On the Admissible Wave Functions

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There seems to be some confusion about the condition for "physically admissible" wave functions. Recently Araki\(^{(1)}\) proposed the condition that the wave functions of a quantum-mechanical system should be included in the domains of all Hermitian operators representing physical quantities of the system. But this condition is too strong to be maintained.

If we assume with v. Neumann\(^{(2)}\) that all self-adjoint (or hypermaximal) operators of the Hilbert space correspond to physical quantities, it is evident that there is not a single function (other than zero) satisfying Araki's condition. Even if we avoid such an extreme point of view, the components of momentum \(p\) of a particle are admitted to be physical quantities simultaneously observable.\(^{(3)}\) Hence all functions of them, above all the kinetic energy \(T=(2m)^{-1/2}p^2\) and its square \(T^2\), are also regarded as physical quantities observable at the same time as \(p\). But the ground state wave function of the hydrogen atom is not in the domain of \(T^2\), so that we should be obliged to exclude it if we accepted Araki's condition.

On the other hand, the condition for admissible wave functions introduced by Kemble\(^{(4)}\) is defined with reference to a particular Hamiltonian \(H\). Roughly speaking, it is equivalent to the requirement that the wave functions should be in the domain of \(H\). Thus it is weaker than Araki's condition and leads no more to contradiction, but it is still cumbersome considerably.

In the writers' opinion, however, these complicated conditions are unnecessary, and all quadratically integrable functions can be regarded as physically admissible. The domain of an operator has no essential physical meaning, although it is quite important from the mathematical standpoint. That a wave function \(\psi\) belongs to the domain of a self-adjoint operator \(A\) representing a physical quantity, is equivalent to the condition that the expectation value of \(A^2\) in the state \(\psi\) is finite.\(^{(6)}\) It is not plausible that such a condition is of any physical significance. Whether this expectation value is finite or not, there is definite probability for each possible value of \(A\) to be observed in the state \(\psi\), provided that \(\psi\) is in the Hilbert space. Thus there seems to be no reason why all functions of the Hilbert space should not be permitted to represent physical states of the system. Perhaps one might object that the wave functions \(\psi\) must be in the domain of the Hamiltonian \(H\) because the Schrödinger equation of motion contains the term \(H\psi\). But this objection is untenable, for the equation of motion generates a one-parameter group of unitary operators and these are naturally defined everywhere in the Hilbert space.

It seems that the cited authors are led to these unnecessary conditions from the necessity to exclude some of the solutions of the wave equation from the family of eigenfunctions, e. g., the
S-state solutions of the hydrogen wave equation which are of the order \( r^{-1} \) at the origin. But this is quite a different question. It is merely a truism that an eigenfunction of an operator \( A \) must be in domain of \( A \), and this is sufficient to exclude the above-stated superfluous solutions, for they are not in the domain of the self-adjoint Hamiltonian operator and hence cannot be its eigenfunctions.(6)