Evolution of Stars of Small Masses in the Pre-Main-Sequence Stages

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The structure of the outer envelope with an H-ionization zone and an H\textsubscript{2}-dissociation zone is investigated for Population I stars of small masses ($2M_\odot \geq M \geq 0.05M_\odot$), which have low luminosities ($L \geq 10^{-3}L_\odot$) and low effective temperatures ($6000 \, ^\circ\text{K} \geq T_e \geq 2500 \, ^\circ\text{K}$), in order to find the surface condition for the internal structure of these stars. The effective temperature of a star which is wholly convective and which has an H\textsubscript{2}-dissociation zone is found to be nearly constant in the wide range of its luminosity.

Using stellar models composed of a radiative core and a convective envelope together with the above surface condition, the evolution of contracting stars is calculated up to the onset of hydrogen burning and the results are compared with the observed red dwarf stars. It is found that the stars on the zero-age main-sequence have radiative cores for $M > 0.26M_\odot$ but they are wholly convective for $0.26M_\odot \geq M \geq 0.08M_\odot$. The stars less massive than $0.08M_\odot$ are found to contract toward the configurations of high electron-degeneracy without hydrogen burning.

§ 1. Introduction

The stars of low surface temperatures are known to have outer convection zones due to the presence of hydrogen ionization zones. Based on the results obtained by Hayashi and Hoshi\textsuperscript{1} on the structure of these outer envelopes, i.e. on the surface condition of the stars, one of the authors\textsuperscript{2} pointed out that the stars in the early pre-main-sequence stages are wholly convective and they evolve downward nearly vertically in the HR diagram. These evolutionary tracks are quite different from the nearly horizontal tracks which were obtained by Henyey et al.\textsuperscript{3} under the assumption that the stars are wholly radiative.

In a recent paper by Hayashi, Hoshi and Sugimoto\textsuperscript{4} the surface condition of the stars was calculated in the mass range $4.0 \, M_\odot$ to $0.05 \, M_\odot$ and the contraction of the stars toward the main-sequence was studied using a sequence of stellar models composed of a radiative core and a convective envelope, which occupy various fractions of the total mass. In this paper, however, the existence of hydrogen molecules near the surface, which was shown by Vardya and Wild\textsuperscript{5} playing an important role in determining the surface condition for the red dwarf stars, was not taken into account and then the calculations were not extended to the region of low effective temperature and low luminosity in the HR diagram.

\textsuperscript{*} This paper will be referred to as HHS in the following.
In the present paper, taking into account the \( \text{H}_2 \)-dissociation zone and the effect of electron-degeneracy, the work of HHS is extended to effective temperature as low as \( 2500^\circ \text{K} \) and luminosity as low as \( 10^{-3}L_\odot \). The chemical compositions are taken as \( X=0.61 \) (concentration by mass of hydrogen), \( Y=0.37 \) (helium) and \( Z=0.02 \) (metal) as in HHS. The structure of the outer envelope is studied first using an adiabatic temperature gradient in order to obtain a simple analytic expression of the surface condition, i.e. curves of constant \( E \) in the HR diagram, and later a small amount of deviation from the adiabat is calculated according to the mixing-length theory of convection. The curves of constant \( E \) are found to be nearly vertical for a star which has the \( \text{H}_2 \)-dissociation zone.

Using the above surface condition and the models of contracting stars, the evolution of the stars in the mass range \( 0.6M_\odot \) to \( 0.05M_\odot \) is calculated up to the onset of \( pp \)-reactions in the same way as in HHS. The stars less massive than about \( 0.08M_\odot \) are found to continue the contraction toward highly degenerate configurations without hydrogen burning. This critical mass for hydrogen burning is nearly in agreement with the value which was found by Kumar without taking into account the surface condition. The stars less massive than about \( 0.26M_\odot \) are found to be wholly convective on the zero-age main-sequence and their positions in the HR diagram are nearly in agreement with observed red dwarfs of known masses if the uncertainty in the opacity is taken into account.

Further, the loci of constant ages of the contracting stars are drawn in the HR diagram to compare with observed young star-clusters, and also the depletion of lithium in the convective envelope during the contraction to the main-sequence is calculated. Finally, discussions are made on the various sources of uncertainties involved in the calculations of the surface condition and the evolution.

