On the Electron-Phonon Interaction in Normal Metals. II

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By the use of Fröhlich's Hamiltonian and the method of Green's functions, the electron-phonon interaction in a normal metal under a steady magnetic field is studied. It is shown that the period of the de Haas-van Alphen effect is not modified by the interaction. The resonance frequency of the Azbel'-Kaner effect, on the other hand, is modified and the cyclotron mass of the electron takes essentially the same value as the thermal mass which is also modified by the interaction as shown previously. The conclusion is in accord with the recent experimental results obtained by Kip and Grimes on Na and K.

The same conclusion, however, is obtained also by applying Landau's theory of Fermi liquids to the inter-electronic Coulomb correlation without introducing the electron-phonon interaction, though the mass shift caused by the former seems too small.

§ 1. Introduction

In a previous paper\textsuperscript{1} we examined how the electron-phonon interaction modifies the density of one-electron states (i.e. the thermal mass of the electron) and electric conductivity of a normal metal. As emphasized there, the physical point is the dynamical character of the interaction. This character is already evident in the lowest order process in the perturbation theory of the electron self-energy. Thus, between emission and reabsorption of a phonon by an electron, there exists retardation which makes the time dependence of the self-energy essential. Indeed, as we have shown, this dynamical part of the self-energy becomes singular when the velocity of sound $v_s$ is much less than the electron velocity $v_F$ at the Fermi surface and appreciably modifies the thermal mass. It results also in some anomalous dispersion of the one-electron excitation spectrum when excitation energies are comparable to the Debye temperature. On the other hand, in the case of electric conductivity, the renormalization of the wave function explicitly comes into play in addition to the mass renormalization mentioned above. These two effects are cancelled by each other both in static and anomalous limits and there appears the bare (band theoretical) electron mass in the expression for conductivity.

The aim of the present paper is to extend the theory so as to include a steady magnetic field. We thus deal with de Haas-van Alphen effect and Azbel'-Kaner effect, taking into consideration the electron-phonon interaction. As for the de Haas-van Alphen effect, Luttinger\textsuperscript{2} and Gor'kov and Bychkov\textsuperscript{3} have already
shown that the period of oscillation of the diamagnetic moment is not modified by the Coulomb interaction between electrons. In the present paper, we show that the period is not modified by the electron-phonon interaction either. The resonance frequency of the Azbel'-Kaner effect, on the other hand, is modified by the electron-phonon interaction and given (in the limit $\omega_c \gg 1$) by

$$\omega_c^* = \frac{eH}{m^* c}.$$  \hspace{1cm} (1.1)

Here the cyclotron mass $m^*$ takes essentially the same value as the thermal mass in the absence of magnetic field and thus given by

$$m^* = m(1 + \lambda),$$  \hspace{1cm} (1.2)

in which $\lambda$ is the dimensionless coupling constant characterizing the electron-phonon interaction. We have already shown that the shift of the thermal mass from its band theoretical value for sodium can reasonably be accounted for by (1.2). We may therefore expect the same shift for the cyclotron mass.

The theoretical conclusion is in accord with the recent experimental results obtained by Kip and Grimes.\(^4\) They found that cyclotron masses in sodium and potassium are isotropic and indeed very close to thermal masses. In the case of these simple metals in which we have no complication of band structure, we can hardly expect that a band theoretical calculation would result in a mass shift of the observed order of magnitude ($20\sim30\%$).

At this point we should mention that such deviations from the simple band theory can arise also from the Coulomb correlation between electrons which is ignored in the band theory. Indeed, applying Landau's formalism of Fermi liquids,\(^5\) we find that the Coulomb correlation also gives the same shift both to thermal and cyclotron masses. There remains therefore the possibility of accounting for the observed mass shift in terms of the Coulomb correlation. Unfortunately it is rather difficult to estimate this effect accurately. As is shown elsewhere,\(^5\) however, it is likely that in sodium and potassium the thermal mass shift caused by the Coulomb correlation amounts only to $3\sim5\%$, though the estimate is not quite conclusive. Apart from this uncertainty, we conclude that the observed mass shift is a genuine many-body effect.

\section*{§ 2. Electron self-energy and de Haas-van Alphen effect}

To simplify mathematics, we adopt Fröhlich's Hamiltonian of the electron-phonon interaction, neglecting all the effects of a periodic lattice. It is not very hard to generalize the theory in this respect and even then our qualitative conclusions will continue to hold.

