
On an example of an $N$-body method for the study of small perturbations in galaxies

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ABSTRACT

This paper presents the first results of a project to apply numerical $N$-body methods to the study of small perturbations in galaxies. The central idea of the project is to make use of $N$-body calculations as realizations of the solutions of the equations that govern perturbations in the Lagrangian representation. Thus, an unperturbed system is described as an independent particle model of the usual kind satisfying Liouville’s equation and Poisson’s equation. Accordingly, the stellar orbits in the gravitational field of the unperturbed system are integrated as $N$ one-body problems. Nevertheless, a full $N$-body calculation is required in the solution of the linearized equations governing the Lagrangian displacements, because the Eulerian perturbation of the acceleration must be represented as a sum of two-body interactions. This paper describes a detailed investigation of applications of such calculations to the radial oscillations of a spherically symmetric galaxy.

Key words: instabilities – galaxies: kinematics and dynamics – galaxies: structure.

1 INTRODUCTION

It has occurred to several investigators to make use of numerical $N$-body methods in order to solve the linearized equations governing small perturbations in collisionless stellar systems (Merritt 1987; Leeuwin, Combes & Binney 1993; Wachlin, Rybicki & Muzzio 1993). The published accounts of this idea develop such methods for the Eulerian representation of small perturbations. In contrast, the present work concentrates on $N$-body schemes for the description of small perturbations in the Lagrangian representation.

Several considerations motivate an interest in $N$-body methods for the study of small perturbations:

(i) $N$-body methods can be applied to the solution of problems in linear perturbation theory which are not easily solved by purely analytic means. Such methods can also be used in order to test approximate analytic solutions of the perturbation equations.

(ii) The representation of linear perturbations with the aid of the $N$-body methods considered here can provide links between analytic work and fully non-linear $N$-body calculations. In particular, the new methods can be used in order to isolate interesting initial conditions for full $N$-body calculations of the evolution of perturbed systems.

(iii) An unperturbed system is represented here in terms of $N$ one-body problems, whereas, the perturbation is represented in terms of one $N$-body problem. Thus, the full power of an $N$-body calculation is concentrated on the perturbation. The perturbation appears in such a calculation as a much stronger signal, relative to statistical fluctuations, than in a conventional $N$-body calculation.

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This paper describes a Lagrangian formulation of $N$-body methods for the study of small perturbations in stellar systems, illustrates such methods with the aid of applications to a family of simple well-understood systems, and discusses the key technical issues that arise in the applications considered. The general formulation of the Lagrangian theory of small perturbations in stellar systems and the $N$-body representation of the general theory are described in Section 2. The formulation of the theory and its $N$-body representation for the radial oscillations of a spherically symmetric system are described in Section 3. Applications to the fundamental mode of radial pulsation in a homogeneous sphere and in a polytropic configuration of index $n = 1$ are the subjects of Sections 4 and 5, respectively. A strategy for exciting radial oscillations with the aid of external perturbations is presented and illustrated in Section 6. Finally, Section 7 contains a summary of the main results of this investigation, a discussion of the potential of such methods for wider applications, and a comparison of $N$-body representations of the Lagrangian and Eulerian perturbation theories.

2 THE LAGRANGIAN REPRESENTATION

2.1 Theoretical background

In the Lagrangian representation of small perturbations in a collisionless stellar system, the fundamental quantity is the Lagrangian displacement $(\Delta x, \Delta v)$ of a star (Vandervoort 1983). If $x$ and $v$ are the position and velocity, respectively, that a star would have had at a given time $t$ in the absence of the perturbation, then, by definition, the position and velocity of the star in the perturbed system at that time are $x + \Delta x$ and $v + \Delta v$, respectively.
For an infinitesimal perturbation, $\Delta x$ and $\Delta v$ are considered to be functions of $(x, v, t)$, and the equations of motion for the perturbation are
\[
\frac{d\Delta x}{dt} = \Delta v \quad \text{and} \quad \frac{d\Delta v}{dt} = \Delta a(\Delta x),
\]
where the total time derivative along the unperturbed trajectory of a star is
\[
\frac{d}{dt} = \frac{\partial}{\partial x} + v \frac{\partial}{\partial x} \frac{\partial V_0}{\partial x},
\]
and the Lagrangian perturbation of the acceleration is
\[
\Delta a(\Delta x) = -\left( \Delta x \frac{\partial}{\partial x} \frac{\partial V_0}{\partial x} \right) - \frac{\partial}{\partial x} \left( G \frac{\partial}{\partial x} \int \frac{\Delta x(x', v', t) f_0(x', v')}{|x' - x|} \, dx' \, dv' \right).
\]
Here the distribution function $f_0(x, v)$ represents the unperturbed density of stars in the six-dimensional phase space of a single star, and $V_0(x)$ represents the gravitational potential in the unperturbed system. Also, $G$ denotes the constant of gravitation, and $m$ denotes the mass of a single star. The second term on the right-hand side of equation (3) represents the Eulerian perturbation of the gravitational acceleration of a star, and the integration there extends over the region $\Omega$ of the phase space that is accessible to stars in the unperturbed system.

Equations (1)–(3) form a complete system of integro-differential equations governing the infinitesimal perturbations of a stellar system. The investigation of such perturbations with the aid of these equations can be formulated in terms of an initial-value problem for the evolution of perturbations or as a characteristic-value problem for the normal modes of oscillation and instability (Vandervoort 1989, 1991).

The Lagrangian formulation of the perturbation theory in galactic dynamics is entirely equivalent to the better known Eulerian formulation. The properties of a perturbation can be determined entirely in terms of its Lagrangian displacement $(\Delta x, \Delta v)$ and the unperturbed distribution function $f_0(x, v)$. In particular, the Eulerian perturbations of the distribution function, the gravitational potential, and the density can be expressed in terms of $(\Delta x, \Delta v)$ and $f_0(x, v)$ (Vandervoort 1983).

### 2.2 An $N$-body realization of the theory of small perturbations

The Eulerian perturbation of the acceleration is expressed in equation (3) in terms of an integral over mass elements $m f_0(x, v) \, dx \, dv$ in the system. Therefore, we can turn equations (1)–(3) into an $N$-body calculation by replacing that integral with a sum over particles. Thus, for a system of $N$ particles, we replace equations (1) with the system of ordinary differential equations
\[
\frac{d\Delta x_\alpha}{dt} = \Delta v_\alpha \quad \text{and} \quad \frac{d\Delta v_\alpha}{dt} = \Delta a_\alpha(\Delta x_\alpha),
\]
where we are now using Greek subscripts in order to identify particles. The Lagrangian perturbation of the acceleration of the $\alpha$th particle is
\[
\Delta a_\alpha(\Delta x) = -\left( \Delta x_\alpha \frac{\partial}{\partial x_\alpha} \frac{\partial V_0(x_\alpha)}{\partial x_\alpha} \right) - \frac{\partial}{\partial x_\alpha} \left( G \frac{\partial}{\partial x_\alpha} \sum_{\beta=1}^{N} m_\beta \Delta x_\beta \right) \left( |x_\beta - x_\alpha| \right).
\]
Here $m_\beta$ denotes the mass of the $\beta$th particle. When the $N$-body calculation is formulated explicitly in this way, the term in which $\beta = \alpha$ must be suppressed in the sum on the right-hand side of equation (5). The unperturbed coordinates $x_\alpha$ and velocities $v_\alpha$ are solutions of the Newtonian equations of motion
\[
\frac{dx_\alpha}{dt} = v_\alpha \quad \text{and} \quad \frac{dv_\alpha}{dt} = -\frac{\partial V_0(x_\alpha)}{\partial x_\alpha} \quad (\alpha = 1, 2, \ldots, N).
\]
Equations (4)–(6) describe, in general, a conceptually straightforward basis for the construction of an $N$-body representation of the theory of small perturbations in collisionless stellar systems. It is to be expected that there would be a variety of choices that could be made and a variety of precautions that should be observed in performing calculations with the aid of such equations. That is certainly the case for conventional $N$-body calculations where, for example, the choices of integration schemes and representations of the 'two-body interactions' are central issues addressed in practice in a number of different ways (see e.g. Sellwood 1987). The approach to these matters that is adopted in what follows is to reduce the analytic equations governing the Lagrangian displacements as much as possible for the intended applications and then to introduce an $N$-body representation of the reduced equations. Thus, we shall apply $N$-body methods to a specific class of problems in the theory of small perturbations and, in that context, address important issues by example. It is in that sense that this paper deals with 'an example' of an $N$-body method for the study of small perturbations. The concluding section of the paper contains a more general discussion of such methods.

### 3 FORMULATION OF THE METHOD FOR THE STUDY OF THE RADIAL OSCILLATIONS OF A SPHERICAL GALAXY

For the sake of simplicity and economy, we concentrate, in what follows, on the calculation of radial oscillations in a static spherically symmetric galaxy.

In such a system, the unperturbed density of stars in the phase space of a single star is a function $f_0(x, v) = f_0(E, L^2)$,
\[
f_0(x, v) = f_0(E, L^2),
\]
where $E = \frac{1}{2} |v|^2 + V_0(r)$ is the energy per unit mass of a star and $L = [x \times v]$ is the total angular momentum per unit mass of a star. The unperturbed density $\rho_0(r)$ and gravitational potential $V_0(r)$ are functions of the radial coordinate $r$ in a system of spherical polar coordinates. The orbit of a star in the unperturbed system is described by a classical central field problem in which the motion in the radial direction is represented in terms of the radial coordinate $r$ and the radial component $v$ of the velocity. The energy per unit mass of a star can be written more explicitly as
\[
E = \frac{1}{2} \frac{1}{r^2} (v^2 + V_0(r) + \frac{L^2}{2r^2}),
\]
and, of course, $E$ and $L$ are integrals of the unperturbed motion. Inasmuch as equation (7) describes a distribution function which is even in the components of $v$, we have $\int cf_0(x, v) \, dv = 0$, and the unperturbed system is indeed static.

#### 3.1 The equations governing a perturbation

Radial oscillations are spherically symmetric perturbations. In particular, the angular momentum $L$ of each star in the perturbed system is conserved. For these reasons, the Lagrangian displacement of a star can be reduced to two components $\Delta x(x, v, t)$ and $\Delta v(x, v, t)$.
resulting equations in equation (9) for the perturbed and unperturbed systems and linearizing the coordinate are considered to be functions of equation (11) is the linearized expression for the change in the entire on the change suffered by the Heaviside function under equation (10) represents the mass of the system that lies within the sphere of radius $r$, and $f(x, v, t)$ denotes the perturbed density of stars in the phase space of a single star. The region of integration $\Omega$ in equation (10) is the region of the phase space accessible to stars in the system. The kernel $H(r - r')$ is the Heaviside function, defined here to have a value of 1 when the argument is positive and a value of 0 when the argument is negative.

We now let $x$ and $v$ represent the position and velocity of the star, respectively, in the unperturbed galaxy at time $t$. Likewise, $r$ and $v$ now represent the radial coordinate and the radial component of the velocity, respectively, in the unperturbed system. The equations of motion governing stellar orbits in the unperturbed system are obtained by replacing the distribution function and the mass variable in equations (9) and (10) with the corresponding unperturbed quantities $f_0(x, v)$ and $M_0(r)$, respectively.