§ 2. The structure of an outer envelope and the surface condition

It was shown in HHS that the relation between pressure and temperature in an outer convection zone of a star, which is less luminous than \( (E/40)^{1.8}L_\odot \) where \( E \) is a parameter describing the surface condition, is approximately given by an adiabat up to the immediate neighborhood of the surface, while the more luminous stars have sizable radiative regions where the convective motion is not sufficient to transport the outward energy flux. The approximation of the adiabatic temperature gradient is more satisfactory in the less luminous stars under consideration, which have dissociation zones of hydrogen molecules.

The regions in the pressure-temperature diagram, where the dissociation of hydrogen molecules and the ionization of hydrogen atoms are incomplete, are illustrated by Fig. 1 together with two adiabats as examples. The concentrations
of hydrogen atomic ions and molecules relative to protons are denoted by \( x = n_{H^+}/n_p \) and \( y = n_{H_2}/n_p \), respectively, where \( n_p = X/m_H \) is the number density of protons.

Fig. 1. Pressure and temperature diagram. The curves \( P_1C_1, P_2C_2 \) and \( P_3C_3 \) are the adiabats which connect the photospheres \( P \) and centers \( C \) of wholly convective stars; (1) \( M=0.1M_\odot \) and \( L=L_\odot \), (2) \( M=0.6M_\odot \) and \( L=0.9L_\odot \), and (3) \( M=0.1M_\odot \) and \( L=10^{-8}L_\odot \), respectively. The pressure ionization prevails in the regions of higher pressure than the line \( XY \), which corresponds to \( \rho = 3m_H/4\pi r_0^3 = 2.7 \text{ g cm}^{-3} \) where \( r_0 \) is the Bohr radius.

The adiabat is a curve of constant entropy and its behavior in the hydrogen ionization zone was studied in HHS. The entropy \( s \) per unit mass of perfect gas in the dissociation zone of hydrogen molecules, which is composed of \( \text{H, H}_2 \) and \( \text{He} \), is given by:

\[
\frac{s}{kn_p} = (1 - 2y) \left\{ \frac{5}{2} + \ln \frac{2(2\pi m_H)^{3/2}(kT)^{3/2}}{\hbar^3 P_{H}} \right\} \\
+ y \left\{ \frac{7}{2} + \ln \frac{(2\pi m_{H_2})^{3/2}(kT)^{3/2}}{\hbar^3 P_{H_2}} \frac{4\pi^3 kT}{h^2 \{1 - \exp(-\hbar v/kT)\}} \right\} + \frac{\hbar v}{kT} \left\{ \exp(\hbar v/kT) - 1 \right\} \\
+ \delta \left\{ \frac{5}{2} + \ln \frac{(2\pi m_{\text{He}})^{3/2}(kT)^{3/2}}{\hbar^3 P_{\text{He}}} \right\},
\]

\( \delta \) We distinguish between \( \ln x = \log_e x \) and \( \log x = \log_{10} x \).
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where \( \delta = n_{\text{He}}/n_e = 0.152 \), and \( I \) and \( \nu \) are the moment of inertia and the frequency of vibration of a hydrogen molecule, respectively,

\[
h^2/8\pi^2kI = 85.4 \text{K}, \quad h\nu/k = 6.10 \times 10^3 \text{ K}.
\]

The formula for dissociation of hydrogen molecules is given by

\[
\frac{P_h^2}{P_{\text{He}}P} = \frac{(1-2y)^2}{y(1+\delta-y)} = \left(\frac{\pi m_kT}{\hbar I_1}\right)^{\delta/2} \{1 - \exp(-h\nu/kT)\} \exp(-z/kT),
\]

where \( z = 4.48 \text{ eV} \) is the dissociation energy.

It is to be noticed that the expressions (1) and (3) are applicable at temperatures \( kT < x \), where both the mean rotational energy and the mean vibrational energy of a molecule are much lower than the dissociation energy. At temperatures below 4000 K where we have used the above expressions, the expression (3) is not different by more than 10 percent from the dissociation formula which was obtained by Vardya\(^7\) taking into account the deviation of molecular vibration from a harmonic oscillator and the deviation of molecular rotation from a rigid rotator.