Now suppose that we apply a steady magnetic field in the $z$-direction. We take the vector potential in the form
In this gauge, the one-electron state is characterized by the set of quantum numbers \( \nu = (n, p_y, p_z) \) and its energy measured from the chemical potential \( \mu \) is given by

\[
\varepsilon_\nu = \left(n + \frac{1}{2}\right) \omega_e + \frac{p_z^2}{2m} - \mu. \tag{2.2}
\]

Here \( \omega_e = (eH/mc) \) and we take \( \hbar = 1 \). The corresponding eigenfunction is given by the well-known expression by the use of Hermite polynomials \( H_n \). We conveniently write it in “momentum” representation:

\[
\langle k | \nu \rangle = \left[ \frac{2\pi}{m \omega_e L^3} \right]^{1/4} \delta_{p_y} \delta_{p_z} \exp \left\{ -i \frac{k_z p_z}{m \omega_e} \right\} \times (-i)^n \chi_n \left( \left[ \frac{2}{m \omega_e} \right]^{1/4} k_x \right) \tag{2.3}
\]

with the normalization volume \( L^3 \) and

\[
\chi_n(x) = [2\pi (n!)^2]^{-1/4} H_n(x) \exp \left\{ -\frac{x^2}{4} \right\}. \tag{2.4}
\]

In terms of creation and destruction operators, we write the Hamiltonian as

\[
\mathcal{H} = \sum_{\nu, \nu'} \varepsilon_\nu a_{\nu}^\dagger a_{\nu'} + \sum_{Q} \omega_Q b_Q^\dagger b_Q + \sum_{\nu, \nu'} \alpha_{\nu, \nu'}(Q) a_{\nu}^\dagger a_{\nu'} (b_Q + b_Q^\dagger). \tag{2.5}
\]

The matrix elements \( \alpha \) are taken from the usual expression by the transformation

\[
\alpha_{\nu, \nu'}(Q) = \sum_k \langle \nu | k + Q \rangle \alpha_Q \langle k | \nu' \rangle, \tag{2.6}
\]

\[
\alpha_Q = g [\omega_Q / 2L^3]^{1/4}.
\]

Strictly speaking, the phonon frequency \( \omega_Q \) and coupling constants \( \alpha_Q \) will be modified by the presence of magnetic field. We neglect such effects since they appear only at very high field which is not attainable in the case of good conductors.

We now calculate the electron self-energy arising from the interaction with phonons. The method is similar to that adopted in the preceeding paper.\(^1\) We apply the perturbational expansion first to the thermal Green’s function, which is then uniquely continued on to the complex energy plane under the condition that the function is regular, say in the upper half-plane including infinity. The lowest order process consists of emission and reabsorption of one phonon by the electron. This leads to the following expression for the self-energy part of the retarded one-electron Green’s function:
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\[ \Sigma(\nu, \varepsilon) = \sum_{\nu', \sigma'} \alpha_{\nu', \sigma'}(Q) \alpha_{\nu', \sigma'}(-Q) \]

\[ \times \left\{ \frac{f(\varepsilon_{\nu'}) + N_Q}{\varepsilon - \varepsilon_{\nu'} + \omega_Q + i\delta} + \frac{1 - f(\varepsilon_{\nu'}) + N_Q}{\varepsilon - \varepsilon_{\nu'} - \omega_Q + i\delta} \right\}. \] (2.7)

Here \( \delta = +0 \) and \( f(\varepsilon) \) and \( N_Q \) are Fermi- and Planck distribution functions, respectively. We confine ourselves to low temperatures, so that we neglect the phonon distribution functions. Then the real part of (2.7) is expressed as

\[ \text{Re} \Sigma(\nu, \varepsilon) = \int_{-\infty}^{\infty} d\varepsilon' \sum_Q I_s(Q; \varepsilon') \]

\[ \times \left( \frac{1 - f(\varepsilon')}{\varepsilon - \varepsilon' - \omega_Q} + \frac{f(\varepsilon')}{\varepsilon - \varepsilon' + \omega_Q} \right), \] (2.8)

in which

\[ I_s(Q; \varepsilon) = \sum_{\nu', \sigma'} \alpha_{\nu', \sigma'}(Q) \delta(\varepsilon - \varepsilon_{\nu'}) \alpha_{\nu', \sigma'}(-Q). \] (2.9)

Near the Fermi surface, as mentioned previously,\(^{11}\) the self-energy (2.8) itself gives a negligible shift of the chemical potential, whereas its derivative with respect to \( \varepsilon \) is of the order of \( (\nu_s/\nu_F)^3 \). We are interested in terms of this order. Thus, taking the derivative which can be transformed into the derivative with respect to \( \varepsilon' \) and then integrating by part,

\[ \lambda = -\left( \frac{\partial}{\partial \varepsilon} \text{Re} \Sigma(\nu, \varepsilon) \right)_{\varepsilon = 0} \]

\[ = \int_{-\infty}^{\infty} d\varepsilon' \sum_Q I_s(Q; \varepsilon') \left( -\frac{\partial f(\varepsilon')}{\partial \varepsilon} \right) \]

\[ \times \left( \frac{1}{\varepsilon' + \omega_Q} - \frac{1}{\varepsilon' - \omega_Q} \right). \] (2.10)