In the perturbed system, the radial coordinate and radial component of the velocity are $r + \Delta r$ and $v + \Delta v$, respectively, where the components $\Delta r$ and $\Delta v$ of the Lagrangian displacement are considered to be functions of $x$, $v$, and $t$. The equations that govern $\Delta r$ and $\Delta v$ are derived by forming the difference of equations (9) for the perturbed and unperturbed systems and linearizing the resulting equations in $\Delta r$ and $\Delta v$. In this connection, we must evaluate the Lagrangian perturbation of the mass variable

$$\Delta M(r, v, L, t) = M(r + \Delta r, t) - M_0(r)$$

In the expression for $\Delta M(r, v, L, t)$ to the integral shown in equation (11), we have made use of the fact that mass elements $m_{i0}(x', v')dxdv'$ are conserved and, hence, invariant under the application of the perturbation. Thus, $\Delta M(r, v, L, t)$ depends entirely on the change suffered by the Heaviside function under the displacement from the unperturbed variables to the perturbed variables. The quantity within square brackets in the integrand in equation (11) is the linearized expression for the change in the Heaviside function. It is to be understood in equation (11) that $\Delta r = \Delta r(x, v, t)$ and $\Delta r' = \Delta r(x', v', t)$. The expression for $\Delta M(r, v, L, t)$ in equation (11) reduces to

$$\Delta M(r, v, L, t) = \frac{\partial M_0(r)}{\partial r} \Delta r - 4\pi \int_0^R (r')^2 dr' \int f_0(x', v') \Delta r(x', v', t) \delta(r - r') dv', \quad (12)$$

where, in the second term, we have recognized the derivative of the Heaviside function as Dirac’s delta function $\delta(r - r')$.

With the aid of this result, we readily bring the equations governing $\Delta r$, $\Delta v$ into the form

$$\frac{d\Delta r}{dt} = \Delta v \quad (13)$$

and

$$\frac{d\Delta v}{dt} = -\frac{\partial}{\partial r} \left[ \frac{GM_0(r)}{r^2} - \frac{L^2}{r^3} \right] \Delta r + 4\pi Gm_{i0}(r) \Delta r \int (x, v, t) f_0(x, v) dv. \quad (14)$$

Here, and in what follows, the integration over velocities extends over the region of the velocity space that is accessible, in the absence of the perturbation, to stars in the neighborhood of the point $x$ in the configuration space. The second term on the right-hand side of equation (14) represents the Eulerian perturbation of the gravitational acceleration. This derivation of equations (13) and (14) makes it clear that the Eulerian perturbation of the gravitational acceleration is simply the radial component of the gravitational acceleration caused by the Eulerian perturbation of the mass variable $M_0(r)$.

In preparation for the solution of equations (13) and (14) with the aid of an $N$-body method, we adopt a representation of the Eulerian perturbation of the acceleration in terms of a ‘hydrodynamical Lagrangian displacement’ $\xi(r, t)$. By definition,

$$\rho_0(r) \xi(r, t) = m \int \Delta r(x, v, t) f_0(x, v) dv. \quad (15)$$

Accordingly, we replace equation (14) with

$$\frac{d\Delta v}{dt} = -\frac{\partial}{\partial r} \left[ \frac{GM_0(r)}{r^2} - \frac{L^2}{r^3} \right] \Delta r + 4\pi G\rho_0(r) \xi(r, t). \quad (16)$$

### 3.2 The smooth field representation

We now express the hydrodynamical Lagrangian displacement as a superposition of orthonormal basis functions $\psi_i(r)$ ($i = 1, 2, \ldots$) in the manner

$$\xi(r, t) = \sum_i a_i(t) \psi_i(r). \quad (17)$$

The displacement $\xi(r, t)$ is the radial component (and the only nonvanishing component) of a vector which will be single valued at the origin only if $\xi(r, t)$ vanishes there, so we require that the basis functions satisfy the boundary condition $\psi_i(r) = 0$ at $r = 0$. We also require that the functions $\psi_i(r)$ satisfy orthogonality relations of the form

$$4\pi \int_0^R \psi_i(r) \psi_j(r) \rho_0(r) r^2 dr = \delta_{ij}, \quad (18)$$

where the unperturbed density $\rho_0(r)$ of the system appears as a weighting function in the integral on the left-hand side and $\delta_{ij}$ denotes the Kronecker delta. The coefficients $a_i(t)$ on the right-hand side of equation (17) are given by

$$a_i(t) = 4\pi \int_0^R \psi_i(r) \xi(r, t) \rho_0(r) r^2 dr = 4\pi \int_0^R \psi_i(r) \left[ m \int \Delta r(x, v, t) f_0(x, v) dv \right] r^2 dr = m \int \psi_i(r) \Delta r(x, v, t) f_0(x, v) dv, \quad (19)$$

in virtue of the orthogonality relations presented in equation (18) and the definition of $\xi(r, t)$ in equation (15). For the purpose of an $N$-body calculation, the important point about this formulation is that the coefficients of the orthonormal functions in the representation of $\xi(r, t)$ can be expressed as integrals of products $\psi_i \Delta r$ over mass elements $m f_0(x, v) dv$. 

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If the adopted system of orthogonal functions is complete in the
sense that
\[ 4\pi r^2 \rho_0(r) \sum_i \psi_i(r)\psi_i(r') = \delta(r-r'), \] (20)
then equations (17) and (19) provide the basis for an exact
representation of the Eulerian perturbation of the acceleration on
the right-hand side of equation (16). In order to verify that this is so,
it suffices in the derivation of the perturbation equations to postpone
the integration over the radial coordinate in the second term on the
right-hand side of equation (12) and to leave the Eulerian perturbation
of the acceleration in the form
\[ 4\pi G \rho_0(r) \xi(r, t) = \frac{Gm}{r^3} \int_0^r \Delta r(x', v', t) \delta(r-r') f_0(x', v') dx' dv' \]
\[ = \frac{4\pi Gm}{r^3} \times \int_0^r (r')^2 dr' \int_0^r \Delta r(x', v', t) \delta(r-r') f_0(x', v') dv'. \] (21)
Upon substituting from equation (20) for Dirac’s delta function in
the integration over the radial coordinate in the second term on the
representation of the Eulerian perturbation of the acceleration on
a star, we again represent the
features to be incorporated into the code. Moreover, the calculation
have properties which must be understood before results can be
interpreted. The following are the main points of interest about the
calculations that have been performed in this investigation. Further
details are provided in subsequent sections.

(i) Equilibrium configurations: the unperturbed systems that have
been chosen for this investigation are homogeneous spheres and
polytropic spheres of index \( n = 1 \). Without loss of generality, we let
all of the stars in each system have the same mass \( m_3 = m \),
inasmuch as the dynamical behaviour of a collisionless stellar system
is independent of the spectrum of stellar masses. For each
equilibrium configuration, the initial values of the coordinates \( r_x \),
the velocities \( v_x \), and the angular momenta \( L_x \) are determined with
the aid of a Monte Carlo process such that the system is a realization
of the distribution function \( f_0(x, v) = f_0(E, L^2) \). The present version
of the Monte Carlo process is similar to that one that was employed
by Miller et al. (1982) in earlier N-body calculations.

(ii) The perturbations: most of these calculations have been
performed for the fundamental mode of radial pulsation. However,
a few experiments were attempts to excite the next higher mode of
radial oscillation.

(iii) Excitation of modes: we can excite perturbations by applying
suitable initial conditions on the Lagrangian displacements. Initial
conditions for the fundamental mode can be derived from an
approximate analytic theory based on the tensor virial equations
(Vandervoort 1983). The theory is exact for the fundamental mode
in a homogeneous sphere. Alternatively, we can perturb the system
by applying a transient external force to the system. We can attempt
to excite individual modes preferentially by adjusting the spatial
structure of the perturbing force.

(iv) The integrators: in most experiments we have integrated the
equations of motion governing the unperturbed coordinates and
velocities and the Lagrangian displacements with the aid of a time-
centred leap-frog scheme. Most of these integrations were performed
in spherical polar coordinates, but a few were done in Cartesian
coordinates as a check. In early experiments, calculations were
performed with the aid of a Runge–Kutta integrator of the fourth
order, also in spherical polar coordinates. In order to initiate
the leap-frog scheme, we advance the velocities and their Lagran-
gian perturbations half a time-step from their initial values with the
aid of the Runge–Kutta integrator.

(v) The smooth-field representation: in the representation of the
Eulerian perturbation of the acceleration, the orthonormal functions
\( \psi_i(r) \) are odd polynomials in the radial coordinate. In practice, only
a finite set of such basis functions can be included in the super-
position on the right-hand side of equation (24). The set should be
small for the sake of computational economy and efficiency, but it
should be large enough to provide an accurate representation of the
perturbations considered. In the calculations reported here, the
right-hand side of equation (24) is truncated to three or fewer
terms. This is sufficient, in part, because \( \psi_i(r) \) alone is a good
approximation to the hydrodynamical Lagrangian displacement for the
fundamental mode of radial pulsation in the equilibrium
configurations considered.

(vi) The calculations: calculations were performed for systems of
100, 1000 and (in two runs) 10 000 bodies. The codes for these
calculations were written in absoft macfortran /020 and run on a
Macintosh IIcx computer.

(vii) Representation of results: if the perturbation of the system is
a stable mode, then the mode should manifest itself as a sinusoidal
oscillation of the coefficients \( a_i(t) \) in the representation of \( \xi(r, t) \) in

3.3 The N-body calculation

For the N-body calculation which represents the radial oscillations
of a spherically symmetric galaxy, we replace equations (13) and
(16) with the equations
\[ \frac{d\Delta r_a}{dt} = \Delta v_a \] (22)
and
\[ \frac{d\Delta v_a}{dt} = - \frac{\partial}{\partial r_a} \left[ \frac{GM_b(r_a)}{r_a^2} - \frac{L_a^2}{r_a^2} \right] \Delta r_a \]
\[ + 4\pi G \rho_0(r_a) \xi(r_a, t), \] (23)
respectively, for a system of \( N \) stars, where \( a = 1, 2, \ldots, N \). At the
radial coordinate \( r_a \) of the \( a \)th star, we again represent the
hydrodynamical Lagrangian displacement as a superposition of
orthogonal basis functions of the form
\[ \xi(r_a, t) = \sum_i a_i(t) \psi_i(r_a) \] (24)
[cf. equation (17)], where we now replace the representation of the
coefficients \( a_i(t) \) as integrals over mass elements in equation (19)
with a representation as sums over particles
\[ a_i(t) = \sum_{\beta=1}^N m_\beta \psi_i(r_\beta) \Delta r_\beta(t). \] (25)
In this formulation of the N-body calculation, we do not suppress
the term in which \( \beta = \alpha \) in the sums over particles. The equations of
motion for the unperturbed orbits are
\[ \frac{dr_a}{dt} = v_a \quad \text{and} \quad \frac{dv_a}{dt} = - \frac{GM_b(r_a)}{r_a^2} + \frac{L_a^2}{r_a^2}, \] (26)
where \( a = 1, 2, \ldots, N \).

3.4 A summary of practical details

The implementation and application of an N-body calculation
involve a number of choices concerning the problems to be
investigated, the strategies to be employed, and the technical
equation (24). Therefore, the essential results of these calculations are presented in plots of the coefficients $a_i(t)$ against the time $t$.

(viii) Parametric resonance: in general, the oscillations of the perturbations in the $N$-body system are not sinusoidal even when the initial conditions represent a normal mode of the theoretical system quite accurately. The amplitude and phase of a typical oscillation vary substantially. This behaviour is a manifestation of a parametric resonance which is a property of the $N$-body realization of the theory of small perturbations and not of the theory itself. With a suitable choice of initial conditions for the unperturbed orbits, the resonance can be suppressed. For homogeneous systems, the resonance is exhibited explicitly in Appendix A, and the procedure for suppressing the effects of the resonance is described there.

