Design and analysis of redshift surveys
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ABSTRACT
In this paper we consider methods of analysis and optimal design of redshift surveys. In the first part, we develop a formalism for analysing galaxy redshift surveys that are essentially two-dimensional, such as thin declination slices. The formalism is a power spectrum method, using spherical coordinates, allowing the distorting effects of galaxy peculiar velocities to be calculated to linear order on the assumption of statistical isotropy but without further approximation. In this paper, we calculate the measured two-dimensional power for a constant-declination strip, widely used in redshift surveys. We present a likelihood method for estimating the three-dimensional real-space power spectrum and the redshift distortion simultaneously, and show that for thin surveys of reasonable depth the large-scale three-dimensional power cannot be measured with high accuracy. The redshift distortion may be estimated successfully, and with higher accuracy, if the three-dimensional power spectrum can be measured independently, for example from a large-scale sky-projected catalogue.

In the second part, we show how a three-dimensional survey design can be optimized to measure the power spectrum, considering whether areal coverage is more important than depth, and whether the survey should be sampled sparsely or not. We show quite generally that width is better than depth, and show how the optimal sparse-sampling fraction $f$ depends on the power $P$ to be measured. For a Schechter luminosity function, a simple optimization $fP \sim 500 h^{-3} \text{Mpc}^3$ is found.

Key words: surveys – galaxies: distances and redshifts – cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION
The measurement of fluctuation power in the galaxy distribution is an important test of galaxy formation models, since the fluctuation spectrum, of mass at least, is predicted readily by such models. Power can be measured from three-dimensional redshift surveys, or from projected catalogues, by numerical inversion techniques. Redshift surveys have the advantage (and disadvantage) that they are distorted by the effects of peculiar velocities, and can therefore be used to extract information on the density parameter, under the assumption that structure grows by gravitational instability. One has then the possibility of measuring three-dimensional power and the density parameter (via $\beta=\Omega^b/\Omega$), where $\beta$ is the bias parameter for the survey in hand) from a galaxy redshift survey (Heavens & Taylor 1995, hereafter HT95; see also Kaiser 1987; Hamilton 1992; Cole, Fisher & Weinberg 1994).

It is clear that the longest wavelength that can be measured is limited by the size of the survey, so it is attractive to consider surveys that are essentially one- or two-dimensional, to maximize at least one dimension without incurring prohibitive cost in observation time (e.g. Broadhurst et al. 1990). The difficulty with such an approach as a method for measuring the power spectrum is that a low-dimensional power measurement at a given wavenumber will have a contribution (which may be dominant) from much smaller scales in three dimensions (e.g. Kaiser & Peacock 1991). The interpretation of the observed power spectrum can therefore be difficult. For surveys that do not correspond to the ‘distant-observer’ approximation (cf. Kaiser 1987), the power spectrum measurement and the redshift distortion become linked, and this further complicates the analysis.

The ease with which the parameters of interest may be extracted depends on the choice of coordinate system and basis functions in which the density field is expanded. For
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2 ONE AND TWO DIMENSIONS: GENERAL CONSIDERATIONS

The fractional overdensity is \( \delta(r) = \frac{\rho(r)}{\bar{\rho}} - 1 \), where \( \bar{\rho} \) is the mean density. Our Fourier transform convention is

\[
\delta(r) = \frac{1}{(2\pi)^3} \int d^3k \delta_3(k) e^{i\mathbf{k} \cdot \mathbf{r}}.
\]

The power spectrum is defined by

\[
\langle \delta_3 \delta_3^* \rangle = (2\pi)^3 P_{3D}(k) \delta^3(k - k'),
\]

so that the correlation function is

\[
\xi(r) = \frac{1}{(2\pi)^3} \int d^3k P_{3D}(k) e^{i\mathbf{k} \cdot \mathbf{r}}.
\]

In an idealized pencil-beam survey, the density field along a line is measured, and the one-dimensional power spectrum estimated. To relate this to the three-dimensional power spectrum, we follow Lumsden, Heavens & Peacock (1989), noting that the correlation function is the same along the line as in three dimensions, by isotropy:

\[
P_{1D}(k) = \int dx \xi(x) \exp(-i\mathbf{k} \cdot \mathbf{x}) = \frac{1}{(2\pi)^3} \int dx d^3k P_{3D}(k) \exp(ikx) \exp(-i\mathbf{k} \cdot \mathbf{x}),
\]

where we assume that the pencil beam lies along \( y = z = 0 \). The integration over \( x \) gives a one-dimensional delta function, and, changing the remaining integration over \( k_y \) and \( k_z \) to a polar integration, we obtain

\[
P_{1D}(k) = \frac{1}{(2\pi)^2} \int |k| d\mathbf{k} P_{3D}(k)|k|.
\]

Hence we see that power in one dimension comes from all shorter wavelengths in three dimensions. This is readily understandable by consideration of the following 2D\( \rightarrow \)1D illustration: imagine looking along a corrugated roof at an angle to the corrugations. The separation of peaks along the line is longer in one dimension than the wavelength by a geometrical factor, so the one-dimensional power (averaged over angles) has contributions from two-dimensional power at all shorter wavelengths.