Note that we have neglected the derivative of \( I(Q; \varepsilon') \) since this leads to a term of higher order in \( (\nu_s/\nu_F) \). We have assumed that the temperature is much lower than the Debye temperature, so that we may put \( \varepsilon' = 0 \) in the denominator of (2.10). Considering the case of the weak field \( (\omega_c/\mu \ll 1) \) and taking the limit of the quantum number \( n \) to be continuous, we can easily see that Eq. (2.10) reduces to Eq. (2.7) of our previous paper.\(^{11}\) The order of magnitude of oscillating terms can be estimated easily when we neglect the Debye cut of phonon momentum. Inserting then explicit expressions (2.2), (2.3), (2.6), (2.9) into (2.10) and carrying out simple summation over quantum numbers, we find

\[ \lambda = \left( \frac{g}{2\pi} \right)^2 (2m^2\omega_c)^\lambda \int_{-\infty}^{\infty} d\varepsilon' \left( -\frac{\partial f(\varepsilon')}{\partial \varepsilon} \right) \]
Here the sum is to be taken over integers \( n' \) which make the radicant positive. We apply the well-known technique\(^7\) to the sum and finally obtain

\[
\lambda = g^2 N(0) \left[ 1 + \sum_{i=1}^{\infty} \left( \frac{2l\omega_e}{\mu} \right)^{1/2} \right] (-1)^i \left( \frac{kT}{\omega_e} \right) \times \{ \cos \left( 2\pi l \left( \mu/\omega_e \right) - (\pi/4) \right) / \sinh \left( 2\pi l (kT/\omega_e) \right) \}. \tag{2·12}
\]

Here \( N(0) \) is the (unrenormalized) density of one-electron states. In the limit \( kT \gg \omega_e \), (2·12) reduces to our previous result except a factor due to the Debye cut. At low temperatures, we would have a de Haas type oscillation, but its amplitude, being proportional to \( (\omega_e/\mu)^{1/2} \), is negligible in the case of a good conductor in a magnetic field attainable at present.

When we take into account the electron-phonon coupling, the one-electron excitation spectrum is determined by

\[
\varepsilon^* - \varepsilon_v = \text{Re} \Sigma(\nu, \varepsilon^*) = 0. \tag{2·13}
\]

Near the Fermi surface we can thus apply the quasiparticle picture with excitation energies

\[
\varepsilon_v^* = \left( n + \frac{1}{2} \right) \omega_v^* + \frac{p_z^2}{2m^*} - \frac{\mu}{(1 + \lambda)}, \tag{2·14}
\]

in which \( \omega_v^* \) and \( m^* \) are given by (1·1) and (1·2), respectively. Since the oscillating term in (2·12) is negligible, the cyclotron mass \( m^* \) takes the same value as the thermal mass given in our previous paper.\(^1\)

An important consequence of the quasiparticle picture is that the period of the de Haas oscillation of the diamagnetic moment is not modified by the presence of the electron-phonon coupling. This can be seen in the following way. In our approximation, the thermal Green's function for electrons near the Fermi surface takes the form

\[
G_v(iz_n) = \frac{a}{iz_n - \varepsilon_v^* + i(\alpha/\tau) \text{sgn} \varepsilon_n}, \tag{2·15}
\]

\[
a^{-1} = 1 + \lambda, \quad \varepsilon_n = 2\pi kT \left( n + \frac{1}{2} \right), \quad n = 0, \pm 1, \pm 2, \ldots,
\]

where \( \tau \) is the mean lifetime which is negligible at low temperatures. As well known the oscillating part of the diamagnetic moment is determined solely by electrons near the Fermi surface. Thus, as far as we are interested in this oscillating part, we may insert the approximate expression (2·15) into the formula for the thermodynamic potential,\(^9\) as if it were valid for all electrons:

\[
\mathcal{Q} = \mathcal{Q}_0 + kT \int_0^q \frac{dg}{g} \sum_{\nu,v} \sum_{i\varepsilon_n} G_v(iz_n) \Sigma(\nu, i\varepsilon_n), \tag{2·16}
\]
where \( \Omega_0 \) is the thermodynamic potential for a free electron gas

\[
\Omega_0 = -kT \sum_{\varepsilon} \ln(1 + \exp(-\varepsilon/kT)).
\]  

(2·17)

Note that in our approximation the parameter \( \lambda \) plays the part of the coupling constant. From (2·15), (2·16) and (2·17), we get the oscillating part of the thermodynamic potential as

\[
\Omega_{osc} = \{-kT \sum_{\varepsilon} \ln(1 + \exp(-\varepsilon^*/kT))\}_{osc}.
\]  

(2·18)*

This expression is exactly the thermodynamic potential for a system of independent particles with the excitation spectrum \( \varepsilon^* \). In comparison with the usual derivation, we at once see that the period of oscillation depends only on the ratio of \( \mu/(1 + \lambda) \) to \( \omega^* \) in which the effect of the electron-phonon coupling is cancelled out. We thus reach the conclusion mentioned above.

