A Soluble Model of the Partial Wave Dispersion Equation and the Bootstrap Approximation

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(Received August 14, 1963)

An exactly soluble model of a $p$-wave dispersion equation is investigated to see the validity of the bootstrap approximation. The model contains a parameter which is related to the coupling constant of the corresponding field theoretical model, the meson pair theory with a generalized separable interaction potential. For a certain range of the parameter, the model exhibits a resonance of positive width and the bootstrap equation can be constructed for the resonance. Its equation can be solved numerically, which gives the position and the reduced width of the resonance as the functions of the parameter. The results are compared with the exact one and the validity of the bootstrap approximation is discussed in detail.

§ 1. Introduction

An idea of the bootstrap mechanism for the strongly interacting particles was first suggested by Chew and Frautschi and it has been applied to many practical problems. The idea is that a set of the "particles", which may be resonant states as well as bound states, would be produced mainly by the "forces" due to the exchange of the "particles" themselves, and the requirement of self-consistency between the input and output quantities, the mass and the coupling constant or reduced width of the "particle", would fix the quantities uniquely. There are two aspects in this approach to study the strong interaction, the first is that the idea suggests a possibility of constructing a self-consistent theoretical scheme for the dynamical $S$-matrix theory. The second is that even when it fails to give such a scheme, it would provide a nice recipe to approximate generally complicated dispersion equations to a manageable form. It is the purpose of the present work to investigate the validity of the bootstrap approximation, from the viewpoint of the second mentioned above, by analyzing the exactly soluble model of the partial wave dispersion equation.

The model considered is the dispersion equation for an elastic $p$-wave scattering amplitude. The amplitude has a simple crossing relation and a non-vanishing finite limit, $\xi$, at infinite energy. The elastic unitarity is modified by introducing a smooth cutoff at high energies. The field theoretical version of this model is the relativistic meson pair theory which describes the $p$-wave interaction of a (complex) scalar boson with a fixed source through a generalized separable potential.

The dispersion equation can be solved exactly by the inverse method and
it gives a resonance with positive width for a certain restricted range of $\xi$.

The bootstrap equation for the resonance can, then, be constructed through the $N/D$ method and solved numerically. The results are compared with the one of the exact solution.

In § 2 we introduce the model, and the exact solution is given and discussed in § 3. The bootstrap equation is constructed in § 4 and the final section is devoted to the presentation of numerical results and the discussion.

§ 2. A model of dispersion equation

We wish to set up in this section a soluble model of dispersion equation for a $p$-wave scattering amplitude. Let us consider an analytic function $h(z)$, $z$ being an appropriate energy variable, with the following properties:

a) $h(z)$ is an analytic continuation of the $p$-wave amplitude $h(x)$, that is

$$h(z = x + i0) = h(x),$$

for $x \geq 1$.\(^{40}\)

b) $h(x)$ satisfies the unitarity condition,

$$\text{Im } h^{-1}(x) = -\rho(x),$$

with

$$\rho(x) = (Vx^2 - 1)^2 R(x) \geq 0$$

for $x \geq 1$, where we introduce a cutoff $R(x)$ which is unity at the threshold and vanishes at infinite energy to ensure the convergence of all the integrals to be considered.

c) $h(x)$ has a nonvanishing finite limit at infinity, i.e.

$$\lim_{x \to \infty} h(x) = \xi = \text{finite} \neq 0.$$  

(4)

d) $h(z)$ satisfies a crossing symmetry,

$$h(-z) = h(z).$$

(5)

e) There are no singularities except the cuts on the real axis indicated by the conditions a) and d) and the poles corresponding to possible bound states.

The condition b) is equivalent to

$$h(x) = \rho^{-1}(x)e^{i\delta(x)} \sin \delta(x),$$

(6)

for $x \geq 1$, where $\delta(x)$ is the real phase shift, the high energy limit of which is restricted to the integer multiple of $\pi$ by means of the condition c). Further, it can be shown that $h(x)$ has the same limit $\xi$ along any direction to infinity.\(^{50}\)

Thus, the above five restrictions uniquely fix the dispersion equation for $h(z)$,

\(^{40}\) We denote, hereafter, the real value of $z$ as $x$. The threshold is normalized to $x=1$. 

\(^{50}\)
except the pole terms which should be added if there exist bound states (see the next section).

§ 3. Exact solution and its properties

In order to obtain an exact solution of (7) without C.D.D. poles, we put

$$h(z) = \xi/A(z).$$

(8)

Here $A(z)$ is an analytic function of $z$ normalized to unity at infinity, the only singularities of which are the cuts indicated by $h(z)$. The discontinuity is given by $\text{Im} A(x) = \mp \xi \rho(\pm x)$ for $x > 1$ (upper sign) and $x < -1$ (lower sign). Now it is straightforward to get the solution

$$A(z) = 1 - \frac{2\xi}{\pi} \int_1^\infty dx \frac{x \rho(x)}{x^2 - z^2},$$

(9)

or

$$h(z) = \xi/A(z) = \xi \left[ 1 - \frac{2\xi}{\pi} \int_1^\infty dx \frac{x \rho(x)}{x^2 - z^2} \right].$$

(10)

This solution corresponds to the relativistic meson pair theory which describes the interaction of a meson with a fixed source through a generalized separable $p$-wave potential (see reference 3).