Notice also that the one-dimensional power spectrum must be a monotonically decreasing function of \( k \) (the amplitudes of the Fourier coefficients may not be monotonic, being drawn from a Rayleigh distribution). Also note that, if the three-dimensional power spectrum has a cut-off at some large wavelength, the one-dimensional power spectrum will be constant (and non-zero) on all larger scales.

For comparison with later analysis in this paper, we define the kernel \( G(k, \hat{k}) \) such that the measured power is

\[
P(k) = \int_0^\infty d\ln |\hat{k}| P_{3D}(\hat{k}) G(k, |\hat{k}|),
\]

from which we see that the kernel for a one-dimensional skewer is

\[
G(k, \hat{k}) = 2\pi |\hat{k}|^2 \Theta(|\hat{k}| - k),
\]
where $\Theta$ is the Heaviside function. This unpleasant convolution function is shown in Fig. 1 [throughout we plot $G(k, \hat{k})/k$, since we use a linear rather than logarithmic $k$-axis]. For a practical pencil-beam survey, the kernel will be suppressed at high $k$, but this calculation illustrates the severe problems in interpreting the power spectrum of pencil-beam surveys—the one-dimensional power may be coming from much larger wavenumbers in three dimensions.

For thin, infinite plane surveys, a similar analysis (Peacock 1991) gives

$$P_{2D}(k) = 2 \int_0^\infty \frac{k}{\sqrt{k^2 - k'^2}} \, dk \, P_{2D}(k') .$$

The associated kernel $G(k, \hat{k}) = 2\hat{k}^2 \Theta(\hat{k} - k)/\sqrt{\hat{k}^2 - k^2}$ is also shown in Fig. 1. Again what one finds is that the kernel feeds a lot of power (if it exists) from high $k$ to low $k$. If the plane survey is of finite thickness, the high-$k$ part of the kernel will be suppressed, and weighting of the data can help still further (see Section 3), but it is still possible that two-dimensional power at $k$ may have little connection with three-dimensional power on such scales.

### 3 TWO-DIMENSIONAL SURVEYS

For surveys that have one dimension considerably smaller than the other two, it is sensible to reduce the dimensionality of the survey by projection onto a two-dimensional surface. We can then reduce the dimensionality of the transform correspondingly. As emphasized in the introduction, there are considerable advantages in using spherical coordinates for the analysis. The survey is almost certainly characterized by a radial selection function, and an angular selection defining the boundary (cf. Fabbri & Natalie 1990; Scharf et al. 1992; Scharf & Lahav 1993; Fisher, Scharf & Lahav 1994a; Fisher et al. 1994b, 1995). Also, the effects of redshift distortion enter only in the radial direction. In the case of a thin-slice survey, one avoids any difficulties that might be apparent in ‘flattening out’ the survey into a plane suitable for Cartesian analysis. One final point is that it is the largest scale modes in which it is most important to treat the radial nature of the distortion correctly.

In this section, we develop a two-dimensional expansion of the density field projected onto a fixed declination, allowing for redshift distortions. We use coordinates $s, \theta, \phi$, where $s = cz/H_0$ is the distance assigned on the basis of the redshift $z$, assuming uniform expansion ($H_0$ is the Hubble constant). This is the normal distance assigned in redshift surveys, and it differs from the true distance $r$ because of peculiar velocities. In this paper we consider only $z \ll 1$, but the spherical coordinate system allows the effects of non-Euclidean geometry and temporal evolution to be included if desired.

Our two-dimensional expansion is based on the three-dimensional Fourier–Bessel expansion of the density field (cf. Lahav 1993; HT95):

$$\delta_{tm}(k) = \frac{2}{\sqrt{\pi}} \int d^3r \, \delta(r) j_t(kr) Y_m^\ast(\theta, \phi),$$

with inverse

$$\delta(r) = \frac{2}{\sqrt{\pi}} \sum_{t, m} \int dk \, k^2 \delta_{tm}(k) j_t(kr) Y_m(\theta, \phi).$$

The statistical properties of $\delta_{tm}(k)$ are derived in the Appendix. We proceed by projecting all galaxies on to the central value $\theta = \theta_m$, and expanding in terms of $m$ and $k$. At this stage, we leave the choice of radial expansion function general, $f(kr)$. We also allow for radial and angular weighting of the data via the functions $w_r(s)$ and $w_\phi(\Omega)$, which may help in optimizing the signal-to-noise ratio and apodizing.