§ 3. The expression for high frequency conductivity

We now proceed to the Azbel'-Kaner effect. For simplicity we only consider the case of transverse resonance. Thus we assume that, in addition to the steady field (2·1), there now exists a small oscillating electromagnetic field polarized in the \( y \)-direction and propagating along the \( x \)-direction. Let the electromagnetic field be represented by the vector potential \( a_y \exp i(k_x x - \omega t) \).

The interaction between electrons and oscillating field is given, to the first order, by

\[
\mathcal{H}' = \frac{1}{c} \int dr j_y(r) a_y(r, t),
\]  

(3·1)

where \( j_y \) is the current density in the absence of the oscillating field:

\[
j_y = \frac{e}{2mi} \left( \psi^+ \frac{\partial \psi}{\partial y} - \text{c.c.} \right) - \frac{e^2}{mc} H_x \psi^+ \psi.
\]  

(3·2)

Note that the total current density is given by

\[
j_y = j_y - \frac{e^2}{mc} \psi^+ \psi a_y.
\]  

(3·3)

Its average can be obtained straightforwardly by the first order perturbation theory,* and written as

\[
J_y(r, t) = -\left\{ \frac{ne^2}{mc} - \frac{1}{c} L(k_x, \omega) \right\} a_y(r, t).
\]  

(3·4)

* It is also possible to derive this result by extending Luttinger's theory to our case. Note that in our approximation, where we assume that the phonon renormalization has already been taken into consideration in the original Hamiltonian, we can simply follow Luttinger's discussions to get (2·18). It is also easy to extend the theory to include the phonon renormalization explicitly.
Here \( L(k, \omega) \) is the Fourier transform of the retarded Green's function

\[
L(k, \omega) = \mathcal{F}\{ \langle j_y(k, t), j_y(-k, 0) \rangle \} = \int_0^\infty \! dt \exp\{i(\omega + i\delta)t\}.
\]

In this expression \( \langle \cdots \rangle \) means taking the average over the grand canonical distribution and \( j_y(k, t) \) is the Heisenberg representation of the operator

\[
j_y(k) = \int dr j_y(r) \exp\{-ik \cdot x\}.
\]

By the use of the electric field \( E_y = (i\omega/c) a_y \), (3·4) can be written as

\[
\begin{align*}
J_y(r, t) &= -\left( \frac{ne^2}{mc} - \frac{1}{c} L(k, 0) \right) a_y(r, t) \\
&\quad + \{ L(k, \omega) - L(k, 0) \} / i\omega E_y(r, t).
\end{align*}
\]

The first term on the right depends only on the instantaneous vector potential and represents the diamagnetic polarization. We are interested in the second term which defines the electric conductivity

\[
\sigma(k, \omega) = \frac{1}{i\omega} \left[ L(k, \omega) - L(k, 0) \right].
\]

We now introduce the expansion

\[
\phi_\nu(x) = \sum_{\nu} \langle r|\nu\rangle a_{\nu\nu},
\]

in which \( \langle r|\nu\rangle \) is the coordinate representation of (2·3). Thus, taking the unit normalization volume, we have

\[
\begin{align*}
\langle r|\nu\rangle &= \phi_\nu(x - \frac{p_y}{m\omega_e}) \exp\{i(p_y y + p_z z)\}, \\
\phi_\nu(x) &= \left[ \frac{m\omega_e}{\pi (n!)^2} \right]^{1/4} H_n(2m\omega_e^{1/2} x) \exp\left\{ -\frac{m\omega_e}{2} x^2 \right\}.
\end{align*}
\]

Inserting (3·9) into (3·2), we obtain (3·6) in the form

\[
\begin{align*}
J_y(k) &= \sum_\nu \omega_e X_{\nu\nu,}^* (k) a_{\nu\nu}, \\
X_{\nu\nu}(k) &= X_{\nu\nu}(k) \delta_{\nu\nu} \delta_{\nu\nu}.
\end{align*}
\]

By the use of the Fourier transform (2·3) and also of the well-known generating function of Hermite polynomials, we obtain the matrix elements

\[
X_{\nu\nu}(k) = \frac{1}{[2m\omega_e]^{1/2}} \exp\left\{ ik \cdot p_y \right\} \left\{ i \frac{\partial}{\partial q} J_{\nu\nu}(q) \right\},
\]

where
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\[ J_{\nu'(q)} = \left[ \frac{n'!}{n!} \right]^{1/2} (iq)^{n-n'} L_n^{(n-n')}(q) \exp(-q^2/2). \]  
(3·13)

Here \( L_{\nu}^{(\nu)} \) are associated Laguerre functions and

\[ q = \frac{k_x}{[2m\omega_c]^{1/2}}. \]  
(3·14)

