25</td>
<td>-0.16</td>
<td>0.18</td>
<td>-6.1</td>
<td>-1.5</td>
</tr>
<tr>
<td>35</td>
<td>-0.012</td>
<td>0.12</td>
<td>-2.5</td>
<td>-1.1</td>
</tr>
<tr>
<td>55</td>
<td>0.019</td>
<td>0.05</td>
<td>-0.6</td>
<td>-0.08</td>
</tr>
<tr>
<td>105</td>
<td>0.028</td>
<td>0.04</td>
<td>-0.04</td>
<td>0.13</td>
</tr>
<tr>
<td>155</td>
<td></td>
<td></td>
<td>-0.15</td>
<td>-0.18</td>
</tr>
</tbody>
</table>

K.E. denotes the kinetic energy of the incident $K^\pm$ meson in laboratory system. $D_{p\pm}(\omega)$ is given from the Dalitz-Tuan (a-) solution: $A_0 = -0.20 + 0.76i$, $A_1 = -1.62 + 0.36i$.

that the $Y^*$ intermediate states are composed of $\pi-A$ or $\pi-\Sigma$ bound systems which will be never affected by the exclusion principle. If it is true, the amplitude will exhibit the same behaviour as that in free space and show repulsive potential effectively strengthened by the many-body effect.

Another situation is provided under the assumption that these $Y^*$ states are the $\bar{K}N$ s-wave bound systems and affected seriously by the exclusion principle.

As an attempt, we assume

$$A_{p\pm}(\omega) - A_{p\pm}^{(N)}(\omega) = \alpha A_{p\pm}(\omega) \quad \text{for } \omega_{as} < \omega < \mu,$$

(21)

where the ratio $\alpha$ is arbitrarily chosen from $1/2$ to $4/5$.*) The evaluated values of $I_{p\pm}(\omega)$ are listed in Table II. The uncertainty of these values corresponds to the restricted interval of $\alpha$. In evaluation of $A_{p\pm}(\omega)$ we have also used the same Dalitz-Tuan (a-) solution.

From Table II, we see that the $\Delta D_{p\pm}(\omega)$ at low energies gets a large contribution from this unphysical spectrum which is close to the physical region. Thus we meet with the notable possibility that large positive contributions from the unphysical region $Y^*$ spectra due to the exclusion principle may change the sign of the amplitude toward the direction of the effectively attractive $K^-p$ force in a nucleus.

With respect to the $K^-p$ scattering in a nucleus, it is concluded that, if the $Y^*$ states have nature not affected by the exclusion principle, the many-body effect serves as a repulsive force (see Table I).

On the other hand in the case of the $\bar{K}N$ bound states, the $\Delta D_{p\pm}(\omega)$ gets a contribution from both $I_{p\pm}^1$ and $I_{p\pm}^f$, which serves effectively also as a repulsive effect.

*) When the $Y_1^*$ state is regarded as the $\bar{K}N$ bound system with binding energy, 47 Mev, the value of $\alpha$ turns out to be about 0.7 in the zero range approximation.
Table II. The contribution to the $\Delta D_{p^+}(\omega)$ from the unphysical region spectra; $D_{p^+}(\omega) = D_{p^+}^{(N)}(\omega) + F_{p^+}$.

<table>
<thead>
<tr>
<th>K.E.</th>
<th>$I_{p^-}^0$</th>
<th>$I_{p^+}^0$</th>
<th>$D_{p^+}^{(N)}(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 Mev</td>
<td>0.82$\pm$1.30f</td>
<td>3.96$\pm$6.33$\times$10$^{-2}$f</td>
<td>$-0.47\sim-0.09f$</td>
</tr>
<tr>
<td>15</td>
<td>0.63$\sim$1.01</td>
<td>3.88$\sim$6.21</td>
<td>$-0.43\sim-0.05$</td>
</tr>
<tr>
<td>25</td>
<td>0.51$\sim$0.81</td>
<td>3.84$\sim$6.13</td>
<td>$-0.24\sim 0.06$</td>
</tr>
<tr>
<td>35</td>
<td>0.43$\sim$0.69</td>
<td>3.81$\sim$6.08</td>
<td>$-0.11\sim 0.15$</td>
</tr>
<tr>
<td>55</td>
<td>0.34$\sim$0.54</td>
<td>3.73$\sim$5.95</td>
<td>$0.04\sim 0.24$</td>
</tr>
<tr>
<td>105</td>
<td>0.23$\sim$0.45</td>
<td>3.53$\sim$5.64</td>
<td>$0.02\sim 0.23$</td>
</tr>
<tr>
<td>155</td>
<td>0.17$\sim$0.27</td>
<td>3.37$\sim$5.38</td>
<td>$0.01\sim 0.09$</td>
</tr>
</tbody>
</table>

The uncertainty of the above values comes from the forbidden ratio $\alpha$ in Eq. (21).