The radial weight may be $k$-dependent. For a constant-declination strip, the obvious orientation of coordinates is to have the centre of the strip at constant $\theta = \theta_m$, and to expand in terms of $m$. We let the thickness be $\Delta \theta$, and the width $\Delta \phi$, centred on $\phi = 0$. Our choice of expansion is

$$\tilde{\beta}_m(k) = \frac{2}{\sqrt{\pi}} \int d^3r \, \rho(s) f(kr) \exp(-im\phi) w_r(s) w_\phi(\Omega).$$

As an important aside, there is an issue over which frame of reference should be used for redshift-space expansions of this sort. Should the redshift be measured in the Local Group frame or the microwave background frame? In either case, the redshift distance is

$$s(r) = r \left[ 1 + \frac{(v - v_o) \cdot r}{H_0 r^2} \right],$$

where $v_o$ is the peculiar velocity of the frame of reference. This relationship is general, but, since we wish to make a perturbation expansion, we must ensure that the second term in the square brackets is always small. Assuming a sufficiently coherent velocity field such that $v$ approaches the Local Group velocity as $r \to 0$, we see that the expansion must be done in the Local Group frame.

The difference between the expansion coefficients and their mean values

$$\tilde{\beta}_m(k) = \frac{2}{\sqrt{\pi}} \int d^3r \rho_0(r) f(kr) \exp(-im\phi) w_r(r) w_\phi(\Omega).$$

can be related to the $\delta_m(k)$ by substituting for $\delta(r)$ from (9), and noting that number conservation implies that $\rho(r) d^3r - \rho(s) d^3s$:

$$D_m(k) = \delta_m(k) A_m(k, k) k^2,$$

where

$$W_r^{nm} = \sqrt{\frac{(2\ell + 1)(\ell - m)!}{4\pi (\ell - m)! (-1)^{m} \ell^{m}}} \times \frac{2 \sin \left[ (\ell - m) \Delta \phi / 2 \right]}{(m - m)} \int_{\cos(\theta_0 + \Delta \beta)}^{\cos(\theta_0)} d\mu P_{ij}^{nm}(\mu),$$

and

$$A_m(k, k) = \Phi_m(k, k) + \beta V_m(k, k),$$

$$\Phi_m(k, k) = \frac{2}{\pi} \int_0^\infty dr \rho_0(r) j_1(\ell k) f(kr) w_s(r),$$

$$V_m(k, k) = \frac{2}{\pi k^2} \int_0^\infty dr \rho_0(r) dJ_1(\ell k) w_s(r) r^2.$$

The signal part of the covariance matrix can be written as

$$\langle D_m(k) D_m^*(k') \rangle = \int d\ln k \int d\Omega P_{2D}(\hat{k}) G_{nm'}(k, k', \ell),$$

where

$$G_{nm'}(k, k', \ell) = (2\pi)^3 \sum_r Z_r^{nm} A_r(k, k) A_r(k', k') \ell^2,$$

and $Z_r^{nm} \equiv \sum_r W_r^{nm} W_r^{nm^*}$. The shot noise contribution to the covariance matrix is

$$\langle D_m(k) D_m^*(k') \rangle_{SN} = \frac{2}{\pi} \int dr d\theta d\phi \times r^3 \sin \theta \rho_0(r) f(kr) f(k'r) \times \exp [i(m - m') \phi] w_s^2(r) w_s^2(\Omega).$$

In practice, one splits $D_m(k)$ into real and imaginary parts, with similar, but more cumbersome, expressions for the covariance matrix elements. Some kernels are shown in Figs 2 and 3 for a survey with a Gaussian selection function $\exp \left[ - (r/r_*)^2 \right]$, with $r_* = 450 h^{-1}$ Mpc, and with survey limits $\Delta \theta = 6^\circ$ centred at declination $30^\circ$, and width $\Delta \phi = 90^\circ$. The radial expansion function is chosen to be a spherical Bessel function, with $\ell = 2$. This choice is motivated in two ways. First, we know that in three dimensions the Bessel functions