As in HHS, the radiative temperature gradient is given by

\[
P_{\text{rad}} = \frac{d\log T}{d\log P} = \frac{\kappa P}{AT^4}, \quad A = \frac{16GM}{3R^2T_e^4},
\]

in a region inside the photosphere which is defined by

\[
\int_0^\infty \kappa \rho \, dr = 2/3,
\]

with its outer region being regarded as isothermal. The opacity \( \kappa \) in the temperature range \( 1.0 \leq \theta = 5040/T \leq 2.0 \), is taken as due to continuous absorption by negative hydrogen ions,\(^9\) which is given by

\[
\kappa = 10^{-1.19} X(1-2y)P_e \theta^{4.3}.
\]

The electron pressure \( P_e \) as a function of pressure and temperature is taken from the table compiled by Vardya\(^10\) and the factor \( (1-2y) \) is calculated with Eq. (3). Then, the opacity is found to be expressed, within an uncertainty of factor 2.5 in the relevant ranges \( \log P = 4-8 \) and \( \theta = 1-2 \), in the form

\[
\kappa = \kappa_0 P^\alpha T^\beta,
\]

with

\[
\begin{align*}
\kappa_0 &= 10^{-13.39}, \quad \alpha = 0.695, \quad \beta = 2.44 \quad (0.1 > y), \\
\kappa_0 &= 10^{-31.36}, \quad \alpha = 0.395, \quad \beta = 7.89 \quad (0.3 > y \geq 0.1), \\
\kappa_0 &= 10^{-45.96}, \quad \alpha = 0.218, \quad \beta = 12.34 \quad (y \geq 0.3).
\end{align*}
\]

Recently, Yamashita\(^11\) made a calculation of the opacity in the atmosphere of \( M \)-type dwarfs taking into account the existence of \( \text{H}^- \), \( \text{H} \), \( \text{H}_2 \), metallic atoms,
CO, H$_2$O, OH and other molecules. In the pressure range $2 \leq \log P \leq 6$ at temperatures 2500$^\circ$K and 3000$^\circ$K, the opacity obtained by him does not differ from the expressions (7) and (8) by more than a factor 2.5. The small effect of this difference on the surface condition will be discussed in § 4.

The convective region with an adiabatic temperature gradient $\rho a$, is assumed to begin at a point where the condition $\rho a = \rho a$ is satisfied, or to begin at the photosphere if $\rho a$ is found to be already larger than $\rho a$ at this point. When electrons are not degenerate in the convection zone, we follow the same procedure as in HHS which connects two points of equal entropy in the convection zone, i.e. its outer boundary and a point $b$ where the ionization of hydrogen atoms is complete. Then, we obtain an expression of $K$ at the point $b$,

$$K = \frac{P_b}{T_b^{\frac{3}{2}}} = (2 + \delta) \left\{ \frac{(2\pi m_e)^{\frac{3}{2}} k^{\frac{3}{2}} (1 - 2y)}{h^2} \right\}^{\frac{1}{2}(2 + \delta)} \left\{ \frac{P}{T^{\frac{3}{2}} \left( 1 + \delta - \frac{1}{y} \right)} \right\}^{\frac{(1 - \delta)(2 + \delta)}{2}} \exp \left[ \frac{y}{2 + \delta} \left\{ \frac{5}{2y} + \frac{3}{2} + \frac{z}{kT} \frac{1}{\exp \left( \frac{h\nu}{kT} \right) - 1} \right\} \right],$$

where $P$, $T$ and $y$ are the values at the outer boundary of the convective region.

When electrons are degenerate, the degree of degeneracy $\psi$ is constant in the inner convective region where the ionization of hydrogen is complete, as was shown by Limber, and the structure of this region is represented again by the solution of polytropic index $3/2$. The equation of state and the definition of $K$ in this region are generalized to

$$P = k \rho T / \mu m_H A(\psi),$$

$$K = \frac{P}{T^{\frac{3}{2}}} = \frac{4\pi}{h^2} (2m_e)^{\frac{3}{2}} k^{\frac{3}{2}} (2 + \delta) \frac{F_{\frac{3}{2}}(\psi)}{A(\psi)},$$

with

$$A(\psi) = (2 + \delta) / \left\{ 1 + \delta + \frac{2F_{\frac{3}{2}}(\psi)}{3F_{\frac{1}{2}}(\psi)} \right\}^{\frac{1}{2}},$$