(ix) Chaotic behaviour: in the experiments on polytropic spheres, which are described in Section 5.1, the Lagrangian displacements initially oscillate sinusoidally, as in a normal mode of the theoretical system, but these oscillations are eventually overwhelmed by an exponential growth of the Lagrangian displacements. For reasons that are explained in Section 5.1, the growth is interpreted as a manifestation of chaotic behaviour closely related to chaotic behaviour in conventional $N$-body calculations (Miller 1964, 1971; Heggie 1991; Goodman, Heggie & Hut 1993; Kandrup & Smith 1991, 1992; Kandrup, Smith & Willmes 1992). The rate of growth is smaller, the greater the number of bodies in the calculation, and the growth is negligible, for practical purposes, when the number of bodies is sufficiently large.

(x) Unreported experiments: many numerical experiments were performed in the course of the investigation described in this paper. For the sake of brevity, we present only the results of a few experiments which illustrate and illuminate the most important issues concerning such $N$-body methods. We are thus left with a considerable body of unpublished results. Many of these concern calculations performed in order to verify the correctness of the code, check the accuracy of the calculations, etc. However, other experiments concerned alternative versions of the method and alternative strategies for its application, which were useful and instructive for the author but would be worth reporting only if subsequent investigations showed them to have a wider interest.

4 HOMOGENEOUS SPHERES

In the first series of experiments in this investigation, the unperturbed configuration is a sphere of radius $R$ and constant density $\rho_0$. The gravitational acceleration is

$$GM_0(r) = -\frac{4\pi}{3} G \rho_0 r = -n^2 r,$$  \hspace{1cm} (27)$$

in the interior of the configuration, where

$$n^2 = \frac{4\pi}{3} G \rho_0.$$ \hspace{1cm} (28)$$

Each star in the unperturbed system is an isotropic, simple harmonic oscillator of frequency $n$.

4.1 The fundamental mode of radial pulsation

The fundamental mode of radial pulsation in a homogeneous sphere of stars can be constructed in the Lagrangian representation either by solving equations (13) and (14) explicitly in the special case that the unperturbed gravitational acceleration is given by equation (27) or by reducing to conditions of spherical symmetry and vanishing rotation the solution obtained elsewhere (Vandervoort 1991) for the pulsation modes in a uniformly rotating, homogeneous spheroid of stars. The Lagrangian displacement ($\Delta r$, $\Delta v$) of the mode is

$$\Delta r = -\frac{2}{3n^2} A[nr \cos(nt) - 2v \sin(nt)]$$ \hspace{1cm} (29)$$

and

$$\Delta v = -\frac{2}{3n^2} A \left[ n^2 r \sin(nt) - nv \cos(nt) - \frac{2L^2}{r^2} \sin(nt) \right],$$ \hspace{1cm} (30)$$

an exact solution of equations (13) and (14), where $A$ is a constant amplitude and the phase of the mode is chosen here so that $\Delta r = -\frac{2}{3n^2} Ar$ and $\Delta v = -\frac{2}{3n^2} Av$ at $t = 0$. The hydrodynamical Lagrangian displacement is

$$\xi(r, t) = -\frac{2}{3n^2} Ar \cos(nt).$$ \hspace{1cm} (32)$$

The frequency of the mode is equal to the common frequency $n$ of the simple harmonic oscillations of the unperturbed stars, so the system is highly degenerate.

Equations (29) and (30) describe the behaviour to be represented in an $N$-body realization of the fundamental mode. One important property of the mode to be noted here is that details of the solution presented in equations (29)–(32) do not depend on the form of the unperturbed distribution function $f_0$; the solution requires only that the configuration is static as described above and that the integral of the unperturbed distribution function over the accessible velocity space is the constant density $\rho_0$ of the configuration.

4.2 The $N$-body calculation

We begin a calculation by specifying the initial values of the unperturbed coordinates $r_{\alpha}$, velocities $v_{\alpha}$, and angular momenta $L_{\alpha}$ of the stars and the components $\Delta r_{\alpha}$ and $\Delta v_{\alpha}$ of the Lagrangian displacements, where $\alpha = 1, 2, \ldots, N$.

The implementation of the Monte Carlo process for the assignment of the initial values of the coordinates, velocities, and angular momenta requires a specification of the unperturbed distribution function. For the calculations described in what follows, our choice for the distribution function is

$$f_0 = \rho_0 \frac{\rho_0}{\pi m^2 R^2} (n^2 R^2 - 2E + \frac{L^2}{R^2})^{-1/2},$$ \hspace{1cm} (33)$$

if $n^2 R^2 - 2E + \frac{L^2}{R^2} > 0$

$$=0,$$ \hspace{1cm} (33)$$

otherwise,

where we are writing the gravitational potential and the energy per unit mass as

$$V_0(r) = \frac{1}{2} n^2 r^2$$ \hspace{1cm} and \hspace{1cm} $E = \frac{1}{2} v^2 + \frac{1}{2} m^2 r^2 + \frac{L^2}{2m^2},$$ \hspace{1cm} (34)$$

respectively. It is readily verified that the integral of this distribution function over the velocity space accessible to a star at $r < R$ is the constant density $\rho_0$ of the unperturbed configuration.

We assign the initial values of the Lagrangian displacements ($\Delta r_{\alpha}$, $\Delta v_{\alpha}$), where $\alpha = 1, 2, \ldots, N$, in accordance with equations (31).

The $N$-body calculation consists of the numerical integration of equations (22), (23) and (26) with the representation of the hydrodynamical Lagrangian displacement $\xi(r, t)$ given by equations (24) and (25). In the calculations reported here, the units of length,

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We have plotted, as functions of the time \( t \), coefficients \( a_i(t) \) derived in accordance with equation (25) and used in the representation of \( \xi(r, t) \) in equation (24). We have truncated the right-hand side of equation (24) to three terms. The three orthogonal functions \( \psi_1(r) \), \( \psi_2(r) \) and \( \psi_3(r) \) are indeed smaller than \( a_i(t) \) by the order of the square-root of \( N \), i.e. negligible at the level of expected noise in an \( N \)-body realization of the system. However, the oscillation of \( a_1(t) \) is clearly not sinusoidal in Fig. 1.

In Appendix A, we show that the variations of the amplitude and phase of the oscillation of \( a_1(t) \) in Fig. 1 can be attributed to parametric resonance and other non-linearities in the \( N \)-body system. The resonance arises as a consequence of the degeneracy of the frequency of the fundamental mode in a homogeneous sphere and the frequency \( n \) of the simple harmonic oscillations of the unperturbed stars. In an \( N \)-body realization of the mode, fluctuations in the complexion of unperturbed stars oscillate with the frequency \( n \) and, hence, with the expected frequency of the mode. The oscillations of quantities derived from the \( N \)-body realization of the unperturbed system are the immediate source of parametric resonance.

It is also shown in Appendix A that we can suppress parametric resonance entirely in the \( N \)-body realization of the fundamental mode with the aid of suitable initial conditions on the unperturbed orbits. We begin by choosing initial sets of values of \( (r_\alpha, v_\alpha, L_\alpha) \) \( (\alpha = 1, 2, \ldots, N) \) for \( N \) particles as before. This complexion of initial conditions is a realization of the density of stars in the phase space of the unperturbed system. We now replace each particle \( \alpha \) with a set of particles distributed at equal intervals of phase around the trajectory in the phase plane of the radial motion which has the assigned value \( L_\alpha \) of the angular momentum and which passes through the initial point \( (r_\alpha, v_\alpha) \). We let \( \mu \) denote the number of particles distributed around each trajectory, and we rescale the mass per particle accordingly. This new complexion of initial conditions for \( \mu \) times the original number of stars is also a realization of the density of stars in the phase space of the unperturbed system.

Fig. 2 shows the behaviour of \( a_1(t) \), \( a_2(t) \) and \( a_3(t) \) in an \( N \)-body calculation of the fundamental mode in a homogeneous sphere composed of \( N = 20 \) orbits with \( \mu = 5 \) stars distributed on each of the \( a_i(t) \) and \( a_3(t) \) are indeed smaller than \( a_1(t) \) by the order of the square-root of \( N \), i.e. negligible at the level of expected noise in an \( N \)-body realization of the system. However, the oscillation of \( a_1(t) \) is clearly not sinusoidal in Fig. 1.

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orbit, uniformly in phase around the trajectory in the phase plane of the radial motion. Here, three orthogonal polynomials \( \psi_0(r), \psi_2(r) \) and \( \psi_3(r) \) are included in the representation of \( \xi(t, r) \). The coefficient \( a_0(t) \) now undergoes a very regular sinusoidal oscillation with a period \( P = 6.137(\pi G\rho_0)^{-1/2} \). According to equation (28), the theoretical period is \( P = 5.541938 \ldots (\pi G\rho_0)^{-1/2} \). The difference between the experimental and theoretical periods is about 13 percent. This is consistent with an expectation that noise in this calculation would be of the order of the inverse square root of the number of orbits. In Fig. 2, the coefficients \( a_2(t) \) and \( a_3(t) \) oscillate with small amplitudes at the level of the noise.

5 POLYTROPIC SPHERES

In subsequent experiments, the unperturbed configuration is a stellar-dynamical counterpart of a gaseous polytrope of index \( n = 1 \). This is an example of a system in which the unperturbed distribution function \( f_0(x, \mathbf{v}) \) is a function only of the energy per unit mass \( E \) of a star. In particular,

\[
    f_0(x, \mathbf{v}) = \frac{(-E)^{-1/2}}{8\sqrt{2\pi G M R^2}},
\]

where \( R \) is the radius of the configuration. The density and mass variable of the configuration are

\[
    \rho_0(r) = \frac{\sin(\pi r/R)}{(\pi/R)},
\]

and

\[
    M_0(r) = 4\pi \rho_0 \left( \frac{R}{\pi} \right)^3 \left[ -\left( \frac{\pi r}{R} \right) \cos \left( \frac{\pi r}{R} \right) + \sin \left( \frac{\pi r}{R} \right) \right],
\]

respectively, where the constant \( \rho_0 \) is the central density.

The fundamental mode of radial pulsation in an inhomogeneous sphere is described approximately by a theory constructed with the aid of the tensor Virial equations of the second order (Vandervoort 1983). In that approximation, the mode is described by equations (29)–(32) with the frequency \( n \) of the mode given by

\[
    n^2 = -\frac{W}{T},
\]

where

\[
    W = -4\pi \int_0^R r^2 \frac{\partial V_0}{\partial r} \rho_0(r) r^2 dr
    = -4\pi G \int_0^R M_0(r) \rho_0(r) r^2 dr
\]

and

\[
    I = 4\pi \int_0^R r^2 \rho_0(r) r^2 dr = 4\pi \int_0^R \rho_0(r) r^4 dr
\]

are, respectively, the gravitational potential energy and the moment of inertia of the unperturbed configuration. [In what follows, \( n \) always denotes the frequency defined by equation (39) except where it is explicitly used in order to denote the polytropic index.] In the polytrope considered here, an explicit evaluation of the right-hand side of equation (39) with the aid of equations (37), (38), (40) and (41) gives

\[
    n^2 = \frac{3\pi G\rho_0}{\pi - 6}.
\]

The \( N \)-body calculations of the fundamental mode were initially organized for the polytropic sphere as they had been for the homogeneous sphere. Thus, the initial conditions on the unperturbed orbits were again chosen with the aid of a Monte Carlo process, and the initial values of the Lagrangian displacements were fixed in accordance with equations (31). The units of length, density and time are \( R, \rho_0 \) and \( (\pi G\rho_0)^{-1/2} \), respectively, where it is to be remembered that \( \rho_0 \) now is the central density of the system. Accordingly, the unit of the amplitude \( A \) is \( \pi G\rho_0 \) in equations (29)–(32). In fixing the initial conditions on the Lagrangian displacements in accordance with equations (31), we set the amplitude \( A \) (which is arbitrary) equal to unity when measured in this system of units.