give narrow kernels, so they seem a good start in two dimensions. A second, related, point is that the function gives little weight to the very nearby part of the survey. Since this is the thinnest part, it is likely to contribute significantly to aliasing difficulties. The weighting scheme chosen also helps in this regard. Fig. 3 shows kernels for plane waves with almost the same wavenumbers as Fig. 2, and with direction along the central $\phi$-value, to correspond as closely as possible to Fig. 2 (cf. the analysis of the Las Campanas survey by Landy et al. 1996). These curves are simply integrals of the squared modulus of the window function transform, calculated via a 200-dimensional Fourier transform (FFT), which accounts for their slightly ragged nature. The comparison between the methods is not quite straightforward, as the two-dimensional modes look rather different. Note that the Fourier modes assume $\beta = 0$; for non-zero $\beta$, the kernels are extremely complicated in the Fourier case. The comparison is most stark if one compares the dimensionality of the objects which one needs to calculate to include redshift distortions. In essence, the two-dimensional coefficients are linear combinations of the three-dimensional coefficients. If no simplification is possible, one needs to calculate a five-dimensional Fourier transform. The two-dimensional coefficients are linear combinations of the three-dimensional coefficients.

Figure 2. Kernel function for constant-declination slice, for modes with $m = 2$, and, from left to right, $k = 0.008, 0.016, 0.025 h$ Mpc$^{-1}$. Solid lines are for $\beta = 1$, dotted lines for $\beta = 0$.

Figure 3. Kernel functions for Fourier modes with the same wavenumbers as Fig. 2.
dimensional object to calculate a range of two-dimensional coefficients. This is required if one uses Fourier modes (Zaroubi & Hoffman 1996, equation 10), but using the spherical modes reduces the dimensionality such that the most complicated objects are only three-dimensional (see equation 13). It is this fact, that the kernels for non-zero $\beta$ can be readily calculated for the spherical modes, which is their major advantage. It arises, of course, from the radial nature of the distortion and selection function, and the use of angular coordinates to delimit the survey.

For high $n$ the spherical kernels are sometimes not centered on $k$, and the two-dimensional power may come principally from shorter wavelengths in three dimensions. The effects of this, and shot noise and cosmic variance, can be accounted for correctly using likelihood techniques, so the three-dimensional power spectrum can be estimated, but it is clear from Figs 2 and 3 that the task is not going to be easy, whichever method is used. The accuracy with which the power and $\beta$ determination can be done with the Fourier–Bessel transform is explored in the next sections.

3.1 Parameter estimation

We can use the analysis method presented in the last section to estimate the real-space power spectrum and $\beta$ by maximizing the likelihood. Symbolically,

$$L(\beta, P) = \frac{1}{(2\pi)^N |C|^{1/2}} \exp \left( -\frac{1}{2} \sum_{ij} D_i C^{-1} D_j \right), \quad (19)$$

where $C$ is the covariance matrix of the $N$ data values, dependent on the (parametrized) $P(k)$ and $\beta$. Once again, the data are the real and imaginary parts of $D_n(k)$. The likelihood method has the advantage that all the aliasing effects are treated correctly, and we do recover three-dimensional power estimates with correct error bars. We illustrate this method by analysing a numerical simulation created with Couchman’s AP$^3$M code (Couchman 1991). A slice between declinations 20° and 40° is projected on to declination 30°, with a right ascension range of 90°. The power spectrum is a power law $P(k) \propto k^{-1}$, and the likelihood for the amplitude of $P(k)$ and $\beta$ is shown in Fig. 4. The details of the analysis are that the non-linear wavenumber [where $k^3 P(k)/(2\pi^2) = 1$] is 183 (units are arbitrary), and the analysis examines modes up to $k_{\max} = 30$. Pushing the maximum analysed wavenumber beyond this pushes $\beta$ down, as the effects of fingers-of-God become apparent. These could be reduced by including the effects of smoothing (cf. HT95), but they have not been incorporated here. The $m$-modes are analysed in steps of 2 from 2 to 20, and the wavenumbers selected are from 6 to 30 in steps of 2. There is no difficulty in principle in taking every $m$- and $n$-mode, but there is a numerical problem as adjacent modes are too strongly correlated and the matrix becomes numerically singular. Note that modes separated by 2 are still correlated, and the correlations are correctly accounted for in (19). We also put a constraint on the wavenumber perpendicular to the line of sight, ensuring that it is not too non-linear, by rejecting modes with $mk/4.0 < k_{\max}$. This ensures that the transverse wavenumber at the peak of $j_r(kr)$ is no more than 4.0/3.3 times $k_{\max}$. Experimentation shows that this gives unbiased estimation of $\beta$ and $P(k)$. The galaxies are weighted with the

Feldman, Kaiser & Peacock (1994) optimized weighting $w_i(r) = [1 + \rho_i(r) P(k)]^{-1}$, and $P(k)$ in the weighting is taken as a constant, comparable to the true power in the simulation. The true parameters are shown by the encircled cross. We see that the method is capable of determining $\beta$ and the power spectrum with somewhat larger errors than a fully three-dimensional survey (HT95) with similar numbers of objects, and also note that here we need to examine larger wavelengths than in the three-dimensional case (up to the non-linear wavenumber/6, as opposed to the non-linear wavenumber/3 in the three-dimensional case). Failure to do this leads to understimation of $\beta$ because of non-linear effects.