Now (3·5) takes the form

\[ L(k_x, \omega) = (e\omega_c)^{\nu} \sum_{\lambda'\nu'} K_{\lambda\nu;\lambda'\nu'}(\omega) X_{\lambda\nu'}(k_x) \]  
(3·15)
in which

\[ K_{\lambda\nu;\lambda'\nu'}(\omega) = i \int_0^\infty dt \exp\{i(\omega + \delta) t\} \langle a^\dagger_{\lambda'}(t) a_{\lambda\nu}(t) a^\dagger_{\nu'} a_{\lambda'\nu'} \rangle. \]  
(3·16)

We denote the set of \((\lambda, \nu, \sigma)\) by a single letter, say \(N\), and regard \(X_N(k_x)\), \(X_N^*(k_x)\) as one-column and one-row matrices, respectively. Then we may write (3·15) compactly as

\[ L(k_x, \omega) = (e\omega_c)^{\nu} X(k_x) K(\omega) X(k_x). \]  
(3·17)

§ 4. Application of Eliashberg's method

We introduce the thermal Green's function

\[ K_{\lambda\nu;\lambda'\nu'}(i\omega_m) = \int_0^\beta du \exp\{-i\omega_m u\} \langle Ta^\dagger_{\lambda'}(u) a_{\lambda\nu}(u) a^\dagger_{\nu'} a_{\lambda'\nu'} \rangle, \]  
(4·1)

where \(a_{\nu}(u)\) are obtained from usual Heisenberg operators by the formal transformation \(t \rightarrow -iu\), \(T\) is Wick's ordering operator with respect to the real parameter \(u\), and

\[ \beta = \frac{1}{kT}, \quad \omega_m = \frac{2\pi}{\beta} m, \quad m = 0, \pm 1, \pm 2, \ldots. \]  
(4·2)

As well known by now, we can apply the perturbational expansion\(^9\) to (4·1) in which each term is represented by a Feynman diagram. With reference to this expansion, we define the four-vertex function \(\Gamma\) by

\[ K_{N;N'}(i\omega_m) = \delta_{N,N'} \frac{1}{\beta} \sum_{\epsilon_n} g_N(i\epsilon_n; i\omega_m) \]
\[ + \frac{1}{\beta^2} \sum_{\epsilon_n} \sum_{\epsilon_{n'}} g_N(i\epsilon_n; i\omega_m) \Gamma_{N,N'}(i\epsilon_n, i\epsilon_{n'}; i\omega_m) g_{N'}(i\epsilon_{n'}; i\omega_m). \]  
(4·3)

Here, by the use of the one-electron Green's function (2·15),

\[ g_{\lambda\nu}(i\epsilon_n; i\omega_m) = G_{\lambda}(i\epsilon_n + i\omega_m) G_{\nu}(i\epsilon_n). \]  
(4·4)

The retarded Green's function (3·16) can be obtained from (4·1) by analytic continuation on to the complex energy plane.
At this point we make some estimate of the order of magnitude. In the case of the Azbel'-Kaner resonance under consideration, even if $H \sim 10^4$ gauss, we have $\omega \sim \omega_c \sim 10^{-4}$ eV which is much smaller than the Fermi energy $\varepsilon_F$. We are thus concerned with the "low frequency" limit. For electrons near the Fermi surface, the quantum number $n$ in (2·14) is of the order of $(\varepsilon_F/\omega_c) \sim 10^4$. Its change $\Delta n$ caused by the interaction with the applied high frequency field is of the order of $(\nu_F k_{\varepsilon}/\omega_c)$ which is much smaller than $n$ itself. We thus deal with the "long wave length" limit. Note that $\Delta n$ itself is not small ($\sim 10^5$), because the cyclotron radius $r_c = (\nu_F/\omega_c)$ is much longer than the skin depth $k_{\varepsilon}^{-1}$:

$$v_F k_{\varepsilon} \gg \omega_c.$$ (4·5)

The skin effect is thus "anomalous".

Under these conditions, the analytic continuation of (4·1) can be done in quite a general way, as shown by Eliashberg. We shall not repeat his deduction here, mainly because of complexity in formulae. We only mention the following point. When analytically continued, (4·4) gives

$$g^{(i)}_{ij}(\varepsilon, \omega) = G^{(i)}_{\varepsilon}(\varepsilon + \omega)G^{(i)}_{\varepsilon}(\varepsilon),$$ (4·6)

in which $G^{(i)}_{\varepsilon}(\varepsilon) = G_{\varepsilon}(\varepsilon + i\delta)$ are retarded and advanced Green's functions, respectively. In the presence of magnetic field, the use of the exact momentum conservation law is not available at the electron-phonon vertex, but the momentum uncertainty $r_c^{-1}$ is much smaller than the Fermi momentum $p_F$. Therefore, whenever (4·6) appear in the perturbational expansion as "cuts", we may replace them by

$$g^{(i)}_{ij} \approx \delta_{ij}[G^{(i)}_{\varepsilon}(\varepsilon)]^2,$$ 

$$g^{(+)}_{ij} \approx \left[\frac{\alpha^2}{\omega - (n' - n)\omega_c + i\pi}\right] \frac{2\pi i\delta(\varepsilon - \varepsilon_c^*)}{\varepsilon - \varepsilon_c^*}.$$ (4·7)