For the polytropic spheres of index \( n = 1 \), as for the homogeneous spheres, we have truncated the right-hand side of equation (24) to three or fewer terms. We have represented the orthonormal basis functions recursively by writing

\[
    \psi_1(r) = N_1 \left( \frac{4\pi}{3} r^3 \right)^{-1/2} \frac{r}{R},
\]

and

\[
    \psi_i(r) = N_i \left[ \sum_{j=1}^{i-1} c_{ij} \psi_j(r) + \left( \frac{4\pi}{3} r^3 \right)^{-1/2} \left( \frac{r}{R} \right)^{2i-1} \right],
\]

in contrast to equation (35), where the constants \( N_i \) and \( c_{ij} \) are determined by the normalization and orthogonalization of the basis functions in accordance with equation (18). The values of the constants required for the calculation of \( \psi_i(r), \psi_2(r) \) and \( \psi_3(r) \) are listed in Table 2.

| \( N_1 \) | 2.896715 |
| \( N_2 \) | 58.207218 |
| \( c_{31} \) | -0.1105212 |
| \( c_{32} \) | -0.0764772 |

The \( N \)-body realization of the fundamental mode in a polytrope of index \( n = 1 \) shows the same tendency to parametric resonance that was observed in the case of the homogeneous sphere. In order to suppress the resonance, we accordingly let the unperturbed configuration consist of \( N \) orbits, selected with the aid of the Monte Carlo process, with \( \mu \) particles distributed at equal intervals of the phase of the radial motion on each orbit.

5.1 Chaotic behaviour

Fig. 3 presents the results of a calculation of the fundamental mode in a polytropic configuration composed of \( N = 200 \) orbits with \( \mu = 5 \) stars distributed around each orbit. As in earlier figures, the coefficients \( a_0(t) \) are plotted in Fig. 3. As would be expected on the basis of the approximate, theoretical representation of the mode in equations (29)–(32), the coefficient \( a_1(t) \) of the polynomial \( \psi_1(r) \) which is linear in \( r \) in equation (24) dominates the initial behaviour of the system and initially oscillates sinusoidally. The measured period \( P = 7.053(\pi G\rho_0)^{-1/2} \) agrees quite well with the prediction \( P = 2\pi/n = 7.13596 \ldots (\pi G\rho_0)^{-1/2} \) based on equation (42). The coefficients \( a_2(t) \) and \( a_3(t) \) initially oscillate irregularly with amplitudes much smaller than the amplitude of the oscillation of \( a_1(t) \). However the oscillations of \( a_2(t) \) and \( a_3(t) \) grow slowly even from the beginning of the calculation, and they grow rapidly in Fig. 3 after 60–80 time units. Eventually, that growth

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Lagrangian displacements in the N-body calculation illustrated in Fig. 3. The initial conditions were chosen to excite the fundamental mode of radial pulsation.

communicates itself to the behaviour of $a_i(t)$, and the initial sinusoidal oscillation of that coefficient is overwhelmed.

The behaviour of the system that underlies the growing oscillations of the coefficients $a_i(t)$ can be described in terms of the long-time behaviour of the quantities $\Delta r_{\text{rms}}$ and $\Delta v_{\text{rms}}$ which are the rms values of the components $\Delta r_a$ and $\Delta v_a$, respectively, of the Lagrangian displacements of all the particles. The behaviour of $\Delta r_{\text{rms}}$ and $\Delta v_{\text{rms}}$ for the calculation illustrated in Fig. 3 is shown in Fig. 4. Evidently, an approximately exponential growth of $\Delta r_{\text{rms}}$ and $\Delta v_{\text{rms}}$ becomes established after about 40 time units.

The quantities $\mu N \Delta r_{\text{rms}}$ and $\mu N \Delta v_{\text{rms}}$ represent the separations of the perturbed and unperturbed systems in the configuration space and velocity space, respectively. It is to be remembered here that the quantity $\mu N$, the product of the number of stars per orbit and the number of orbits, is the number of stars in the system. The separation of the two systems in the 6N-dimensional phase space is therefore composed of the quantities $\mu N \Delta r_{\text{rms}}$ and $\mu N \Delta v_{\text{rms}}$. The result in Fig. 4 indicates that the growth of the separation of the perturbed and unperturbed systems in the phase space is ultimately exponential.

An exponential growth of the separation in the phase space of two systems with nearly identical initial conditions is a well-known manifestation of chaos in conventional N-body calculations (Miller 1964; Heggie 1991; Goodman et al. 1993; Kandrup & Smith 1991, 1992; Kandrup et al. 1992). For the sake of definiteness here, it must be understood that we are considering an infinitesimal separation of two systems and adopting a linearized description of its growth. For a system composed of more than a few bodies, the investigations of Goodman, Heggie & Hut and of Kandrup, Smith & Willmott show that the e-folding time for the growth of the separation is a fraction of the crossing time of the system, i.e. of the order of the dynamical time-scale $(\pi G p_0)^{-1/2}$. These authors agree that fluctuations in the gravitational accelerations of the particles, attributed to particle noise and analysed in terms of the theory of two-body encounters, are an important source of such chaotic behaviour in N-body systems. However, Goodman et al. argue that particle noise is the dominant source of chaos whereas Kandrup et al. claim that collective effects also make a significant contribution to chaotic behaviour. It is beyond the scope of this paper to try to resolve that difference. In any case, a resolution seems unnecessary for present purposes.

The question arises as to whether or not the exponential growth of $\Delta r_{\text{rms}}$ and $\Delta v_{\text{rms}}$ in the present calculations is also a manifestation of the chaotic behaviour of N-body systems. Equations (1)–(3) are the counterparts in the Lagrangian perturbation theory of the linearized variational equations employed in studies of chaos in N-body systems (Miller 1971; Heggie 1991; Goodman et al. 1993). The particle noise that leads to chaotic behaviour in the solution of the variational equations for an N-body system is suppressed in the Lagrangian perturbation equations. However, particle noise and the associated fluctuations in the Eulerian perturbations of the accelerations are restored when the perturbation equations are replaced with their N-body realizations, equations (4) and (5). Particle noise and associated fluctuations in the accelerations are not restored, however, in equations (6) which govern the unperturbed motions of the stars. The restriction to spheres in radial oscillation and the use of the smooth-field representation further modify the fluctuations associated with particle noise in the perturbations of the accelerations. Thus, the particles in a system described by equations (22)–(26) experience accelerations which are subject to temporal fluctuations through the coefficients $a_i(t)$ but which have regular spatial structures determined by the basis functions $\psi_i(r)$. In contrast, the fluctuations in an N-body system are both temporal and spatial. Notwithstanding these differences, it appears that the exponential growth of $\Delta r_{\text{rms}}$ and $\Delta v_{\text{rms}}$ in the present calculations is also a manifestation of the chaotic behaviour of N-body systems.

Equations (22)–(26) describe the behaviour of two dynamical systems. Equations (26) describe an unperturbed system as an independent-particle model in terms of N one-body problems. Inasmuch as the one-body problems are central field problems, they are integrable, and the unperturbed system is an integrable system. Equations (22) and (23) describe the motion of a perturbed system relative to the unperturbed system. In other words, equations (22)–(26) describe a kind of ‘model’ of a perturbed system infinitesimally separated in the phase space from an unperturbed system in which the motion is integrable. If the growth of $\Delta r_{\text{rms}}$ and $\Delta v_{\text{rms}}$ in this model is indeed a manifestation of chaos, then that behaviour must be attributed to the perturbed system. To the extent

![Figure 3](https://academic.oup.com/mnras/article-abstract/303/2/393/1068163/1068163.1)

**Figure 3.** Variations of the coefficients $a_i (i = 1, 2, 3)$ with time in an N-body realization of a radial oscillation of a polytropic sphere of index $n = 1$ consisting of 1000 stars (200 orbits with five stars distributed uniformly around each orbit). The initial conditions were chosen to excite the fundamental mode of radial pulsation.

![Figure 4](https://academic.oup.com/mnras/article-abstract/303/2/393/1068163/1068163.2)

**Figure 4.** Growth of the rms values $\Delta r_{\text{rms}}$ and $\Delta v_{\text{rms}}$ of the components of the Lagrangian displacements in the N-body calculation illustrated in Fig. 3.
that particle noise plays a role, it does so through the representation of the Eulerian perturbations of the accelerations in equations (23)--(25).

This interpretation is supported by the absence of exponential growth of the Lagrangian displacements of the particles in the N-body realizations of the fundamental mode in homogeneous spheres. The analysis in Appendix A shows that a truncated form of the perturbation equations is integrable in that case. If the more general perturbation equations described in Section 4 are also integrable or nearly so, then chaotic behaviour should not occur in the perturbed homogeneous spheres. In the run illustrated in Fig. 2, the oscillations of the coefficients $a_i(t)$ are very regular, and there is no sign of growth of the Lagrangian displacements of the kind found in Fig. 3 for a polytropic sphere. Such evidence of integrability persists in runs even longer than that illustrated in Fig. 2. On the other hand, the equations governing perturbations in the polytropic spheres are not integrable or, at least, not obviously integrable. Thus, a consistent interpretation of the results here and in Section 4 would be that, for the fundamental mode in the homogeneous spheres, the motion described by the N-body equations is integrable and, hence, regular, whereas, for the fundamental mode in the polytropic spheres, the motion is nonintegrable and, hence, chaotic.

In saying in the preceding paragraph that exponential growth of the Lagrangian displacements is absent in the case of the homogeneous spheres, we are excluding from consideration the behaviour that underlies the exponentially growing oscillations of $a_i(t)$ in the system described in Appendix A (see Fig. A1) in which parametric resonance is not suppressed. The parametric instability of that N-body realization of the fundamental mode is clearly a collective effect which is distinct from chaotic behaviour. This is a case in which the collective behaviour of the system described by the Lagrangian perturbation equations is modified by particle noise in the N-body realization.

The growth of $\Delta r_{\text{rms}}$ and $\Delta v_{\text{rms}}$ in polytropic spheres is much more rapid in a calculation in which the three lowest orthogonal polynomials are included on the right-hand side of equation (24) than in a calculation in which only the linear function $\psi_0(r)$ is retained in that representation of the hydrodynamical Lagrangian displacement. Figs 5 and 6 represent the result of a calculation in which the initial conditions are the same as in the preceding calculation but only the linear term is retained in the representation of $\Xi(r,t)$. Fig. 5 includes plots of $a_2(t)$ and $a_3(t)$ even though these coefficients are now not included on the right-hand side of equation (24). Evidently, the growth of $\Delta r_{\text{rms}}$ and $\Delta v_{\text{rms}}$ is much slower in Fig. 6 than in Fig. 4.