3.2 Errors on $\beta$ and $P(k)$

In this section, we calculate the expected errors in a deep, thin-slice survey, such as might be achievable in the first year of the Anglo-Australian Telescope Two-Degree Field (AAT 2dF) survey. The error on the parameters is readily estimated using the Fisher information matrix (Tegmark, Taylor & Heavens 1997).

For a set of parameters $\theta_i, i = 1, N$ (e.g. $\beta$ and the power spectrum in $N - 1$ wavenumber bins), the covariance matrix of the parameter estimates is

$$F = \text{Tr} \{C^{-1} \partial \theta_i \partial \theta_j \}, \quad (20)$$

where $\partial \theta_i \partial \theta_j$ is the Fisher information matrix. $C$ is the covariance matrix of the data $\langle \mathbf{D} \mathbf{D}^\top \rangle$, and $C_{ij} = \partial \theta_i \partial \theta_j$. The Fisher matrix is obtained from the data covariance matrix (16). To illustrate this, we calculate the parameter covariance matrix for a thin slice, 6° × 90°, with a Gaussian selection function

$$\rho_i(r) = \rho_\star \exp (-r^2/r_\star^2).$$

We take $\rho_\star = 0.02 h^{-3} \text{ Mpc}^{-3}$ and $r_\star = 450 h^{-1} \text{ Mpc}$, broadly comparable to an optical survey to a limit $b = 19.5$ (cf. the forthcoming AAT and Sloan galaxy surveys).

Figure 4. Likelihood function for the amplitude of the real-space power spectrum and $\beta$ for a numerical simulation whose true parameters are shown by the cross. The contours are separated by 0.5 in ln(likelihood).
We analyse modes from $m = 2$ to 20, once again separated by 2 to avoid the covariance matrix becoming numerically singular. The $k$-values are spaced by $0.0167\, h\, \text{Mpc}^{-1}$, and the modes are analysed up to $k = 0.05\, h\, \text{Mpc}^{-1}$, consistent with our previous numerical experiments for unbiased results. The summations extend to $\ell = 60$, and the $k$ integrations extend to $k = 0.165\, h\, \text{Mpc}^{-1}$. The galaxies are weighted with the Feldman et al. (1994) optimized weighting $w(r) = [1 + p_0(r)P(k)]^{-1}$, and $P(k)$ in the weighting is taken to be $2700\, h^{-3}\, \text{Mpc}^3$. The expected error on $\beta$ from such a slice is 0.236, and the expected fractional error in $\beta$ is shown in Fig. 5, for a power spectrum assumed to be smooth on a scale of $0.0167\, h\, \text{Mpc}^{-1}$. Increasing the width of these $k$ bins decreases the error. Note how the error increases at the high-$k$ end beyond the maximum wave-number analysed (4.0/3.3 times 0.05 $h\, \text{Mpc}^{-1}$ ± 0.06), and at the low-$k$ end, where the size of the survey becomes comparable to the wavelength ($2\pi/r_s \approx 0.014$). The correlation matrix for the parameters is shown in Table 1. This analysis takes only a matter of minutes on a workstation, once the matrices $D$ and $V$ have been calculated. These take a few hours, but are calculated once only for a given survey. What is apparent from this example is that, even for a deep survey with many objects, $P(k)$ is detected on scales of the survey $k \sim 2\pi/450 \sim 0.014\, h\, \text{Mpc}^{-1}$, but not with good accuracy. $eta$-estimation is actually not bad (error 24 per cent), but this could be improved noticeably (to 15 per cent) if the three-dimensional power spectrum is determined independently, from a sky-projected catalogue such as the APM survey. An application of this method to the Las Campanas survey (Shectman et al. 1995) is in progress.

### Table 1. Correlation matrix for the parameters, in the order $\beta$ and the five fractional power spectrum measurements in order of increasing $k$.