Now we write down the expression for the conductivity (3·8) obtained by Eliashberg's method. It takes the form

$$\sigma(k_\varepsilon, \omega) = \frac{e^2\omega}{i\omega} \int \frac{d\varepsilon_1 d\varepsilon_2 d\varepsilon_3 d\varepsilon_4}{(4\pi i)^4} \bar{X}Z_1(\varepsilon_1, \varepsilon_2) \Pi(\varepsilon_2, \varepsilon_3; \omega) Z_1(\varepsilon_3, \varepsilon_4) X,$$ (4·8)

$$\Pi(\varepsilon, \varepsilon'; \omega) = \frac{\beta\omega}{2} \text{ch}^{-1} \left(\frac{\beta\omega}{2}\right) g^{(+)}(\varepsilon; \omega) \left[4\pi i\delta(\varepsilon - \varepsilon_c) + \Gamma(\varepsilon, \varepsilon'; \omega) g^{(-)}(\varepsilon'; \omega) \right].$$ (4·9)

In (4·8), $Z_1$ represents the charge renormalization arising from the vertex function $\Gamma$. Their explicit expressions are rather complicated, but, as shown in the previous paper, the charge renormalization arising from the electron-phonon coupling is of higher order in $(\nu_F/\nu_p)$, so that we may simply put
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\[ Z_{\varepsilon} \equiv 4\pi i \delta (\varepsilon - \varepsilon'). \] (4·10)

In the anomalous limit (4·5), the vertex function \( \Gamma \) satisfies

\[ \Gamma(\varepsilon, \varepsilon'; \omega) = \Gamma^{(0)}(\varepsilon, \varepsilon') \]
\[ + \int \frac{d\varepsilon''}{4\pi i} \Gamma^{(3)}(\varepsilon, \varepsilon') g^{(s-s')}(\varepsilon''; \omega) \Gamma(\varepsilon'', \varepsilon'; \omega), \] (4·11)

where \( \Gamma^{(0)} \) is the irreducible part which does not contain any of cuts (4·7).

By the use of (4·10), (4·11), we can write the conductivity (4·8) as

\[ \sigma(k_n, \omega) = a^2(e\omega)^{3/2} \sum_{\chi} \left( -\frac{\partial f(\varepsilon_*)}{\partial \varepsilon} \right) \Phi_{\nu'}; \chi \lambda X_{\lambda'\chi}, \] (4·12)

in which \( \Phi \) plays a role of the distribution function and indeed satisfies the kinetic equation

\[ i \left( \omega - (n' - n)\omega_* \right) \Phi_{\nu'; \chi \lambda} = \delta_{\nu'\nu}; \chi \lambda + \frac{i a^2}{2} \sum \Gamma^{(3)}(\varepsilon_0; \nu'; \nu; \omega) \Phi_{\nu'; \chi \lambda} + \frac{1}{\tau} \Phi_{\nu'; \chi \lambda}. \] (4·13)

As we have seen in the previous paper, the last two terms on the right give the usual collision term in the absence of magnetic field when we adopt the ladder approximation for \( \Gamma^{(0)} \). To be realistic in discussing the resonance at low temperatures, we should also include the impurity scattering into \( \Gamma^{(0)} \). We then encounter difficult problems in the theory of magnetoresistance. Since we are interested in the resonance frequency in the limit \( \omega_* \tau \gg 1 \), we do not intend to solve (4·13) exactly. Instead, we take the simple approximation

\[ \Phi_{\nu'; \chi \lambda} = \frac{\delta_{\nu'\nu}; \chi \lambda}{i \left( \omega - (n' - n)\omega_* + i/\tau \right)}, \] (4·14)

in which \( \tau \) is now to be taken as an effective relaxation time due to collisions. Then (4·12) gives

\[ \sigma(k_n, \omega) = i \sum (a e\omega)^3 |X_{n,n+a}(k_n)|^2 \left( \frac{\partial f(\varepsilon_*)}{\partial \varepsilon} \right) \frac{1}{(\omega - \alpha \omega_* + i/\tau)}. \] (4·15)

From definitions of the renormalization factor \( a \) and of the effective mass \( m^* \), we at once see that in (4·15) \( a \omega_* = \omega_* \). Further, in the explicit expression for matrix elements (3·13) and in the definition of the parameter \( q \), (3·15), \( m \omega_* \) is actually independent of the electron mass and therefore can be written as \( m^* \omega_* \). Thus

\[ \sigma(k_n, \omega) = i \left( \frac{e^2 \omega_*}{2m^*} \right) \sum \left( \frac{\partial f(\varepsilon_*)}{\partial \varepsilon} \right) \frac{\partial}{\partial q} J_{n+n,a}(q) \left| \frac{1}{\omega - \alpha \omega_* + i/\tau} \right|. \] (4·16)

This expression has the same form as the usual one except that the electron

\[ * \] There were typographical errors in the previous paper. In Eq. (3·3) there should replace \( 1 - f(E_p) \) by \( f(E_p^*) \), and in Eq. (3·5), \( f(E_p) \) on the right by \( f(E_p^*) \).
mass is replaced everywhere by the renormalized mass $m^*$. Hence it is not surprising to obtain $\omega_c^*$ as the resonance frequency. That this is indeed the case can be seen in the following way.