It is interesting to investigate the possible bound and resonant states of the system. For this purpose, we first study the imaginary part of the inverse amplitude $A(z)$,

$$\text{Im} A(z) = -\frac{4\xi}{\pi} \int_1^\infty dx \frac{x \rho(x)}{|x^2 - z^2|^2} \text{Re} z \cdot \text{Im} z,$$

(11)

which shows that there are no zeros except on either the real or the imaginary axis. From (10) and (11), we see that for

$$\xi \geq \xi'$$

(12)

with

$$\xi' = \left[ \frac{2}{\pi} \int_1^\infty dx \frac{\rho(x)}{x} \right]^{-1},$$

(13)

there is only one (double) zero on the imaginary axis, which would correspond to a ghost bound state with an imaginary mass. For the range of $\xi$ limited
by $\xi'$ and $\xi_c$, defined by
\[
\xi_c = \left[ \frac{2}{\pi} \int_1^\infty dx \frac{x^2 \rho(x)}{x^2 - 1} \right]^{-1} < \xi', \tag{14}
\]
there is only one zero on the positive real axis below the threshold, which represents a stable bound state.

In case of the monotonically decreasing cutoff of actual interest, there always exists a positive $\xi = \xi_M$ (smaller than $\xi_c$), for which $y = \text{Re} A(x)$ just osculates to the horizontal axis at a certain point $x = x_M > 1$. That is, there exists a $\xi_M$ and a $x_M$ defined by
\[
d \text{Re} A(x_M, \xi_M) / dx_M = 0. \tag{15}
\]
And then, we have a resonance with a positive width at $x_0 (> 1)$ below the critical value $x_M$ for any $\xi$ restricted by
\[
\xi > \xi > \xi_M. \tag{16}
\]
The position $x_0$ and the reduced width $\gamma$ are given by
\[
\text{Re} A(x_0) = 0, \tag{17}
\]
and
\[
\gamma = -2\xi \left[ \frac{\partial \text{Re} A(x)}{\partial x} \right]_{x=x_0}. \tag{18}
\]
Near the resonance the amplitude can be approximated to
\[
h(x) = \frac{-\gamma/2}{x - x_0 + i\rho \gamma/2}. \tag{19}
\]

Besides the physical resonance mentioned above, there appears an unphysical resonance at $x_1 > x_M$, which is due to the artificial modification of the elastic unitarity through the introduction of the cutoff. As in the nonrelativistic potential theory, the phase shift crosses down the value $\pi/2$ at $x_1$ and vanishes at infinite energy.

§ 4. Bootstrap equation

Now we turn to construct the bootstrap equation for the physical resonance. According to the standard $N/D$ method, we put
\[
h(x) = N(x) / D(x), \tag{20}
\]
assuming that $D(z)$ has no poles and nonvanishing finite limit at infinity. Now it follows immediately the coupled integral equations,
\[
N(x) = n_0 + \frac{1}{\pi} \int_0^1 dx D(x) \frac{\text{Im} h(x)}{x(x - z)}, \tag{21}
\]
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and

\[ D(z) = 1 - \frac{x}{\pi} \int_1^\infty dx' \frac{\rho(x)N(x)}{x(x-z)}, \quad (22) \]

where both equations are subtracted at the origin to make the expressions low energy sensitive. If we identify the subtraction constant \( n_0 \) with \( \xi /A(0) \) of (10), it is easy to see that the solution (10) satisfies the coupled equations (21) and (22) through (20), regardless the existence of the bound state.\(^6\) We assume, however, the uniqueness of the solution, since it is likely but yet unproved.\(^7\)

The bootstrap equation can be set up, if we approximate the “force”, \( \text{Im} \ h(x) \) in (21), to the contribution due to the exchange of the resonance itself. Because of the crossing symmetry (5) and the resonant formula (19), we have\(^*)

\[ \text{Im} \ h(x) = -\text{Im} \ h(-x) = -(\gamma/2) \pi \delta(x+x_0), \quad (23) \]

for \( x < -1 \). Substituting (23) into (21), we get

\[ N(x) = n_0 - \frac{\gamma}{2} \frac{D(-x)}{x_0} \frac{1}{x+x_0}, \quad (24) \]

and further from (22), we have

\[
\text{Re} \ D(x) = 1 - \frac{n_0 x}{\pi} \int_1^\infty dx' \frac{\rho(x')}{(x'-x)} + \frac{\gamma x D(-x)}{2\pi x_0} \int_1^\infty dx' \frac{\rho(x')}{(x'-x)(x'+x_0)}, \quad (25)
\]

with

\[ D(-x_0) = \left[ 1 + \frac{n_0 x_0}{\pi} \int_1^\infty dx \frac{\rho(x)}{x(x+x_0)} \right] \left[ 1 + \frac{\gamma}{2\pi} \int_1^\infty dx \frac{\rho(x)}{(x+x_0)^3} \right], \quad (26) \]

wherein principal integration is understood for the singular integral. It immediately follows the bootstrap equations for \( x_0 \) and \( \gamma \);

\[ \text{Re} \ D(x_0) = 0; \quad (27) \]

and

\[ \gamma = -2N(x_0) \left| \frac{\partial \text{Re} \ D(x)}{\partial x} \right|_{x=x_0}, \quad (28) \]

by imposing the requirement that the input characteristics of the resonance must be equal to the output ones.