<table>
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<tr>
<th>$\beta$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
</tr>
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<td>-0.43</td>
<td>-0.21</td>
<td>0.10</td>
</tr>
<tr>
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<td>0.21</td>
<td>0.03</td>
<td>-0.04</td>
</tr>
<tr>
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<td>-0.24</td>
<td>0.12</td>
<td>-0.03</td>
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<td>-0.24</td>
<td>1.00</td>
<td>-0.12</td>
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<td>1.00</td>
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</tr>
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<td>-0.04</td>
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<td>-0.24</td>
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<td>1.00</td>
</tr>
</tbody>
</table>


## 4 THREE-DIMENSIONAL SURVEYS:

**OPTIMIZING FOR POWER ESTIMATION**

We have shown in the previous section how the expected error on a parameter may be estimated in advance for a given analysis method and survey design. The conclusion that two-dimensional surveys are not particularly good for determining large-scale three-dimensional power suggests that genuine three-dimensional surveys may be more profitable. The issues of design and analysis are also relevant in three dimensions, and here we consider the problem of optimizing the design of a galaxy redshift survey to measure the power spectrum on some particular scale. The typical decisions to be made are whether to go for a deep survey over a small area of sky, or a shallower survey over a wider area of sky. We also consider whether it makes sense to sample the galaxies sparsely, or to observe every one. This section is essentially a Fourier analogue of Kaiser’s (1986) treatment of sparse-sampling to estimate the two-point correlation function, generalized to account for a radial selection function, and with the proper power error estimate of Feldman et al. (1994) incorporated.

For simplicity, we make the following assumptions: we assume that the effects of redshift-space distortions are small, and we assume that the observing time for each galaxy is proportional to the inverse square of its flux. The former is motivated by earlier studies (HT95) where the optimal weighting was found to be insensitive to the degree of redshift distortion (see also Hamilton 1997). The latter assumption is an example; different constraints, for fibre systems for example, could be incorporated if desired. Our prime constraint is that the total duration of observing is taken to be fixed. In considering sparse-sampling, we restrict attention to a sparsely sampled fraction which is constant for all galaxies in the parent sample. Thus, for example, we do not consider a variable sparse-sampling rate which depends on flux.

It was shown by Feldman et al. (1994), HT95 and Hamilton (1997) that the optimal weighting of galaxies in the survey is

$$w(r) = \frac{1}{1 + fn(r)P(k)},$$

where $P(k)$ is the (prior estimate of the) power to be measured, and $n(r)$ is the mean number density of galaxies at position $r$. We introduce the possibility of sparse-sampling by multiplying this number density by a factor $f$.

Feldman et al. demonstrated that this weighting gives rise to an error in the power of $\sigma_P^2/P^2 = (2\pi)^3/(V_f I)$, where $V_f$ is the volume of $k$-space over which the power is averaged, and

$$I(f, S, \Omega) = \int dr \frac{r^2}{1 + fn(r)},$$

Here $S$ and $\Omega$ are the flux limit and solid angle of the survey. Our problem then reduces to maximizing $I$ with respect to $f$, $S$ and $\Omega$, subject to the constraint that the total observing time is fixed.
To do this optimization, we need the luminosity function $\Phi(L)$, from which the number density is obtained:

$$n(r, S) = n_0(X) \equiv \int_0^\infty dL \Phi(L),$$

(24)

where $X = 4\pi^2 S$. If the time to observe an object of flux density $S'$ is $\lambda/(16\pi^2 S'^2)$ for some constant $\lambda$, then the time to observe a fraction $f$ of all objects to a flux limit $S$ in a solid angle $\Omega$ reduces to

$$t(f, S, \Omega) = \frac{\Omega \lambda f}{2(4\pi S)^{3/2}} \int_0^\infty dXX^{3/2} n_2(X),$$

(25)

where

$$n_2(X) \equiv \int_0^\infty dL \Phi(L) L^{-2}.$$

The time constraint then simply yields $S \propto (\Omega f)^{3/7}$, and the error is minimized when

$$\frac{\Omega^{3/7}}{f^{3/7}} \int dX \left[ \frac{X^{1/2}}{1 + \frac{1}{f \Phi n_2(X)}} \right]^2$$

(26)

is maximized. $\Omega$ and $f$ may be chosen freely, apart from the obvious limits, with the depth of the survey $S$ being dependent on the choice. We see immediately that the error is minimized if $\Omega$ is made as large as possible. This is a quite general result, consistent with the general knowledge that surveys should be wide before being deep. If we fix the solid angle of the survey (as large as is convenient), then we can straightforwardly solve for $f$ to optimize the error. The analysis is readily generalized for observing times that are proportional to $S^{\alpha - \gamma - \beta}$ ($\alpha = \beta = 2$ might be appropriate for fixed-width slit spectroscopy). In this case, one maximizes

$$f^{-1/(1 + 2\alpha - \beta - \beta)} \int dX \left[ \frac{X^{1/2}}{1 + \frac{1}{f \Phi n_2(X)}} \right]^{\alpha - 1/2},$$

(27)