Table III. The contribution of the many-body effect to the $K^+p$ scattering amplitude in a nucleus.

<table>
<thead>
<tr>
<th>K.E.</th>
<th>$\Delta D_{p^+}(\omega) = I_{p^+}^1 + I_{p^+}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(b)</td>
</tr>
<tr>
<td>5 Mev</td>
<td>$-4.9\sim-2.5\times10^{-2}f$</td>
</tr>
<tr>
<td>15</td>
<td>$-4.5\sim-2.1$</td>
</tr>
<tr>
<td>25</td>
<td>$-3.5\sim-1.2$</td>
</tr>
<tr>
<td>35</td>
<td>$-2.3\sim-0.02$</td>
</tr>
<tr>
<td>55</td>
<td>$1.2\sim3.4$</td>
</tr>
<tr>
<td>105</td>
<td>$3.4\sim5.0$</td>
</tr>
<tr>
<td>155</td>
<td>$3.3\sim5.3$</td>
</tr>
</tbody>
</table>

force (see Table III). These values of the $\Delta D_{p^+}(\omega)$ are however so small that the amplitude for the $K^+p$ scattering in a nucleus will show the same behavior as that in free space. On the other hand the effective range analysis of the $K^+p$ process in the isotopic spin $T=1$ state has actually shown the repulsive $K^+p$ force in free space, and confirms the above statement:

$$\frac{k_{\text{c.m.}} \cot \delta_i}{\alpha_i} = -\frac{1}{\alpha_i} + \frac{1}{2} \frac{k_{\text{c.m.}}^2}{r_i},$$

(22)

where $\alpha_i = 0.34 \times 10^{-13}$ cm, $r_i = 0.50 \times 10^{-13}$ cm, and $k_{\text{c.m.}}$ represents the centre-of-mass momentum of the $K^+$ meson.

§ 5. Conclusion

At the present stage of our study of the $K^-p$ process, we cannot give a decisive answer to the question; which interpretation of the $Y^*$ resonant states is correct? the $K-N$ bound systems, or the $\pi-Y$ bound systems. In our previous result, there seemed to remain two possible interpretations to account for the
attractive $K^-p$ interaction force in a nucleus at low energies.

The first interpretation is as follows: If the $Y^*$ resonant states are regarded as the $\pi$-$N$ bound systems in a good approximation, they would be little affected by the exclusion principle in a nucleus. In this case, the $K^-p$ scattering amplitude will get a rather small negative contribution only through the effect due to the exclusion of the final states below the Fermi surface; i.e. $I_p^+$ in Eq. (18). Thus the attractive $K^-p$ force in a nucleus will require the Dalitz-Tuan (+) solutions.

The second interpretation is as follows: If the $Y^*$ resonant states are regarded as consisting mostly of the $\bar{K}$-$N$ bound systems, the $K^-p$ scattering amplitude would get a large positive contribution through the effect of the forbidden $Y^*$ states in a nucleus, i.e. $I_p^-$ in Eq. (18). Thus, in this case, the resultant $K^-p$ interaction force in a nucleus could appear to be effectively attractive, even if the free scattering amplitude is given by the Dalitz-Tuan (−) solutions.

This last situation is consistent with the fact that the interpretation of the $Y^*$ resonant states in terms of the $\bar{K}$-$N$ bound states requires the (−) solutions.

It is concluded that the (−) solutions cannot necessarily be excluded, even if the $K^-p$ force in a nucleus appears to be attractive as that in the low energy $K^-$-nucleus data suggested from the optical model potential analysis.

In the above analysis we have concentrated our attention only on the Dalitz-Tuan solutions, but the method developed may also be applied to the case of the Humphrey-Ross solutions\(^{11}\) which imply the difficulty in explaining the $Y^*$ in terms of the $s$-wave $\bar{K}$-$N$ bound state in the framework of the zero-range approximation. It can be inferred that, whether the $Y^*$ is the $K^-N$ system or not, the solution II is consistent with the attractive $K^-p$ potential in nuclei except at very low energy regions (K.E. ≤ 50 Mev) in which the value of the $I_p^+$ in Table I can be negative and of a comparable magnitude to the free amplitude.

In the $K^+p$ scattering process in a nucleus, at low energies, scattering amplitude gets rather small negative contribution due to the exclusion principle. Therefore this amplitude will be shifted toward the repulsive direction furthermore as compared with that of the free space scattering. This effect will not be large enough to change the behaviour of this amplitude qualitatively.