As mentioned already, $n \gg 1$ for electrons near the Fermi surface, so that we may regard $n$ as a continuous variable, say,

$$n = \left( \frac{\rho^2/2m^*\omega_e^*}{\sin^2 \theta} \right) \sin^2 \theta, \quad (0 < \theta < \pi/2)$$

and also make use of the asymptotic expression

$$J_{n+a,n}(q) \sim J_{a}([4n+2a+2]^{1/3} q),$$

where $J_{a}$ is the Bessel function. The sum over $\nu$ in (4·16) is now replaced by the integration over the momentum and we obtain

$$\sigma(k_x, \omega) = \frac{3\pi e^2 \omega_e^*}{m^*} \sum_{\omega \omega_e^*} \frac{i}{\omega - \alpha \omega_e^* + i/\tau} \sin \omega \sin \theta [J_{a'}(\sin \theta)]^2. \quad (4·17)$$

The expression agrees with that obtained by Cohen, Harrison, and Harrison.\(^{11)}\)

In particular, in the anomalous limit (4·5), it reduces to

$$\sigma(k_x, \omega) = \frac{3\pi e^2 \omega_e^*}{m^* \nu_F^* k_e^*} \frac{i}{\omega - \omega_e^* + i/\tau} \frac{1}{\cosh \left( \frac{\pi}{\omega_e^* \tau} \right)}. \quad (4·19)$$

Note that $m^* \nu_F^*$ is independent of the electron mass. Except for the mass renormalization (and also a possible modification of $\tau$), (4·19) thus agrees with that obtained by Mattis and Dresselhaus.\(^{12)}\)

§ 5. The effect of the Coulomb correlation

Of course it is possible to apply the Green's function method to discuss the effect of the inter-electronic Coulomb interaction on the cyclotron resonance along the same line as in the case of the electron-phonon coupling. However we here take another approach to this problem, using the kinetic equation of Landau's theory of Fermi liquids.\(^{13)}\) We thus neglect any quantum oscillation at all, which may appear in the self-energy or the “correlation function”. This may be justified certainly for the field strength now attainable in metals.

The field configuration is the same as in the previous sections. We write the oscillatory field as

$$E_y = E_k \exp (i k x - i \omega t). \quad (5·1)$$

We also assume the distribution function in the form
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\[ n = n_0 + \partial n = n_0 + n_0' \phi_k \exp(ikx - i\omega t), \]  

(5·2)

where \( n_0' = \partial n_0/\partial \varepsilon \) and \( n_0 \) is the equilibrium Fermi distribution function. Then the (linearized) kinetic equation is written as

\[ -i\omega \phi_k(p) + \left\{ -ikv_{p'}^* \sin \theta \cos \varphi + \omega_e^* \frac{\partial}{\partial \varphi} + \frac{1}{\tau} \right\} \cdot \{ \phi_k(p) - \int f(p, p') n_0' \phi_k(p') \, dt \} \]

\[ = -eE_kv_{p'}^* \sin \theta \sin \varphi, \]

(5·3)

where \( p \) is the quasimomentum of the quasiparticle, \( dt = 2d^3p/(2\pi)^3 \), \( v_{p'}^* = (\partial \varepsilon_0/\partial \rho)_{p'} \) is the group velocity of the quasiparticle at the Fermi surface, \( \omega_e^* = eH/m^*c \), \( m^* = p_F/v_{p'}^* \), \( p_F \) is the Fermi momentum, \( \tau \) is the relaxation time (introduced phenomenologically), and \( f(p, p') \) is the so-called correlation function characterizing the inter-electronic interaction, which can be expanded as

\[ f(p, p') |_{p = p' = p_F} = N^*(0)^{-1} \sum_i F_i \int (\cos \theta p') \]

(5·4)

\( N^*(0) \) being the renormalized density of states at the Fermi surface. In the isotropic case, \( m^* \) is related to \( F_i \) by the relativity principle as

\[ m^*/m = 1 + F_1. \]

(5·5)

The coordinate system is taken as shown in Fig. 1. (\( \rho, \theta, \varphi \) are the polar coordinates in quasimomentum space.)