Fig. 6 illustrates the effect for a Schechter luminosity function $\Phi(L) dL = \phi^* (L/L^*)^{-\alpha} \exp(-L/L^*) dL/L^*$, with $\phi^* = 0.013 h^3$ Mpc$^{-3}$. The optimal sampling occurs at $fP \approx 500 h^{-3}$ Mpc$^3$, although of course $f$ itself is bounded above by unity. To the left of the minimum, shot noise becomes dominant, whereas, to the right, the extra sampling reduces the volume observable, so that cosmic variance dominates. To estimate $f$, the power at a wavenumber $k = 0.01$ to 0.1 h Mpc$^{-1}$ is about 1000–10 000 $h^{-3}$ Mpc$^3$, depending on the galaxy type and theoretical prejudice (e.g. Baugh & Efstathiou 1993, 1994; Ballinger, Heavens & Taylor 1995), which motivates a sparse-sampling strategy of $f = 0.1$. The error rises rapidly if $fP \lesssim 100$, so one must take care not to undersample. If the power spectrum on large scales has the Zel’dovich form $P \propto k$, a survey to measure very large-scale power should be sampled fully.

5 CONCLUSIONS

In this paper we have presented a new method for analysing thin, near-constant-declination slice surveys, using a two-dimensional projection and expansion in radial and angular functions. We have also considered the optimization problem of depth and sparse-sampling for three-dimensional surveys. There are two main advantages of using spherical coordinates for analysis. The first is that the survey is usually defined by a fixed areal coverage, and a flux limit that leads to a selection function which is purely radial. The second advantage is that the effect of redshift distortion is radial, so it is straightforward to include it in the analysis. By expanding in spherical and angular functions, one can treat linear redshift distortions without further approximations, and this allows, in particular, analysis of long-wavelength modes which do not subtend small angles on the sky. By using carefully chosen radial functions for the analysis, one can ensure that the modes one analyses essentially include only three-dimensional modes which are still linear.

For an all-sky survey, the formalism leads to very simple analysis, and this method is clearly the best that we have to date. What was not clear was whether the method could be adapted for surveys of relatively small areal coverage, since the mixing of modes of different $\ell$ and $m$ makes the expansion more cumbersome. This paper shows that, even with thin, essentially two-dimensional surveys, one can retain the advantages of the spherical expansion without severe additional complexity. Our error analysis shows that three-dimensional power can be estimated from two-dimensional surveys, properly including the effects of aliasing, mode–mode correlations, shot noise and cosmic variance. However, the reduction in dimensionality means that the errors achievable are unlikely to be very small. This is in contrast to three-dimensional surveys, where small errors on the real-space power spectrum can be achieved from the Fourier–Bessel technique (Ballinger et al. 1995). In two dimensions,
\( \beta \) can be determined with reasonable accuracy (about 25 per cent), but the best approach will probably be to use a sky-projected catalogue to estimate the three-dimensional power independently, and then to use spherical harmonics with the slice to measure \( \beta \) with higher accuracy (about 15 per cent).

We also show in this paper that three-dimensional surveys may be optimized for measuring three-dimensional power, given a constraint on total observing time, by choosing as wide an area of sky as possible, and by sparse-sampling at a rate which is dependent on the expected power to be measured.

REFERENCES


Broadhurst T. J., Ellis R. S., Koo D. C., Szalay A. S., 1990, Nat, 343, 726


APPENDIX A

In this appendix, we calculate the covariance matrix for the continuous spherical transform. We transform the density field

\[
\delta(r) = \frac{1}{(2\pi)^3} \int d^3k \delta(k) e^{i\mathbf{k} \cdot \mathbf{r}}
\]

and expand the exponential as a sum of Bessel functions:

\[
\delta(r) = \frac{1}{2\pi^2} \int dk \, d\Omega_k \, \delta_k
\]

\[
\times \sum_m i^n (kr) Y_m^*(\theta_k, \phi_k) Y_m(\theta, \phi) k^2,
\]

\[(A1)\]

where \( \Omega_k = (\theta_k, \phi_k) \) is the direction of the \( k \)-vector, and \( (\theta, \phi) \) is the direction of \( r \). The definition of the spherical harmonics used in this paper is that found in Binney & Tremaine (1987):

\[
Y_m^*(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \frac{\Gamma(\frac{\ell+1}{2})}{\Gamma(\frac{\ell-m}{2})} P_\ell^m(\cos \theta) \times \exp(im\phi) \times \begin{cases} \frac{1}{2} & m \geq 0 \\ \frac{1}{2} & m < 0 \end{cases}.
\]