Finally we remark the point that a multiple-scattering correction is not significant for rather high energy part, viz., $\omega > \omega_F$, because if we consider the quantity,
\[
\sigma_p^+ (\omega) + \sigma_p^- (\omega) - \sigma_p^{(Y^*)^+} (\omega) - \sigma_p^{(Y^*)^-} (\omega),
\]
we will find that the increase of the $\sigma_p^{(Y^*)^+} (\omega)$ by the multiple scattering process is just cancelled out with the decrease of the $\sigma_p^{(Y^*)^-} (\omega)$. Thus the above quantity
will effectively near to zero over a certain energy range.

Acknowledgments

The author would like to express his sincere thanks to Professor H. Miyazawa for his kind guidance and continual encouragement.

Appendix

We show the difficulty in deriving, in a simple form, the dispersion relation for the forward $K$-nucleus scattering amplitude which bears the complicated nature of the target.

Employing the mass variable $\nu = k^2 - \omega^2 > 0$ for the incident kaon, where $k$ and $\omega$ are the momentum and the energy corresponding to the unphysical mass in the rest system of a nucleus, we can derive a dispersion relation for the amplitude $T^{(S)}(\omega)$ which is the function only of the energy $\omega$ and the magnitude of the Fermi momentum of the nucleus $p_F$.

$$T^{(S)}(\omega) = \frac{1}{\pi} \int_{-\omega}^{\omega} \frac{\text{Im} T^{(S)}(\omega')}{\omega' - \omega} \, d\omega'.$$

In perturbation calculation, it is obvious that the branch cuts of this unphysical amplitude are located only along the real axis when it is analytically continued into the complex $\omega$-plane. Among these branch cuts, we find several extended branch cuts coming from the one-particle intermediate states, i.e. $A$, $\Sigma$ and $Y^*$ states. As nucleons have finite momenta below the Fermi level, these cuts extend over the interval $[\omega_o, \omega_i']$, where

$$\omega_i = \frac{M^2 - m^2 + \nu}{2m} \left(1 + \frac{p_F^2}{m^2}\right)^{1/2} - \frac{p_F}{m} \left(\frac{M^2 - m^2 + \nu}{2m}\right)^{1/2}$$

(A.2)

and

$$\omega_i' = \frac{M^2 - m^2 + \nu}{2m} \left(1 + \frac{p_F^2}{m^2}\right)^{1/2} + \frac{p_F}{m} \left(\frac{M^2 - m^2 + \nu}{2m}\right)^{1/2}$$

(A.3)

corresponding to $M$'s which represent the masses $M_A$, $M_\Sigma$ and $M_{Y^*}$ of intermediate hyperons $A$, $\Sigma$ and $Y^*$, respectively.

Although our desire is to perform the limiting procedure, $\nu \rightarrow -\nu$, to get a dispersion relation for the actual amplitude, as $\nu \rightarrow -\nu$ arises several complicated branch cuts which extend over the corresponding regions in the complex plane illustrated in Fig. 1. This is due to the fact the energies of the one-particle states,

$$\omega_M = \frac{M^2 - m^2 - \mu^2}{2m}$$

(A.4)
are smaller than the kaon threshold \( \mu \). In Fig. 1, one of these regions corresponding to the particle with mass \( M \) is indicated by the shadowed region surrounded with the circle \( C \) having its centre at

\[
C_F \left( \frac{1}{2} \mathsf{M} + \mathsf{M} \left( 1 + \frac{\mathsf{F}^2}{m^2} \right)^{-1/2} \right), \quad 0
\]

and the radius \( \frac{1}{2} \mathsf{M} + \mathsf{M} \left( 1 + \frac{\mathsf{F}^2}{m^2} \right)^{-1/2} \),

and the hyperbola \( \frac{a^2}{\mathsf{M}^2} - \frac{b^2}{m^2} = 1 \); \( a = \text{Re} \, \omega \) and \( b = \text{Im} \, \omega \).

![Fig. 1. Region of the unphysical branch cuts of the amplitude \( T^{(N)}(\omega) \) in \( \omega \)-plane. The centres of circles are indicated by the points \( C_p, C_p \) and \( C_p \), and the intersections of the cuts and the real axis by \( \omega_M, \omega_p \) and \( \omega_F \); for \( 0 < \omega_p < \omega_F, \omega_p = \omega_M \left( 1 + \mathsf{F}^2/m^2 \right)^{-1/2} \), \( \omega_F = \omega_M \left( 1 + \mathsf{F}^2/m^2 \right)^{-1/2} \) and \( C_p \left( \frac{1}{2} \mathsf{M} \left( 1 + \mathsf{F}^2/m^2 \right)^{-1/2} + \right) \), \( \mathsf{M} \left( 1 + \mathsf{F}^2/m^2 \right)^{-1/2} \).

References

Application of Dispersion Relations


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