It thus appears that the exponential growth of $\Delta r_{\text{rms}}$ and $\Delta v_{\text{rms}}$ is sensitive to particle noise in the N-body representation of the Eulerian perturbation of the potential, i.e. in the representation of $\Xi(r,t)$, and that fluctuations in the coefficients $a_2(t)$ and $a_3(t)$ are the major contributors to that process in Figs 3 and 4. Another experiment with substantially the same implications consists of integrating equations (22) and (23) with the Eulerian perturbation of the acceleration imposed as if it were an external force. Thus, we substitute expressions $a_i(t) = a_{i0} \cos (\omega t + \phi)$, where $a_{i0}$, $\omega$ and $\phi$ are constants, for the coefficients on the right-hand side of equation (24). The test for self-consistency in the calculation then takes the form of a comparison of the runs of the coefficients $a_i(t)$, computed from the complexion of particles in the system in accordance with equation (25), with the imposed expressions for those coefficients. As part of this investigation, such calculations have been performed for realizations of a polytrope of index $n = 1$ consisting of 1000 particles (200 orbits with five particles on each orbit). In these calculations, the values of the constants $a_{i0}$, $\omega$ and $\phi$ were adjusted until there was satisfactory agreement of the computed coefficients $a_i(t)$ with the imposed expressions for those coefficients. An effect of this procedure is to suppress particle noise and, consequently, to suppress fluctuations in the representation of the hydrodynamical Lagrangian displacement in the present calculation includes only the lowest orthogonal function.
growth of the Lagrangian displacements. The present investigation includes experiments of two kinds in which fluctuations are present, collective effects are suppressed and, nevertheless, the Lagrangian displacements eventually grow exponentially.

Experiments of the first kind consist of runs with initial conditions in which the Lagrangian displacements are assigned randomly. In other words, the initial conditions are imposed and the calculations are run in a manner designed to avoid, as well as possible, the excitation of modes or other collective disturbances. In these runs, \( \Delta r_{\text{rms}} \) and \( \Delta v_{\text{rms}} \) ultimately grow with exponential growth rates comparable with the growth rate in the run illustrated in Figs 3 and 4. Although modes of instability can grow from particle noise in a conventional \( N \)-body calculation, it is unlikely that the growth of the Lagrangian displacements in the present calculations can be attributed to unstable modes. Only a mode whose spatial structure can be represented at least approximately by the basis functions included on the right-hand side of equation (24) is treated self-consistently in the smooth field representation. Consequently, most collective modes should be suppressed in these calculations. A mode which is not suppressed should reveal itself as a coherent oscillation, possibly growing exponentially, of one or more of the coefficients \( a_i(t) \). In these calculations, a small oscillation of \( a_1(t) \) signals the presence of the fundamental mode, but the mode is quickly overwhelmed by the growth of the Lagrangian displacements. Otherwise, collective motions appear to be absent in these runs.

In experiments of the second kind, we integrate equations (22), (23) and (26) simultaneously for a system of ‘field’ stars, oscillating in the fundamental mode of radial pulsation, and a system of ‘test’ stars moving in the gravitational field of the system of field stars. For the system of field stars, the Eulerian perturbations of the accelerations are specified as imposed forces in the manner described above. For the test stars, however, the Eulerian perturbations of the accelerations are computed from the comparison of the field stars in accordance with equations (24) and (25). Thus, fluctuations and extraneous collective effects in the motions of the field stars are suppressed, and the motions of the test stars are disturbed only by fluctuations in the Eulerian perturbations of the accelerations. The Lagrangian displacements of the individual test stars exhibit long-time exponential growth in these experiments. Thus the Lagrangian displacements of the test stars are individually sensitive to fluctuations in the Eulerian perturbations of the accelerations. The sensitivity of individual trajectories to small perturbations is a feature of chaotic behaviour in ordinary \( N \)-body systems (Heggie 1991; Goodman et al. 1993).

The e-folding times for the exponential growth of the Lagrangian displacements depend on the number of bodies included in a calculation. Fig. 7 compares e-folding times for 10 experiments with 100 bodies (20 orbits with five particles on each), 10 experiments with 1000 bodies (200 orbits with five particles on each), and one experiment with 10 000 bodies (2000 orbits with five particles on each). The initial values of the coordinates and velocities in the different runs are different \( N \)-body realizations of the same unperturbed polytrope. Each realization is distinguished by the value of the seed that is assigned at the first call on the random number generator in the Monte Carlo process for the specification of the initial conditions. The observed e-folding times are represented by filled circles in Fig. 7. Although there is considerable scatter among the e-folding times for different realizations of the system, for a fixed number of bodies, there is a significant tendency for the e-folding times to increase with the number of bodies. A least-squares fit of a straight line to a log-log plot of the data would imply that the e-folding time is equal to \( 2.45N^{1/4} \), but the present experiments do not suffice to make this a firm result. Moreover, it is a result that refers to a special representation of the perturbations of the accelerations, so it would be premature to attach a general significance to it. Nevertheless, the dependence of the growth rates on the number of bodies indicates that the growth is an effect of particle noise distinct from a collective mode of instability in a collisionless stellar system.

An e-folding time proportional to \( N^{1/4} \) in the present calculations is to be contrasted with an e-folding time essentially independent of \( N \) in a conventional \( N \)-body calculation (Heggie 1991; Goodman et al. 1993; Kandrup & Smith 1991, 1992; Kandrup et al. 1992). Evidently, the chaotic behaviour of a full \( N \)-body system is significantly suppressed in the present \( N \)-body representation of Lagrangian perturbation theory.

Two conclusions may be drawn from the experiments described above. (i) The long-time growth of the Lagrangian displacements of the particles is a manifestation of the chaotic behaviour of \( N \)-body systems. (ii) Such chaotic behaviour in these experiments is driven by fluctuations (i.e. by particle noise) in the Eulerian perturbations of the accelerations.

### 5.2 A 10 000-body experiment

Figs 8 and 9, respectively, show the runs of the coefficients \( a_1(t) \), \( a_2(t) \) and \( a_3(t) \) and the rms displacements \( \Delta r_{\text{rms}} \) and \( \Delta v_{\text{rms}} \) in the experiment with 10 000 bodies represented in Fig. 7. The suppression of chaotic behaviour as the number of bodies increases is evident in the comparison of these results with those in Figs 3 and 4.

In an experiment with a system of as many as 10 000 bodies, we can begin to explore the structure of a perturbation. In the present work, a calculation of the hydrodynamical Lagrangian displacement \( \xi(r,t) \) provides an example. In order to estimate \( \xi(r,t) \) at a given time \( t \) consistently with the definition in equation (15), we bin the complexity of particles in the unperturbed system at that time in equal mass shells. For the particles in each bin, we evaluate the
average values of \( r_a \) and \( D_r a \). The average value of \( D_r a \) in each bin is our estimate of the value of \( \xi(r, t) \) at \( r \) equal to the average value of \( r_a \) in that bin.

We have run this calculation in segments, and we have stored a check file containing the unperturbed coordinates and velocities, and the components of the Lagrangian displacements at the end of each segment. Each check file thus contains the data required for the calculation of \( y(r, t) \) at the end of a segment as described above. We have binned the particles in 10 mass shells in the calculations of \( y(r, t) \) described here.

The run of \( y(r, t) \) in the 10 000-body system is illustrated in Fig. 10. In each of these figures, the dependence of \( y \) on \( r \) is shown at different phases of the oscillation of \( a_1(t) \) shown in Fig. 8. Here the phase is defined to be \( 2\pi t/P \) where \( P \) is the period of the fundamental mode. From the run of \( a_1(t) \), the period \( P \) is estimated in the present experiments to be equal to \( 7.178(\pi G\rho_0)^{-1/2} \). To within fluctuations of the order of \( N^{-1/2} \), this experimental value agrees with the prediction \( P = 2\pi/n = 7.13596 \ldots (\pi G\rho_0)^{-1/2} \) based on equation (42).

The values of \( \xi(r, t) \) derived from the complexion of particles as described above are represented by filled circles in Fig. 10. The solid lines there are derived from equation (32), the representation of \( \xi(r, t) \) that underlies the virial estimate of the frequency of the mode. In applying this ‘theoretical’ representation of \( \xi(r, t) \), we have set \( A = 1 \), consistently with the assignment of the initial Lagrangian displacements for the run, and we have derived the value of the frequency \( n \) from the experimental value of the period \( P \). The value of \( n t \) is, of course, the phase at which \( \xi(r, t) \) is to be evaluated in each case. Otherwise, the virial representations of \( \xi(r, t) \) are unconstrained by the results of the \( N \)-body calculation.

The spatial structure and oscillation of \( y(r, t) \) derived here from the 10 000-body experiment and illustrated in Fig. 10 agree quite well with equation (32). The hydrodynamical displacement is strictly linear in \( r \), as represented in equation (32), only for the fundamental mode in a homogeneous sphere. In centrally concentrated configurations, one would expect to find curvature in the run.
of $\xi(r, t)$. However, the curvature in the experimental runs of $\xi(r, t)$ with $r$ illustrated in Fig. 10 appears to be very slight.

Inasmuch as our initial conditions for the fundamental mode of radial pulsation are only approximate in this experiment, the Lagrangian displacements ($\Delta r_{\alpha}, \Delta v_{\alpha}$) would be expected to describe a superposition of the fundamental mode and other radial modes. The perturbations other than the fundamental mode would be most conspicuous in the run of $\xi(r, t)$ with $r$ when the phase of the oscillation is equal to $\pm \pi/2 \mod 2\pi$ and the contribution of the fundamental mode to $\xi(r, t)$ vanishes. From the plots of $\xi(r, t)$ in Fig. 10 when the phase is equal to 2.51 radians and 15.60 radians, it appears that the contributions of higher modes to $\xi(r, t)$ are insignificant.

The results described in the preceding two paragraphs indicate that the virial representation of the fundamental mode is relatively accurate in the present application. This is not entirely unexpected, because the central concentration of a polytrope of index $n = 1$ is very modest.

It is an encouraging measure of the power of the present $N$-body representation of the fundamental mode of radial pulsation in a polytropic sphere that a consistent representation of $\xi(r, t)$ is obtained over 10 periods of the mode.

6 EXCITATION OF MODES WITH EXTERNAL PERTURBATIONS

In the preceding examples, we have imposed initial conditions on the Lagrangian displacements with the aid of a theory of the fundamental mode. It was an advantage to do so, because we could then compare the behaviour of the $N$-body systems with well-established expectations. The results provided unambiguous tests of the $N$-body methods and of the procedures and codes with which these methods have been implemented in this investigation. More generally, however, we cannot always expect to know the initial conditions for perturbations that we might wish to study. How, nevertheless, might we excite the perturbations of interest?

An alternative strategy for the excitation of a mode in the present calculations is to apply a suitable external perturbation to a system in which the Lagrangian displacements ($\Delta r_{\alpha}, \Delta v_{\alpha}$) of the particles initially all vanish. In the application of this strategy to the excitation of a mode of radial oscillation, we integrate equations (22), (23) and (26) as described in Sections 3.3 and 3.4, but we replace equation (24) with a representation of $\xi(r_{\alpha}, t)$ which includes the contribution of the external perturbation.

6.1 Excitation of the fundamental mode

In an experiment devised in order to excite the fundamental mode, we have replaced equation (24) with

$$\xi(r_{\alpha}, t) = \sum_i a_i(t) \psi_i(r_{\alpha}) + a_{Ext}(t) \psi_i(r_{\alpha}).$$  \(45\)

Here the term $a_{Ext}(t) \psi_i(r_{\alpha})$ represents the external perturbation. The function $\psi_i(r_{\alpha})$ imposes a spatial structure consistent with our expectations for the mode. The coefficient $a_{Ext}(t)$ is a ‘pulse’ which lasts for a finite interval of time and subsequently vanishes. The duration of the pulse is chosen to be comparable with the period that might be expected for the mode.