where $F_n(\psi)$ is the Fermi-Dirac function and $A(\psi)$ is unity in the non-degenerate case, $\psi = -\infty$. Further, the entropy $s$ per unit mass is given by

$$\frac{s}{kn_p} = \frac{5}{2} (1 + \delta) + \ln \left\{ \frac{V \pi}{4} \left( \frac{m_H}{m_e} \right)^{\frac{3}{2}} \right\} + \delta \ln \left\{ \frac{V \pi}{4} \left( \frac{m_H}{m_e} \right)^{\frac{3}{2}} \right\} + \frac{5}{3} \frac{F_{\frac{3}{2}}(\psi)}{F_{\frac{1}{2}}(\psi)} - \psi - (1 + \delta) \ln F_{\frac{1}{2}}(\psi),$$

where the zero-point of entropy has been taken to be the same as in the expression (1).

Equating the entropies (13) with (1) and following the same procedure as
in the non-degenerate case, the expression of $K$ in the degenerate case is found to be given by Eq. (9) whose right-hand side is multiplied by a correction factor

$$
\frac{1}{\mathcal{A}(\phi)} \left\{ \frac{2F_{1/2}(\phi)}{\sqrt{\pi}} \right\}^{1/(a+1)} \exp \left\{ \frac{1}{2+\bar{a}} \left( \frac{5}{3} \frac{F_{3/2}(\phi)}{F_{1/2}(\phi)} - \phi - \frac{5}{2} \right) \right\},
$$

which reduces to unity in the non-degenerate case. This expression of $K$ together with Eq. (11) determines the value of $K$ when the stellar mass, luminosity and radius are given.

Generally, in the non-degenerate and degenerate cases, the value of $K$ is connected with the characteristic parameter $E$

$$
E = 4\pi G^{3/2} (\mu m_H/k)^{1/2} K M^{1/3} R^{5/3} A^{1/2}(\phi) ,
$$

which describes the degree of central condensation of the internal structure. The Emden solution of polytropic index 3/2 and then the wholly convective structure correspond to $E=45.48$, and a smaller value of $E$ corresponds to a structure composed of a radiative core and a convective envelope.

![Fig. 2. Curves of $E=45.48$ and evolutionary tracks of contracting stars in the HR diagram. The solid curves show the tracks of contracting pre-main-sequence stars. The dashed curves together with the solid curves above the open and closed circles represent the surface condition for $E=45.48$. The closed circles represent the zero-age main-sequence stars which are wholly convective. The dotted curve $\phi=0$ represents the stage of incipient degeneracy in the stellar interior.](https://academic.oup.com/ptp/article-abstract/30/4/460/1881953)
Based on the above formulations, calculations are made to find curves of constant $E$ in the HR diagram for stars of masses in the range $2M_\odot$ to $0.05M_\odot$. Here, it is convenient to use a formula which is obtained from Eqs. (11) and (15),

$$
M^{1/3} R^{8/3} = \frac{h^8}{(4\pi)^3 (2m_e G)^{1/2} m_H^{5/3}} \frac{E}{\mu m_e F_{1/3}(\psi) A^{1/3}(\psi)}.
$$

For given $E$ and $\psi$, $R$ is first found from Eq. (16) and then $L$ is calculated using Eq. (9) with the factor (14) and Eq. (15). The results for $E=45.48$ are shown in Fig. 2. It has been found that the photosphere is the outer boundary of the convection zone in every case shown in Fig. 2.

In order to show the basic features of the results, curves of $E=45.48$ and 20.77 for a star of $0.2M_\odot$ are illustrated by Fig. 3. The dotted line $c$ is an extension of the result in HHS where the existence of hydrogen molecules was neglected. The deviation from it occurs when the photosphere enters into the $H_2$-dissociation zone shown in Fig. 1, and the curves of constant $E$ go down

![Fig. 3. Curves of constant $E$ in the HR diagram for a star of $0.2M_\odot$. All the curves correspond to $E=45.48$, except for the curve $f$ ($E=20.77$). As compared with the solid curve $a$, the dotted curves represent the following cases: $b$, the constant of opacity is 2.5 times as large; $c$, the presence of $H_2$-molecules is neglected; $d$, the effect of degeneracy is neglected; and $e$, the star is completely degenerate. The dots on the curve $a$, $b$ and $f$ represent the stages of incipient degeneracy, $\psi=0$, in the stellar interior.](https://academic.oup.com/ptp/article-abstract/30/4/460/1881953)
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nearly vertically in the HR diagram since $F_{ad}$ is very small in the H$_2$-dissociation zone. As the luminosity decreases further, electrons become degenerate in the inner region of the star and the curve of $E=45.48$ approaches asymptotically a line of constant radius corresponding to the completely degenerate configuration, which is given by Eq. (16) with $\psi \to \infty$. Further, the case when the constant of opacity in Eq. (7) is taken 2.5 times as large as in Eq. (18) is shown by the dotted curve $b$ in order to illustrate the effect of the uncertainty involved in the opacity formula.