The spherical expansion is

\[
\begin{aligned}
\delta_m(k) &= \frac{3}{2\pi} \int d^3r \delta(r) j_0(kr) Y_m^*(\theta, \phi), \\
\text{which becomes}
\delta_m(k) &\equiv (2\pi)^{-3/2} i \int d\Omega_k \delta_k Y_m^*(\theta_k, \phi_k) \delta^0(k - \tilde{k}),
\end{aligned}
\]

\[(A4)\]

where we have used the orthogonality (Binney & Quinn 1991)

\[
\int d\Omega j_0(k'r) j_0(kr) R_m^*(\theta, \phi) R_m(\theta, \phi) = \frac{\pi}{2kk'} \delta^0(k - k') \delta_\omega \delta_{\omega m},
\]

\[(A5)\]

where \( \delta^0 \) and \( \delta^\omega \) are Dirac and Kronecker delta functions respectively. The covariance matrix is

\[
\langle \delta_m(k) \delta^*_{m'}(k') \rangle = (2\pi)^{-3/2} \delta_\omega \delta_{\omega m} \delta_{\omega m'},
\]

\[
\times \int \delta_\omega \delta_{\omega m} \delta_{\omega m'} Y_m^*(\theta_k, \phi_k) Y_{m'}(\theta_k, \phi_k) \delta^0(k - \tilde{k}) \delta^0(k' - \tilde{k}')
\]

\[(A6)\]

Defining the power spectrum by

\[
\langle \delta_\omega \delta^*_{\omega} \rangle = (2\pi)^3 P(k) \delta^0(k - \tilde{k})
\]

\[
= (2\pi)^3 P(k) \delta^0(k - \tilde{k}) \delta^0(\mu_k - \mu_k) \delta^0(\omega_k - \omega_k)
\]

\[(A7)\]

\( (\mu \equiv \cos \theta) \), the analogous expression of orthogonality for the Fourier–Bessel modes is

\[
\langle \delta_m(k) \delta^*_{m'}(k') \rangle = P(k) \frac{\delta^0(k - \tilde{k})}{k^2} \delta_\omega \delta_{\omega m} \delta_{\omega m'}.
\]

\[(A8)\]
The power is evenly divided between real and imaginary parts, except for \( m = 0 \) modes, which are real.

### A1 Redshift distortions

In a redshift-space map, galaxies are placed at a position \( s = (s, \theta, \phi) \), where the distance coordinate \( s \) is the recession velocity divided by the Hubble constant \( H_0 \). In general this is not the true distance because the galaxy may have a peculiar velocity \( v \). The redshift-space position is then related to the real-space position \( r \) by

\[
s - r = \frac{v \cdot \hat{r}}{H_0}
\]

where \( v_0 \) is the peculiar velocity locally.

To expand the spherical expansion to linear order, we first note that \( \rho(r) \, d^3r = \rho(s) \, d^3s \), and make a Taylor expansion of the resulting integrand to first order in \( s - r \):

\[
j_{\ell}(ks)w_{\ell}(s) \approx j_{\ell}(kr)w_{\ell}(r) + (s - r) \frac{d}{dr} [j_{\ell}(kr)w_{\ell}(r)].
\]

To obtain an expression for \( s - r \), we assume potential flow \( v = -V\Phi \) (valid for linear, growing-mode perturbations), where \( \Phi(r) \) is the velocity potential. The effect of the Local Group velocity \( v_0 \) is to add an extra term to the mean of the transform coefficients, and will be treated separately. Expanding \( \Phi \) in terms of \( \Phi_{\ell m}(k) \), we find

\[
\int d\Omega v_0(\Omega) \exp(-im\phi)\hat{b}_0 \cdot \hat{r},
\]

where \( \hat{r} \) and \( \hat{b}_0 \) are unit vectors.

The peculiar Poisson equation \( \nabla^2 \Phi = \beta \delta(r) \) relates the potential to the galaxy overdensity field, which leads to \( \Phi_{\ell m}(k) = -\beta \delta_{\ell m}(k)/k^2 \). From this we find (choosing units such that \( H_0 = 1 \))

\[
s - r = \frac{1}{H_0} \sum_{\ell m} \int dk \Phi_{\ell m}(k) \frac{dj_{\ell}(kr)}{dr} Y_{\ell m}^*(\theta, \phi) k^2.
\]

This leads to the \( V \) matrix terms in the main text. The effect of the Local Group velocity is to add the following to the mean value of \( D_m(k) \):

\[
\frac{v_0}{H_0} \sqrt{\frac{2}{\pi}} \int dr \left( \frac{d\rho_0}{dr} - 2r \rho_0 \right) w_{\ell}(r) f(kr)
\]

\[
\times \int d\Omega v_0(\Omega) \exp(-im\phi)\hat{b}_0 \cdot \hat{r},
\]

where \( f \) and \( \hat{b}_0 \) are unit vectors.