Figs 11 and 12 provide an illustration of this strategy for a system of 1000 bodies (200 orbits with five bodies on each). The initial conditions on the unperturbed coordinates and velocities are the same as in the experiments represented in Figs 3 and 4. The pulse on the right-hand side of equation (45) is given by

$$a_{Ext}(t) = \begin{cases} a_p [1 - \cos(2\pi T_p/t)] & (0 \leq t \leq T_p) \\ 0 & (t > T_p), \end{cases}$$  \(46\)

where the constants $a_p$ and $T_p$ are the amplitude and duration of the pulse, respectively. The duration $T_p$ of the pulse in the experiment illustrated in Figs 11 and 12 is 8 time units, a duration slightly longer than the theoretical period of 7.136 time units for the mode.

In the experiment illustrated in Figs 11 and 12, we have indeed excited the mode that was prepared with the aid of initial conditions in the experiment illustrated in Figs 3 and 4. The period of the mode in the present experiment is found to be $P = 7.046(\pi G \rho_0)^{-1/2}$, a value in excellent agreement with the result of the earlier experiment. The amplitudes of the mode in the two experiments cannot be

Figure 11. Variations of the coefficients $a_i (i = 1, 2, 3)$ with time in an $N$-body realization of a radial oscillation of a polytropic sphere of index $n = 1$ consisting of 1000 stars (200 orbits with five stars distributed uniformly around each orbit). An external perturbation was imposed, as described in the text, in order to excite the fundamental mode of radial pulsation.

Figure 12. Growth of the root-mean-square values $\Delta r_{\text{rms}}$ and $\Delta v_{\text{rms}}$ of the components of the Lagrangian displacements in the $N$-body calculation illustrated in Fig. 11.
6.2 An attempt to excite a higher mode

The final experiment in this investigation is an attempt to excite a mode of radial oscillation higher than the fundamental mode. Again, the strategy is to apply a suitable external perturbation to a system in which the Lagrangian displacements \((\Delta r_a, \Delta v_a)\) of the particles initially all vanish. Our expectation is that the hydrodynamical Lagrangian displacement for the lowest of the higher modes will have one node between the centre and boundary of the configuration. In other words, the spatial structure of \(\xi(r, t)\) should be qualitatively similar to the orthogonal function \(\psi_2(r)\). Accordingly, we now integrate equations (22), (23) and (26) as described in Sections 3.3 and 3.4, but we replace equation (24) with

\[
\xi(r, t) = \sum a_i(t)\psi_i(r_a) + a_{Ext}(t)\psi_2(r_a)
\] (47)

This experiment has been performed on the system of 10000 bodies (2000 orbits with five bodies on each orbit) described in Section 5.2. The pulse \(a_{Ext}(t)\) in this experiment is also given by equation (46), but the duration \(T_p\) is now chosen to be 2 time units.

The runs of the coefficients \(a_1(t), a_2(t)\) and \(a_3(t)\) are shown in Fig. 13. The coefficient \(a_2(t)\) oscillates with a period approximately equal to \(2.44(\pi G P_0)^{-1/2}\). On the other hand, \(a_1(t)\) and \(a_3(t)\) oscillate irregularly at a level not significantly above noise of the order of \(N^{-1/2}\). The amplitude of the oscillation of \(a_2(t)\) waxes and wanes on a timescale of the order of 10–12 time units. The source of this modulation of the oscillation of \(a_2(t)\) has not been identified.

We have calculated the hydrodynamical Lagrangian displacement of this perturbation in the manner described in Section 5.2. Fig. 14 presents a comparison of the tabulated displacements with the orthogonal function \(\psi_2(r)\), multiplied by an amplitude of suitable magnitude and algebraic sign, to the Lagrangian displacement.

These results provide some basis for an identification of the oscillation, at least tentatively, as a second mode of radial oscillation. This cannot be regarded as a firm identification until the source of the modulation of the oscillation in Fig. 13 is established. The investigation of this issue continues.

7 DISCUSSION

The example of an \(N\)-body method for the study of small perturbations in galaxies that is described in this paper successfully represents radial oscillations in models of spherically symmetric systems. In particular, the \(N\)-body representation of the fundamental mode of radial pulsation reproduces exact theoretical results for that mode in homogeneous spheres. For polytropic configurations of index \(n = 1\), the present calculations also confirm an estimate of the period of the fundamental mode derived with the aid of the tensor virial equations (Vandervoort 1983). Moreover the approximately linear dependence of the hydrodynamical Lagrangian displacement on the radial coordinate in these calculations is consistent with the representation of the perturbation that underlies the virial estimate of the period [see equation (32) as well as equations (29) and (30)].

7.1 Spurious behaviour

In the present investigation we replace one dynamical system, the system described by the Lagrangian perturbation equations, with another dynamical system, the system described by the adopted \(N\)-body equations. We seek to recognize in solutions of the latter equations behaviour which is common to the two systems. Thus, it is important to ignore (or suppress) behaviour in the \(N\)-body representation which is not characteristic of the theory of small perturbations. Likewise, it is important to know when an \(N\)-body representation fails to exhibit behaviour that would be characteristic of a collisionless system described by the analytic theory.

The present investigation leads to two examples of behaviour in the \(N\)-body representation of perturbations which are spurious.

![Figure 13](https://example.com/fig13.png)

**Figure 13.** Variations of the coefficients \(a_i(t = 1, 2, 3)\) with time in an \(N\)-body realization of a radial oscillation of a polytropic sphere of index \(n = 1\) consisting of 10000 stars (2000 orbits with five stars distributed uniformly around each orbit). An external perturbation was imposed, as described in the text, in an attempt to excite a higher mode of radial pulsation.

![Figure 14](https://example.com/fig14.png)

**Figure 14.** Dependence of the hydrodynamical Lagrangian displacement \(\xi(r, t)\) on the radial coordinate \(r\) at two phases of the oscillation of the polytropic sphere of index \(n = 1\) excited in the experiment represented in Fig. 13. Dashed curves represent fits of the orthogonal function \(\psi_2(r)\), multiplied by an amplitude of suitable magnitude and algebraic sign, to the Lagrangian displacement.

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within the framework of linear perturbation theory. These are parametric resonance and chaotic behaviour.

As explained in Appendix A, degeneracy or near degeneracy of the orbital frequencies with the frequency of the fundamental mode of radial pulsation is the source of parametric resonance in the present experiments. The simple harmonic motions of the stars make this degeneracy very strong in the homogeneous spheres considered. Even in the polytropic spheres of index $n = 1$, the anharmonic components of the stellar motions are too weak to lift this degeneracy significantly. We have suppressed parametric resonance in these calculations by suitably distributing stars around the unperturbed orbits. If the orbital motions were highly anharmonic, as they should be in more centrally concentrated systems, then the degeneracy of the orbital motions with the mode should be lifted and the effects of parametric resonance suppressed independently of the way in which we populate the unperturbed orbits with stars. In fact, the manifestations of parametric resonance in the $N$-body representation of the fundamental mode persist in the more centrally concentrated configurations described below.

The fundamental mode of radial pulsation in spheres is a special case, and it seems doubtful that parametric resonance would be a generic example of spurious behaviour in calculations of the kind considered here. In particular, one would not expect parametric resonance to appear in the $N$-body representations of higher modes of radial oscillation in spheres, in non-radial modes in spheres, or in modes in nonspherical systems. Tests of this expectation remain to be made in future investigations. In cases in which parametric resonance does not arise, the population of each orbit with a number of stars is a measure which can be omitted for the sake of economy.

On the other hand, the analysis in Section 5.1 suggests, in contrast, that chaotic behaviour is a generic example of spurious behaviour in an $N$-body representation of Lagrangian perturbation theory. Such behaviour can be suppressed in these calculations, relative to the oscillations under study, either by increasing the number of bodies included in the calculations or by dropping higher terms in the superposition of basis functions used in the representation of the Eulerian perturbation of the acceleration. The benefit of this connection of increasing the number of bodies is not entirely surprising. However, truncating the right-hand side of equation (24) from three terms to one is a more economical method for suppressing chaotic behaviour in these calculations.

Because the unperturbed configurations are homogeneous or nearly so in the present experiments, the hydrodynamical Lagrangian displacement is linear in $r$ or nearly so for the fundamental mode of radial pulsation. Therefore, the polynomials $\psi_2(r)$ and $\psi_3(r)$ make relatively insignificant contributions to this perturbation. The coefficients of $\psi_2(r)$ and $\psi_3(r)$ in equation (24) do introduce fluctuations in the present calculations, and these fluctuations are a source of chaotic behaviour. This experience suggests that, as a general practice in such calculations, the representation of two-body interactions should be just general and flexible enough to provide a valid and accurate representation of the Eulerian perturbation of the acceleration. Terms which prove to be superfluous in this representation should be omitted in order to suppress chaotic behaviour.

Finally, a calculation that is free of chaotic behaviour can be performed, as explained in Section 5.1, by adopting an analytic model of the coefficients $a_i(t)$ in equation (24) and requiring that the model be consistent with the runs of $a_i(t)$ derived from the complexion of the Lagrangian displacements in the calculation with the aid of equation (25).

### 7.2 Generalizations

The example of an $N$-body method that is formulated in this paper and the applications that are described here are rather special. In this and the following subsections, we consider the potential of such methods for wider applications and the limitations on such methods that might be encountered in wider applications. We also compare $N$-body methods in the Lagrangian and Eulerian representations. It should be emphasized that most of what is known about $N$-body representations of Lagrangian perturbation theory is based on the calculations described in this paper. Therefore, any discussion of generalizations and wider applications involves elements of conjecture and speculation.

The organizing principle of the present investigation is that $N$-body methods are to be adapted to the study of problems in the analytic theory of small perturbations in stellar systems. Accordingly, a system of ‘$N$-body equations’ is introduced only after the equations of the theory have been reduced as much as possible for the intended application. Thus, although equations (4)–(6) are generic, the equations appropriate to the present investigation of radial oscillations in spherically symmetric systems are equations (22)–(26). Calculations with the aid of the adopted equations should be regarded not as conventional $N$-body calculations but as $N$-body representations of the underlying analytic theory.

Two questions arise concerning the potential of such $N$-body representations of Lagrangian perturbation theory for wider applications. Can other modes be represented with the aid of $N$-body methods in formulations of the particular kind presented here? And can more general versions of such $N$-body methods be formulated for less specific applications?

In work begun after the submission of the earlier version of this paper, the author is extending the present calculations of the fundamental mode of radial pulsation to more centrally concentrated spheres. In systems more centrally concentrated than the systems considered in this paper, the observed frequency of the fundamental mode of radial oscillations in spherically symmetric systems are equations (22)–(26). Calculations with the aid of the adopted equations should be regarded not as conventional $N$-body calculations but as $N$-body representations of the underlying analytic theory.

The agreement of the virial representation of the mode with the $N$-body representation does diminish with increasing central concentration. In a configuration in which the central density is 45 times the mean density, for example, the observed frequency of the mode in an $N$-body realization is about 20 per cent less than the virial estimate of the frequency. Moreover, the coefficient $a_2(t)$ oscillates with a relatively large amplitude in such cases, the basis function $\psi_2(r)$ is now a significant component of the hydrodynamical Lagrangian displacement $\xi(r, t)$, and $\xi(r, t)$ is no longer a linear function of $r$, even approximately.