§ 3. Contraction of stars in the pre-main-sequence stages

As was shown in HHS, a star of small mass which is contracting through quasi-static equilibria in the early pre-main-sequence stages is wholly convective

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<th>$\log L/L_\odot$</th>
<th>$\log T_e$</th>
<th>$\log R/R_\odot$</th>
<th>$\log T_e$</th>
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and it evolves downward along a curve of $E=45.48$ in the HR diagram. As the contraction proceeds, a radiative region appears at the center of the star and it grows up gradually. Then, the value of $E$ decreases and the evolutionary track in the HR diagram changes its direction to the right. These evolutionary sequences were calculated in HHS for stars of $2 M_\odot$, $1 M_\odot$ and $0.6 M_\odot$ using the surface conditions and the models of a contracting star, which is composed of a radiative core and a convective envelope and in which the energy generation is proportional to the temperature and the opacity follows the Kramers law.

The surface conditions obtained in § 2 are somewhat different from those in HHS for the star of $0.6 M_\odot$, especially in a region of low luminosity in the HR diagram, but the difference is small for the stars of $2 M_\odot$ and $1 M_\odot$ in the relevant regions of their luminosities. Then, using the same stellar models and the new surface conditions, calculations along the line in HHS have been made
for stars of 0.6 \( M_\odot \) and 0.4 \( M_\odot \). The guillotine factor \( \frac{t}{\bar{g}} \) has been taken from Morse's table\(^{13}\) for compositions of metals, \( \text{O:Mg:Si:Ca:Fe}=4:1:1:1:1 \), and a geometric mean of its values at the center and at the outer boundary of a radiative core has been used. The results are tabulated in Table I, where the subscript \( a \) denotes the bottom of the convective envelope, \( E=45.5 \) correspond to the stages when the radiative core appears, and the ages are measured from a stage of infinite stellar radius. The evolutionary tracks in the HR diagram are shown in Fig. 2.

In stars of still smaller masses, hydrogen begins to burn at their centers before the radiative cores appear there, and these stars settle themselves on the main-sequence with wholly convective structures. As the central temperature of these stars is lower than \( 8 \times 10^6 \, ^\circ \text{K} \), the \( pp \)-chain reaction stops at \( \text{He}^3 \) and then the rate of energy generation\(^{14}\) in the neighborhood of \( 5 \times 10^6 \, ^\circ \text{K} \) is given by

\[
\varepsilon = \frac{10^{-1.95}}{X^* \rho(T/5 \times 10^6)^{6.4}} \text{erg/g sec.} \quad (17)
\]

In these stars, electrons are partially degenerate and \( \psi \) is constant in the region of a star where the ionization of hydrogen is complete. For the Emden solution of polytropic index 3/2 which corresponds to \( \psi = \text{constant} \), we have

\[
\rho = 1.430 \frac{M}{R^3}, \quad T = 0.538 \frac{\mu m G M}{k} A(\psi), \quad (18)
\]

where \( A(\psi) \) is given by Eq. (12). Using Eqs. (17) and (18) together with Eq. (4D.4) in HHS, the total energy generation by hydrogen burning is expressed as

\[
\frac{L}{L_\odot} = 10^{-1.44} \left( \frac{M}{M_\odot} \right)^{8.4} \left( \frac{R}{R_\odot} \right)^{-9.5} A^{0.6}(\psi). \quad (19)
\]

Thus, the stage of the zero-age main-sequence is determined as a solution of Eq. (19) and the surface condition for \( E=45.48 \).