\(^*\) In the relativistic problem, the exchange of the “particle” in the crossed channel generally yields logarithmic cuts for the \( N \) function instead of the pole in this case. See, however, the discussion in the final section.
§ 5. Numerical results and concluding discussions

The bootstrap equations (27) and (28) can be solved numerically, if the cutoff function is fixed reasonably. \( x_0 \) and \( \gamma \) are, then, evaluated as functions of \( n_0 \), which should be considered as the parameter of the model equivalent to \( \xi \). Numerical correspondence between \( \xi \) and \( n_0 = h(0)_{\text{exact}} = \xi / A(0) \) is given in Fig. 1. Here, we assume the following naive function as the cutoff:

\[
R(x) = A' / (A' + (x^2 - 1)^3).
\]  

\( A \) gives a measure in unit of the particle mass where the pure elastic unitarity breaks down. We assign temporarily 10 and 20 to \( A \). With (29) the kinematical factor \( \rho(x) \) has a peak near \( x = A \), and all integrals to be considered are well convergent.

In the actual computation, \( n_0 \) and \( \gamma \) are calculated for each of the prefixed \( x_0 \)'s with appropriate spacing. The iterative procedure is adopted in evaluating \( D(-x_0) \) of (26), starting from \( D(-x_0) = 1 \). The results are shown in Figs. 2 and 3, where \( x_0 \) and \( \gamma \) are shown against \( n_0 \), respectively. The result with only the first iteration is given besides the one with sufficient iteration. The exact solution is also presented for comparison.

In the following we summarize several points of interest:

i) The solution of the bootstrap equation with sufficient iteration is quite satisfactory in obtaining the position of the resonance, irrespective of the value of \( A \), except for the case where it is very near the threshold or near the critical value \( x_M \).

ii) As for the reduced width, the solution gives a nice result for the larger value of \( A \), but it is less satisfactory for the smaller \( A \). The bootstrap equation yields lower values for the reduced width.

iii) The iterative procedure is very rapidly convergent.

iv) The effect of the repeated iterations is not remarkable except for the case where the resonance is near the threshold. This exception is expected since we fixed the subtraction point at the origin and the departure of \( D(-x_0) \) from
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Fig. 2. The position $x_0$ of the resonance as a function of $n_0$. The results with the first iteration (broken line) and the sufficient iterations (dash-dotted line) are shown together with the exact solution (solid line) for both cases of $A=10$ (a) and 20 (b).

Fig. 3. The reduced width $\gamma$ of the resonance as a function of $n_0$. Notations are the same as in Fig. 2.
unity would not be negligible in case of \( x_0 \) near the threshold.

v) With the exception mentioned above, the effect of repeated iterations does not always improve the results. In fact, the reduced width of the first iteration is even better than the one with sufficient iterations for \( \Lambda = 10 \).

It is not clear to what extent the above conclusions hold for the practical problems, where the crossing relation is more complicated and the bootstrap approximation yields generally logarithmic branch cuts instead of poles. Further, the inelastic process may be important. It is, however, our feeling that the above conclusions would still be valid qualitatively, since the left-hand discontinuity contributes to the dispersion equation only through an integrated form, and the detailed structure of the discontinuity is expected to be insensitive for the qualitative feature of the solution. As regards the inelastic process, it is partially taken into account phenomenologically through the cutoff.

It is finally suggested that, in a cutoff theory, it would be better to take into account also the "force" due to the exchange of the unphysical resonance besides the physical one. It does not imply to introduce any additional parameter, if we require self-consistency for the unphysical resonance, too. The recipe would be expected to give a sensible result for the low energy phenomena, even when the dispersion equation is rather high energy sensitive.

In conclusion we express our thanks to Prof. D. Ito for his interest to the present work.

References

   A. Kanazawa and N. Tokuda, Prog. Theor. Phys. 30 (1963),
   It is easy to extend their results to the case where the separable potential is given by
\[ \langle x|\hat{V}|x'\rangle = \delta(x_0 - x_0') \frac{\partial U(x)}{\partial x} \cdot \frac{\partial U^*(x)}{\partial x}, \]
instead of Eq. (2) of the above reference. The correspondence of the resulting scattering amplitude with ours (Eq. (10) in the text) can be established by putting \(-\iota/4\pi = \bar{\xi} \) and \( k^3 U(k)^2/3 = \rho(\sqrt{k^2 + 1}) \). Further, it can be generalized to the case of a complex field.\(^6\)
7) The uniqueness for a given unphysical (left-hand) singularities is proved by G. Frye and