In preparation for a generalization of the present $N$-body methods for the representation of non-radial modes in spheres, the author has carried out a separation of the Lagrangian perturbation equations in spherical polar coordinates. The analysis is based on an earlier separation of the Eulerian perturbation equations in spherical polar coordinates by Polyachenko (1987; see also Chapter 3 and Appendix 2 in Fridman & Polyachenko 1984). In both cases, the perturbation equations separate into three angles, the usual polar and azimuthal angles in the configuration space and an azimuthal angle (measured around the radial direction) in the velocity space. This development makes possible the formulation of an $N$-body realization of the solution of the separated perturbation equations along lines described in this paper. Again, the generic equations...
(1)–(3) are replaced with a highly reduced system of perturbation equations before the introduction of an \( N \)-body representation. Elimination of the three angular coordinates reduces the phase space of the problem from six dimensions to three and concentrates the power of the \( N \)-body calculation on these three non-trivial dimensions.

In such applications, the smooth field representation appears to be the natural choice for the calculation of the Eulerian perturbation of the acceleration. Where such applications are to be made in conjunction with analytic solutions of the perturbation equations, the representation of the Eulerian perturbation in terms of a set of basis functions might be adopted as a representation common to both the \( N \)-body and the analytic formulations. With a complete set of basis functions, the representation is exact. In the study of particular modes, for which this approach is designed, and with an appropriate choice of the basis functions, it should be possible to produce accurate representations of the Eulerian perturbations of the acceleration with relatively small numbers of basis functions. In general, as in the experiments described in Section 5.1, it should be advantageous for the suppression of chaotic behaviour to use sets of basis functions just large enough for accurate representations of the Eulerian perturbations of the acceleration.

In wider applications, alternative representations of the Eulerian perturbation of the gravitational acceleration might be preferred. A very general representation would be required, for example, in exploratory studies such as general tests for the stability of a system. The use of the smooth field representation with a small set of basis functions could be too restrictive for such explorations. In such extreme cases, one might resort to a direct integration of equations (4)–(6) as a conventional \( N \)-body calculation with the two-body interactions on the right-hand side of equation (5) softened appropriately. In situations intermediate between those requiring a direct integration of equations (4)–(6) and those in which an integration of a specialized set of equations such as equations (22)–(26) would suffice, one could adopt a version of any of the methods (e.g. the methods described by Sellwood 1987) that have been applied to full \( N \)-body systems. In principle, any \( N \)-body method that can be formulated for a full \( N \)-body system could be formulated for a realization of Lagrangian perturbation theory.

It is possible, however, that a direct integration of equations (4)–(6) would be an unsatisfactory procedure even if the two-body interactions on the right-hand side of equation (5) were softened. The two-body interactions involve two derivatives of the kernel \( |x_3 - x_n|^{-1} \), so the interaction of two bodies at a small separation varies as the inverse cube of the separation. One might expect that the strong short-range character of this interaction would make the integration of a softened version of equations (4)–(6) more sensitive to the length of the softening parameter than conventional \( N \)-body calculations. Moreover, one must be aware of the possibility that manifestations of chaotic behaviour might be more severe in a direct integration of equations (4)–(6) than they have been found to be in the present integrations of equations (22)–(26). Indeed, the sensitivity of the Lagrangian displacements to chaotic behaviour may set the practical limits to the general use of \( N \)-body representations of Lagrangian perturbation theory.

### 7.3 Comparison of Lagrangian and Eulerian schemes

The Lagrangian and Eulerian perturbation equations represent two physically equivalent but complementary approaches to the study of small perturbations in stellar systems. Whereas the Lagrangian equations describe the evolution of the system in terms of the perturbations of the locations of stars in the phase space, the Eulerian equations describe this evolution in terms of the perturbations of the density of stars in the phase space. In \( N \)-body methods of both kinds, in their linear formulations, the unperturbed orbits of the stars are integrated as \( N \) one-body problems in the unperturbed gravitational field of the system, whereas the perturbation equations are integrated as single \( N \)-body problems. In an Eulerian scheme, the perturbations considered are perturbations of the masses contained in elements of volume in the phase space. For this reason, Eulerian schemes are called ‘perturbation particle methods’ by their authors (Leeuwin et al. 1993; Wachlin et al. 1993).

The choices to be made and the precautions to be observed in \( N \)-body calculations describing small perturbations in the Lagrangian and Eulerian representations and in conventional \( N \)-body calculations are generally similar. The similarities do not require review here, inasmuch as they concern matters that are well known (see e.g. Sellwood 1987). However, there are important respects in which Lagrangian and Eulerian schemes are to be contrasted.

In the published accounts of the Eulerian approach (Leeuwin et al. 1993; Wachlin et al. 1993) it is emphasized that regions of the phase space initially empty of perturbation particles will generally acquire positive or negative mass perturbations in the course of the evolution of a system. This development must be anticipated in the formulation of an Eulerian method. Thus, the choice of the sampling density in the Monte Carlo scheme described by Leeuwin et al. (1993) for the calculation of the perturbation forces must cover regions of phase space that include substantial regions which are initially empty. The Eulerian scheme described by Wachlin et al. (1993) requires that the calculation follows all of the volume elements in phase space that will eventually be occupied by the (positive or negative) phase density of the perturbation. There are elements of uncertainty and arbitrariness in the procedures described by these authors for dealing with these problems, and the authors warn that this is an aspect of an Eulerian scheme that requires vigilance. A Lagrangian scheme does not suffer such a complication, because the focus of the scheme is on the trajectories of the particles and their Lagrangian displacements. A Lagrangian scheme accounts automatically for the displacement of particles into regions of phase space that are initially empty.

According to the (unverified) conjecture at the end of the preceding subsection, a direct \( N \)-body integration of the Lagrangian perturbation equations may prove to be very sensitive to the strong short-range character of the two-body terms in the representation of the Eulerian perturbation of the gravitational acceleration. In contrast, it appears that the softening of the two-body interactions can be done as effectively in an \( N \)-body integration of the Eulerian perturbation equations as in a full \( N \)-body calculation. In this respect, the advantage may lie with the Eulerian formulation in the most general applications.

The consequences of the chaotic behaviour of \( N \)-body systems must be considered in the case of an Eulerian scheme as in the case of a Lagrangian scheme. As explained by Leeuwin et al. (1993), large errors in the computed distribution function should accumulate rapidly in an \( N \)-body representation of a collisionless stellar system. Leeuwin et al. argue on the basis of an investigation by Quinlan & Tremaine (1991) that, notwithstanding the accumulation of errors in a numerically integrated trajectory of a system in phase space, this trajectory is ‘shadowed’ by the exact trajectory of a system with a slightly different Hamiltonian. Accordingly, they suggest that a numerical \( N \)-body representation of a perturbation in a collisionless stellar system will be a valid physical representation, albeit a representation that applies to a system slightly different
from the system envisaged. Leeuwin et al. also argue that the accumulated errors in the distribution function mainly affect the fine structure in the distribution function and that the moments of the distribution function that are of principal interest are insensitive to such fine structure. These arguments are reinforced by experience that N-body calculations give consistent and reproducible representations of stellar systems under modifications of integration schemes, representations of the gravitational forces, initial conditions, etc. Thus it appears that the consequences of chaos are 'hidden' in an Eulerian scheme. In contrast, the sensitivity of the Lagrangian displacements to chaotic behaviour makes the consequences of chaos more visible in a Lagrangian scheme.

The foregoing comparisons refer to N-body formulations of linear treatments of small perturbations. When it is of interest to follow unstable perturbations to large amplitudes, Eulerian formulations accommodate the required non-linear treatments very well (Leeuwin et al. 1993; Wachlin et al. 1993). A non-linear version of the Lagrangian perturbation equations can be written, but these equations seem only to be equivalent to a comparison of two conventional N-body calculations with slightly different initial conditions. This is what would be expected in a scheme that focuses on the particle trajectories. The emphasis in the present investigation is on N-body representations of the linear theory of perturbations. If interesting perturbations grow, then the unper- turbations, coordinates and velocities and their Lagrangian displace- ments accommodate the required non-linear treatments very well (Leeuwin et al. 1993; Wachlin et al. 1993). A non-linear version of the Lagrangian perturbation equations can be written, but these equations seem only to be equivalent to a comparison of two conventional N-body calculations with slightly different initial conditions. This is what would be expected in a scheme that focuses on the particle trajectories. The emphasis in the present investigation is on N-body representations of the linear theory of perturbations. If interesting perturbations grow, then the unper-

7.4 Concluding remarks

The results of the present investigation indicate that a Lagrangian formulation of N-body methods for the representation of small perturbations in stellar systems provides a simple but powerful basis for future studies of the oscillations and the stability of stellar systems. The question will naturally arise as to whether Eulerian or Lagrangian schemes are preferable. Notwithstanding their physical equivalence, the Lagrangian and Eulerian schemes do appear to have different strengths and weaknesses in practical applications. Both Eulerian and Lagrangian methods have now been made to work on simple problems chosen for the purpose of investigating and illustrating the methods. Therefore, a preference for one approach or the other is largely a matter of taste at the present time. Eventually, preferences may be based on the success of these methods in the solution of more challenging problems.

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APPENDIX A: PARAMETRIC RESONANCE IN THE RADIAL PULSATION OF A HOMOGENEOUS SPHERE

An example of parametric resonance in an N-body calculation of the kind described in this paper can be constructed with the aid of the exact solution of a truncated form of equations (22), (23) and (26) for the fundamental mode of radial pulsation in a homogeneous sphere. According to equation (32), $\xi(r, t)$ is a linear function of $r$ when a homogeneous sphere oscillates in the fundamental mode. Thus the fundamental mode would be a solution of equations (13), (16), (17) and (19) in the smooth-field representation even if the right-hand side of equation (17) were truncated to a single term involving the linear function $\psi_1(r)$.

Accordingly, we consider a model of the N-body system described by a solution of equations (22)–(26) in which we truncate the right-hand side of equation (24) to a single term involving the lowest orthogonal function

$$\psi_1(r) = \left(\frac{4\pi}{3} \rho_0 R^3\right)^{-1/2} \left(\frac{5}{3}\right)^{1/2} \frac{r}{R}$$  \hspace{1cm} (A1)

[see equation (35) and Table 1]. With the aid of equation (24), truncated to the single term in $\psi_1(r)$, and equation (A1), we reduce equation (23) to

$$\frac{d\Delta v_a}{dr} = -\frac{\partial}{\partial r_a} \left[ \frac{GM_0(r_a)}{r_a} - \frac{L_a^2}{r_a^3} \right] \Delta r_a + 4\pi G\rho_0 \left(\frac{5}{4\pi\rho_0 R^3}\right)^{1/2} a_1(t) r_a,$$  \hspace{1cm} (A2)

where $\alpha = 1, 2, \ldots, N$. We are able to evaluate $M_0(r)$ with the aid of equation (27), and $a_1(t)$ is to be determined from the complexion of particles in the system in accordance with equation (25).