In order that this be a real solution, it is necessary that \( \rho_{\text{rad}} \) given by Eq. (9) is larger than 2/5 not only at the center but throughout the stellar interior. This has been investigated using the tables by Morse\(^{13}\) for Kramers’ opacity and by Lee\(^{15}\) for electron-conduction opacity. Stars in the mass range 0.26 \( M_\odot \) to 0.08 \( M_\odot \) are found to be wholly convective on the zero-age main-sequence. The results for these stars are tabulated in Table II where the ages are measured from stages of infinite radii. The positions of these stars in the HR diagram are shown by the closed circles in Fig. 2.

Stars more massive than 0.26 \( M_\odot \) have radiative cores on the main-sequence, while stars less massive than 0.08 \( M_\odot \) are unable to release enough energy by hydrogen burning since their central temperatures begin to decrease after the electron-degeneracy proceeds beyond a certain limit. As an example of these wholly convective stars which evolve toward highly degenerate configurations,
an evolutionary sequence of a star of $0.07 \, M_\odot$ is shown in Table III. The above critical mass for hydrogen burning has been investigated for different chemical compositions of the stars. Its value $0.08 \, M_\odot$ is not altered even if we adopt the opacity which is five times as large (see § 4 for the change in the surface condition). This shows that the critical mass is nearly independent of the metal content. However, it depends more strongly on the helium content through the mean molecular weight. For instance, if we take the composition $X=0.90$, $Y=0.08$ and $Z=0.02$, the critical mass is about $0.12 \, M_\odot$.

Using the above results and those in HHS for the more massive stars, the loci of constant ages of the pre-main-sequence stars of different masses are drawn in the HR diagram, as illustrated by Fig. 4. During the contraction of the stars, lithium is depleted in the convective region due to its burning at the bottom of this region. The mean reaction time of $\text{Li}^7(p, \alpha)\text{He}^4$ in the neighborhood of $3 \times 10^6 \, ^\circ\text{K}$ is given by

$$\tau = 10^{1.33} (\rho X)^{-1} (T/3 \times 10^6)^{-1.8} \text{ years.} \quad (20)$$

In a wholly convective star, the mass of a core, in which the reaction time is shorter than $e$ times the central value, is about 2.5 percent of the total mass.
As is seen in Table I, the temperature $T_a$ at the bottom of the convective core attains its maximum value nearly when a radiative core appears at the center, and this is the stage when the depletion of lithium in the convective region is most active. Then, the depletion of lithium on the stellar surface is given by

$$\ln \left( \frac{\text{Li}}{\text{Li}^{(0)}} \right) = -0.025 \int dt/\tau_a, \quad (21)$$

where Li and Li$(0)$ are the final and initial concentrations of lithium in the pre-main-sequence stages and $\tau_a$ is the reaction time at the bottom of the convection zone. The factor 0.025 in Eq. (21) was omitted in HHS.

The amount of lithium depletion is very sensitive to $T_a$, as shown by Eq. (20), and then to the radius or the luminosity at the stage when the radiative core appears at the center. This luminosity in turn depends on the constant of opacity, i.e. the metal content and the guillotine factor and also on the helium content through the mean molecular weight, as was shown by Eq. (10A·3) in HHS.

Calculations of the integral in Eq. (21) are made for the two metal contents, $Z=0.02$ and 0.04 ($X=0.61$ being the same), using the results for $0.6 \, M_\odot$ and $0.4 \, M_\odot$ in Table I and the result for $1 \, M_\odot$ in Table 10-2 in HHS. In the case of $Z=0.04$, the changes in the surface condition are taken into account. The results are shown in Table IV. Further, for a hydrogen-rich composition $X=0.90$, $Y=0.08$ and $Z=0.02$, log(Li/Li$(0)$) is found to be as large as $-17$ in the star of $0.6M_\odot$. It is to be noticed that the above results show only the general tendency of the strong dependence of lithium depletion on the stellar mass and the chemical composition, in view of the uncertainties involved in the opacity and in the surface conditions (especially in the efficiency of energy transport by convection in the case of $1M_\odot$).

Table IV. Amounts of lithium depletion in the pre-main-sequence stages ($X=0.61$).