Assuming that the \( l=0 \) and 1 components of \( F_i \) are dominant, (which can be justified for the screened Coulomb type interaction\(^6\)), we can solve Eq. (5·3) formally as

\[ \phi_k(\theta, \varphi) = \frac{1}{\omega_e^* \tau} \int d\varphi' \exp \left\{ (\omega_e^* \tau)^{-1} \left[ (\varphi' - \varphi) - ikv_{p'}^* \sin \theta (\sin \varphi' - \sin \varphi) \right] \right\} \]
\[ \times \left\{ eE_k \nu_p^* \bar{r} \sin \theta \sin \phi' + 3F_1 \left( -ik\nu_p^* \bar{r} \sin \theta \sin \phi' + \omega_e^* \bar{r} \frac{\partial}{\partial \phi'} + 1 + i\omega \bar{r} \right) \right\}, \] (5.6)*

where \( \bar{r} = \frac{r}{1 - i\omega \phi} \) and \( d\Omega = \sin \theta d\theta d\phi \). The boundary condition was taken as \( \phi_k(\phi) = \phi_k(\phi + 2\pi) \). If we define the following notations

\[ \{ \phi_k^x, \phi_k^y \} = \int \frac{d\Omega}{4\pi} \sin \theta \left\{ \cos \phi \right\} \phi_k(\theta, \phi), \] (5.7)

Eq. (5.6) leads us to the coupled equations for \( \phi_k^x \) and \( \phi_k^y \),

\[ \phi_k^x = -F_1 \phi_k^y - \sum_{n=-\infty}^{\infty} \frac{1}{1 - i\alpha \omega_e^* \bar{r}} \left[ s_n(X) \left\{ eE_k \nu_p^* \bar{r} + i3F_1 \omega \bar{r} \phi_k^y \right\} - i \frac{3}{2} F_1 \left( \frac{\omega}{\nu_p^* k} \right) g_a'(X) \phi_k^y \right], \] (5.8)

\[ \phi_k^y = -F_1 \phi_k^x - \sum_{n=-\infty}^{\infty} \frac{1}{1 - i\alpha \omega_e^* \bar{r}} \left[ \frac{1}{2} g_a'(X) \left\{ eE_k \nu_p^* \bar{r} + i3F_1 \omega \bar{r} \phi_k^y \right\} / \nu_p^* k \right] + i3F_1 \omega \bar{r} \left( \frac{\omega_e^*}{\nu_p^* k} \right)^3 \alpha^2 g_a(X) \phi_k^x, \] (5.9)

where \( X = \nu_p^* k / \omega_e^* \), \( s_n(X) \) is given by Eq. (4.17) and \( g_a(X) \) and \( g_a'(X) \) are defined as

\[ g_a(X) = \int_0^{\pi/2} d\theta \sin \theta [J_a(X \sin \theta)]^2, \] (5.10)

\[ g_a'(X) = (d/dX) g_a(X), \] (5.11)

with the \( \alpha \)-th order Bessel function \( J_a(x) \).

Now we are interested in the extreme case of \( \nu_p^* k / \omega_e^* \gg 1 \) and \( \nu_p^* k / \omega_e^* \gg 1 \). Then, using the asymptotic expression for the above functions, we get

\[ \phi_k^x = (1 + F_1)^{-1} \left\{ eE_k \omega_e^* \bar{r} \right\} \sum_{n=-\infty}^{\infty} \frac{1}{1 - i\alpha \omega_e^* \bar{r}} \] (5.12)

The function \( \phi_k^x \) gives the higher order terms in \( X^{-1} \). From this result the current density is easily obtained as

\[ J_k^x = \frac{e}{m} \int \rho_p \nu_p^* \phi_k^x d\tau_p = -3 \cdot \frac{en}{m \nu_p^*} \phi_k^x = \frac{3\pi}{4} \cdot \frac{ne^2 E_k}{m^2 \nu_p^* k} \coth \frac{\pi}{\omega_e^* \bar{r}}. \] (5.13)

* As we are looking for the solution of the form of (5.7), the term involving \( F_0 \) can be discarded.
Eq. (5·13) is identical with Eq. (4·19). Note that \( m^*v_F^* \) does not contain the correlation effect. Thus, the only effect of correlation is in the cyclotron frequency \( \omega_c^* \), and the effective mass appeared there is the same as the thermal mass.

§ 6. Concluding remarks

As was shown previously,\(^1\) it seems that, at least in the case of sodium, the shift of the thermal mass from the free electron value can be accounted for as the self-energy effect arising from the coupling with phonons. We have shown in the present paper that one may expect the same shift for the cyclotron mass. The theory is consistent with experimental results.\(^4\) Although the inter- 


electronic Coulomb correlation gives the same shift both to thermal and cyclotron masses, this part of the shift seems small.\(^6\) At the present stage of the theory, however, we cannot entirely exclude possible contribution of the Coulomb interaction and also of cross effects of Coulomb and electron-phonon interactions. The relative importance of the electron-phonon coupling might be clarified by detecting the predicted anomalous dispersion in the infrared region.

References