<table>
<thead>
<tr>
<th>Stellar mass</th>
<th>$1.0M_\odot$</th>
<th>$0.6M_\odot$</th>
<th>$0.4M_\odot$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log (\text{Li}/\text{Li}^{(0)})$</td>
<td>$Z=0.02$</td>
<td>$-0.02$</td>
<td>$-0.17$</td>
</tr>
<tr>
<td>$Z=0.04$</td>
<td>$-1.0$</td>
<td>$-1.6$</td>
<td>$-580$</td>
</tr>
</tbody>
</table>
§ 4. Discussions and comparison with observations

The observational values and their probable errors of the masses, luminosities and radii of three red dwarfs, which were obtained by Limber by reexamining the observational data, are shown in Fig. 5. The effective temperature of the zero-age main-sequence stars calculated in the last section, which are denoted by the closed circles in Fig. 5, are higher than the observational values by $\Delta \log T_e = 0.05$. This difference will arise more or less from uncertainties involved in the calculations of the surface condition. Various causes for the displacement of the curves of constant $E$ will be discussed in the following.

![Fig. 5. Wholly convective stars on the zero-age main-sequence as compared with observations. The three crosses represent the observational values which were reexamined by Limber. The closed circles and open circles represent the results when the constant of opacity is $\kappa_0$ and $5\kappa_0$, respectively, where $\kappa_0$ is given by Eq. (8).](image)

1) **Effect of a superadiabatic gradient.** In § 2 the convection zone is treated as adiabatic, i.e. $\mathcal{F} = \mathcal{F}_{ad}$, but there will be a deviation from it in the neighborhood of the surface where the energy transport by convection will not be so efficient as in the inner region. According to the mixing-length theory of convection, the superadiabatic temperature gradient is determined by\(^{(17,18)}\)

$$
\left( \frac{\mu m_\text{H}Q}{32k} \right)^{1/8} \left( \frac{I}{H} \right)^3 c_p \mathcal{F} \mathcal{T}^{1/4} (\mathcal{F} - \mathcal{F}_{ad})^{3/2} + \frac{ac}{4} T_e \left( \frac{\mathcal{F}}{\mathcal{F}_{rad}} - 1 \right) = 0,
$$

(22)

with
where \( c_p \) is the specific heat per unit mass at constant pressure, \( l \) is the mixing length, and \( H \) is the pressure scale height. This scale height has been found to be nearly equal to the density scale height in the cases studied in this paper.

Using Eq. (22), the effect of the superadiabatic gradient on the surface condition has been calculated for stars of \( 0.26 \, M_\odot \) and \( 0.1 \, M_\odot \). The results show that \( F - F_{\text{ad}} \) is about one tenth of \( F_{\text{ad}} \) at the photospheres and the curves of \( E = 45.48 \) in the HR diagram are displaced in the direction of lower effective temperature by \( \Delta \log T_e = 0.001 \) for \( 0.26 \, M_\odot \) and 0.003 for \( 0.1 \, M_\odot \). Further, it has been found that these results are not altered if, according to Vitense,\(^{17}\) we include in Eq. (22) a term which represents the exchanges of heat by radiation between the convective elements and their surroundings.

2) Dependence on the constant of opacity. If we take the constant of opacity \( \kappa_0 \) which is 2.5 times as large as Eq. (8), the curve of \( E = 45.48 \) for \( 0.2 \, M_\odot \) is displaced to the right in the HR diagram by \( \Delta \log T_e = 0.015 \), as shown in Fig. 3. If the constant of opacity is taken 5 times as large, the positions of the zero-age main-sequence stars are nearly in agreement with observations, as shown in Fig. 5. It is to be noticed that the opacity formulae (7) and (8) have an uncertainty of factor 2.5 and further the opacity is proportional to the metal content through the electron pressure.

3) Presence of convection and temperature gradient in the atmosphere. In § 2 the photosphere has been defined by Eq. (5) as a point of optical depth \( 2/3 \) and its outer region has been assumed as isothermal. It has been found, however, that the convection reaches above the photosphere and the energy is partially transported by it. In order to investigate the structure of the atmosphere in more details, we assume that the net flux \( F \) of radiation at a point of optical depth \( \tau ( = \frac{m\rho}{\kappa_0} d\tau) \) is expressed in a form

\[
F = F_\odot \frac{(1 + e^{-b})}{(1 + e^{ar-b})},
\]

where \( a \) and \( b \) are two constants and \( F_\odot \) is the total energy flux. Then, using Eddington's approximation\(^{19}\) that the flow of radiation is independent of the direction except for a general distinction between inward and outward flows, the equation of radiative transfer is easily integrated to give

\[
T^4 = \frac{T_\odot^4}{4} \left( 2 + 3(1 + e^{-b}) \left( \tau + \frac{1}{a} \ln \frac{1 + e^{-b}}{1 + e^{ar-b}} \right) \right),
\]

from which the optical depth of the photosphere \( (T = T_\odot) \) is found to be

\[
\tau_p = \frac{2}{3(1 + e^{-b})} - \frac{1}{a} \ln \{1 - e^{-b}(e^B - 1)\},
\]

\[
B = 2a/3(1 + e^{-b}).
\]
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It is to be noticed that the minimum value of \( \tau_p \) is 2/3.

The inner region of the photosphere has been investigated with Eqs. (22) and (23) for a star of \( 0.1 \, M_\odot \), \( 10^{-3} L_\odot \) and \( \log T_\star = 3.49 \). The result shows that the flux of radiation in this region is well represented by Eq. (24) with \( a = b = 4 \).

If we adopt these values of \( a \) and \( b \) in the atmosphere, \( \tau_p \) is found to be 0.72 from Eqs. (26) and (27), which shows that the previous definition of the photosphere by Eq. (5) is still satisfactory.

Further, if we assume a relation \( P \propto T^{n+1} \) in the atmosphere \( (\tau \leq 2/3) \), where the constant \( n \) is the polytropic index at the photosphere, the integrated form of Eq. (5) with the opacity formula (7) is given by

\[
\frac{16 \kappa_p P_p}{3AT_\star^4} \frac{1}{1 + \alpha + \beta/(n+1)} = \frac{2}{3},
\]

where the subscript \( p \) denotes the photospheric values, and \( \alpha \) and \( \beta \) are the constants in the opacity formula. The correction factor \( (1 + \alpha + \beta/(n+1))/(1 + \alpha) \) due to the non-isothermal character of the atmosphere is found to be about 2 for the stars considered in this paper. Then, the corrected surface condition is the same as the result of the isothermal case \( (n = \infty) \) in which the constant of opacity is reduced by the above factor 2. Thus, the amount of correction is found from the result in the above section 2).

In the above we have not taken into account the effects of line-blanketing and non-grayness of the atmosphere. Their effects to the surface condition will be equivalent to change, to some extent, the optical depth of the photosphere, i.e. the value in the right side of Eq. (28) and then to change the constant of opacity accordingly.

4) **Thickness of the region where ionization of hydrogen is incomplete.**

Two adiabats in a star of \( 0.1 \, M_\odot \) are shown in Fig. 1 for two cases, \( L = L_\odot \) and \( 10^{-8} L_\odot \). The ionization is incomplete in a bulk of the star of \( L_\odot \) (curve \( P_1 C_1 \) in Fig. 1), and the assumption of polytropic index 3/2 in determining the stellar radius will not be satisfactory. In the case of the low luminosity \( 10^{-8} L_\odot \), however, the thickness of a region of incomplete ionization is fairly small as compared with the total radius. Calculations show that the curve of \( E = 45.48 \) moves in the direction of lower effective temperature by less than \( \Delta \log T_\star = 0.003 \) for the star of \( 0.26 \, M_\odot \) and \( 10^{-8} L_\odot \), if this thickness is taken into account.

5) **Deviations of the equation of state and the entropy from those of free electron gas.**

When the electrons are partially degenerate, the Coulomb interactions between the charged particles will affect the equation of state and the entropy. This effect will be large especially in the neighborhood of the line XY in Fig. 1, where the pressure ionization of hydrogen atoms is incomplete.

In order to obtain the results which are independent of the above effect in the calculation of \( K \) in $\S$ 2, we have compared the entropies at two points
in a star, i.e. a photosphere in a state of perfect gas and a point in the inner region where the approximations of free electron gas will be allowed. This approximation will be satisfactory for a star whose central temperature and pressure are far distant from the line XY, as the point C3 in Fig. 1. However, when the centers lie in the neighborhood of the line XY, the approximation will break down. In these cases, to obtain more reliable results for the surface condition it is necessary to know more accurately, for instance by means of a quantum-mechanical calculation, the thermodynamical properties of hydrogen gas in the pressure-temperature region, where the complicated phase transitions such as the dissociation of molecules, the ionization of atoms and the degeneracy of electrons occur simultaneously.

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References

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