Equations (22) and (A2) admit a solution of the form

$$\Delta r_a = \alpha_a + b v_a,$$  \hspace{1cm} (A3)

and

$$\Delta v_a = \left(\frac{da}{dr} - n^2 b\right) r_a + \left(\frac{db}{dr} + a\right) v_a + b \frac{L_a^2}{r_a^3}.$$  \hspace{1cm} (A4)

where, in virtue of equations (26), the functions $a(t)$ and $b(t)$ must satisfy the pair of ordinary differential equations

$$\frac{d^2 a}{dr^2} - 2n^2 \frac{db}{dr} = 4\pi G\rho_0 \left(\frac{5}{4\pi\rho_0 R^3}\right)^{1/2} a_1(t)$$  \hspace{1cm} (A5)

and

$$\frac{db}{dr} + 2a = 0.$$  \hspace{1cm} (A6)
Before we can solve these equations, we must evaluate the coefficient \(a_1(t)\) on the right-hand side of equation (A5) in accordance with equation (25). For that purpose, we must determine the unperturbed coordinates and velocities \((r_\alpha, v_\alpha)\) as solutions of equations (26) with the unperturbed gravitational acceleration given by equation (27). The solution for the unperturbed coordinates and velocities must satisfy the initial conditions of the \(N\)-body calculation. In other words, the initial values \(r_{\alpha 0}, v_{\alpha 0}\), and \(L_{\alpha 0}\) of \(r_\alpha, v_\alpha\), and \(L_\alpha\), respectively, where \(\alpha = 1, 2, \ldots, N\), are to be determined with the aid of the appropriate Monte Carlo process. (Bear in mind here that \(L_\alpha = L_{\alpha 0}\) in virtue of the conservation of angular momentum.) Inasmuch as the unperturbed stellar motions are simple harmonic oscillations, the solutions for \(r_\alpha\) and \(v_\alpha\) can be written exactly and in closed form. For the sake of brevity, we shall not write the solutions explicitly in this paper.

In order to evaluate \(a_1(t)\), we substitute from equation (A1) for \(\psi_1(r)\) and from equation (A3) for \(\Delta r_\alpha\) on the right-hand side of equation (25) and then make use of our solutions for \(r_\alpha(t)\) and \(v_\alpha(t)\) in the resulting expression. After some reduction of that result, we find that the right-hand side of equation (A5) is

\[
4\pi G \rho_0 \left( \frac{5}{4\pi \rho_0 R^2} \right)^{1/2} a_1(t)
\]

\[
= \frac{5Gm}{2R^2} \sum_{n=1}^{N} \left[ r_{\alpha 0}^2 + n^{-2} \left( v_{\alpha 0}^2 + \frac{L_{\alpha 0}^2}{r_{\alpha 0}^2} \right) \right] a 
\]

\[
+ \frac{5Gm}{2R^2} \sum_{n=1}^{N} \left[ r_{\alpha 0}^2 - n^{-2} \left( v_{\alpha 0}^2 + \frac{L_{\alpha 0}^2}{r_{\alpha 0}^2} \right) \right] a \cos(2nt)
\]

\[
+ \frac{5Gm}{2R^2} \sum_{n=1}^{N} n^{-1} r_{\alpha 0} v_{\alpha 0} a \sin(2nt)
\]

\[
- \frac{5Gm}{2R^2} \sum_{n=1}^{N} \left[ r_{\alpha 0}^2 - n^{-2} \left( v_{\alpha 0}^2 + \frac{L_{\alpha 0}^2}{r_{\alpha 0}^2} \right) \right] nb \sin(2nt)
\]

\[
+ \frac{5Gm}{2R^2} \sum_{n=1}^{N} n^{-1} r_{\alpha 0} v_{\alpha 0} nb \cos(2nt),
\]

(A7)

where, for the sake of simplicity, we let all of the stars have the same mass \(m_\beta = m\).

We can write the right-hand side of equation (A7) in a more compact form by introducting constants \(\Delta \alpha, \epsilon, \phi\) such that

\[
\frac{5Gm}{2R^2} \sum_{n=1}^{N} \left[ r_{\alpha 0}^2 + n^{-2} \left( v_{\alpha 0}^2 + \frac{L_{\alpha 0}^2}{r_{\alpha 0}^2} \right) \right] = 4n^2 - (n + \Delta \alpha)^2,
\]

(A8)

\[
\frac{5Gm}{2R^2} \sum_{n=1}^{N} \left[ r_{\alpha 0}^2 - n^{-2} \left( v_{\alpha 0}^2 + \frac{L_{\alpha 0}^2}{r_{\alpha 0}^2} \right) \right] = en^2 \cos(2\phi),
\]

(A9)

and

\[
\frac{5Gm}{2R^2} \sum_{n=1}^{N} n^{-1} r_{\alpha 0} v_{\alpha 0} = -en^2 \sin(2\phi).
\]

(A10)

Indeed, after rewriting the right-hand side of equation (A7) with the aid of equations (A8)–(A10), replacing the right-hand side of equation (A5) with the result, and eliminating \(dt/dt\) in equation (A5) with the aid of equation (A6), we obtain the equation

\[
\frac{d^2 a}{dt^2} + (n + \Delta \alpha)^2 a = en^2 \cos(2nt + 2\phi)a - en^3 \sin(2nt + 2\phi)b.
\]

(A11)

Upon solving equations (A6) and (A11) simultaneously for appropriate initial conditions, we complete the required solution of equations (22)–(26) for our truncated model of the \(N\)-body system.

It is useful to represent the solution of equations (22)–(26) in terms of \(\Delta \alpha, \epsilon, \phi\) in this way, because \(\Delta \alpha, \epsilon, \phi\) are quantities of the order of \(N^{-1/2}\) which characterize fluctuations in the \(N\)-body realization of the perturbation. In other words, we can suppress the fluctuations in equation (A11) by setting \(\Delta \alpha = 0\) and \(\epsilon = 0\) there. In this case, we can find a solution of equations (A6) and (A11) which makes equations (22) and (A2) for \(\Delta r_\alpha\) and \(\Delta v_\alpha\) consistent with the exact solution for the mode represented by equations (29) and (30). For this purpose, we must impose initial conditions on \(a(t)\) and \(b(t)\) such that \(\Delta r_\alpha, \Delta v_\alpha, \Delta r_{\alpha 0}\) for each particle, satisfy initial conditions of the form of equations (31). The required conditions are

\[
a = -\frac{2}{3n^3} A, \quad \frac{da}{dt} = 0 \quad \text{and} \quad b = 0 \quad \text{at} \quad t = 0,
\]

(A12)

and the corresponding solution for \(a(t)\) and \(b(t)\) is

\[
a = -\frac{2}{3n^3} A \cos(nt) \quad \text{and} \quad b = \frac{4}{3n^3} A \sin(nt).
\]

(A13)

Equations (A6) and (A11) are a system of ordinary differential equations of the third order describing an oscillator in a phase space of the variables \(da/dt, a,\) and \(b\). For an \(N\)-body realization of the mode (i.e. in the general case that \(\Delta \alpha \neq 0\) and \(\epsilon \neq 0\)), this system of equations is closely related to Mathieu’s equation. Indeed equation (A11) formally reduces to Mathieu’s equation if we set \(b = 0\). Parametric amplification occurs in an oscillator described by equations (A6) and (A11) much as it occurs in an oscillator described by Mathieu’s equation.

In Fig. A1, we compare an \(N\)-body calculation of the solution of equations (22) and (A2), in which \(a_1(t)\) is derived from equations (25) and (A1), with the corresponding solution derived from equations (A3) and (A4) and the solution of equations (A6) and (A11). The system contains \(N = 100\) particles. The initial conditions on the Lagrangian displacements of the stars are fixed in accordance with equations (31);
this requires that \( a(t) \) and \( b(t) \) satisfy equations (A12). Fig. A1 shows plots of \( a_1(t) \) against time. The solid curve shows the result of the N-body calculation whereas the open circles are derived from equations (A3) and (A4) and the solution of equations (A6) and (A11). Asymptotically, the growth of the perturbation illustrated in Fig. A1 is exponential.

Thus, the truncated system of equations solved in this appendix represents an exactly soluble model of the N-body calculation considered in this paper. The model shows clearly that fluctuations in the N-body representation of the unperturbed configuration can be a source of parametric resonance in the calculation. The comparison in Fig. A1 of the exact solution of the truncated N-body equations with the representation of the solution derived from the N-body calculation provides a useful check of the code used in these calculations.

In Fig. 1, it appears that the inclusion of terms in \( \psi_2(r) \) and \( \psi_3(r) \) in the representation of \( \xi(r, t) \) suppresses the exponential growth of the N-body realization of the mode. This is understandable within the framework of the present analysis, because the inclusion of the higher orthogonal functions would introduce terms in \( r_{\alpha}^2 \) and \( r_{\alpha}^3 \) in the Eulerian perturbation of the acceleration on the right-hand side of equation (A2). The additional terms would change the linear oscillator described by equations (A6) and (A11) to a non-linear oscillator. Effects of the resonance persist, nevertheless, in Fig. 1 in the form of modulations of the amplitude and phase of the oscillation of \( a_1(t) \).

We can suppress parametric resonance entirely in the N-body realization of the fundamental mode with the aid of suitable initial conditions on the unperturbed orbits. We begin by choosing initial sets of values of \( (r_\alpha, v_\alpha, L_\alpha) \) \( (\alpha = 1, 2, \ldots, N) \) for \( N \) particles as before. This complexion of initial conditions is a realization of the density of stars in the phase space of the unperturbed system. We now replace each particle \( \alpha \) with a set of particles distributed at equal intervals of phase around the trajectory in the phase plane of the radial motion which has the assigned value \( L_\alpha \) of the angular momentum and which passes through the initial point \( (r_\alpha, v_\alpha) \). We let \( \mu \) denote the number of particles distributed around each trajectory, and we rescale the mass per particle accordingly. This new complexion of initial conditions for \( \mu \) times the original number of stars is also a realization of the density of stars in the phase space of the unperturbed system.

We conclude this appendix by showing explicitly that this prescription for assigning initial conditions suppresses parametric resonance in the solution of the truncated system of equations. With the multiplication of the number of stars in the calculation, as described in the preceding paragraph, and with the modification of the initial conditions on the unperturbed orbits, we must modify the solution of equations (22) and (A2) represented in equations (A3)–(A7) by replacing equation (A7) with

\[
4\pi G \rho_0 \left( \frac{5}{4\pi \rho_0 R^2} \right)^{1/2} a_1(t) \\
= \frac{5 G m}{2 R^2} \sum_{b=1}^{N} n^{-1} \left[ r_{\beta_0}^2 - n^{-2} \left( v_{\beta_0}^2 + \frac{L_{\beta_0}^2}{r_{\beta_0}^3} \right) \right] a \\
+ \frac{5 G m}{2 R^2} \sum_{b=1}^{N} \sum_{p=0}^{\mu-1} n^{-1} \left[ r_{\beta_0}^2 - n^{-2} \left( v_{\beta_0}^2 + \frac{L_{\beta_0}^2}{r_{\beta_0}^3} \right) \right] \\
\times a \cos \left[ 2n \left( t + \frac{2\pi p}{n\mu} \right) \right] \\
+ \frac{5 G m}{2 R^2} \sum_{b=1}^{N} \sum_{p=0}^{\mu-1} n^{-1} r_{\beta_0} v_{\beta_0} a \sin \left[ 2n \left( t + \frac{2\pi p}{n\mu} \right) \right] \\
- \frac{5 G m}{2 R^2} \sum_{b=1}^{N} \sum_{p=0}^{\mu-1} n^{-1} r_{\beta_0} v_{\beta_0} n b \cos \left[ 2n \left( t + \frac{2\pi p}{n\mu} \right) \right].
\]  

(A14)

The sums of trigonometric terms on the right-hand side of equation (A14) vanish in virtue of the identity

\[
\sum_{p=0}^{\mu-1} \exp \left( \frac{4\pi i p R^2}{\mu} \right) = 0.
\]  

(A15)

We are left with

\[
4\pi G \rho_0 \left( \frac{5}{4\pi \rho_0 R^2} \right)^{1/2} a_1(t) \\
= \frac{5 G m}{2 R^2} \sum_{b=1}^{N} \left[ r_{\beta_0}^2 - n^{-2} \left( v_{\beta_0}^2 + \frac{L_{\beta_0}^2}{r_{\beta_0}^3} \right) \right] a \\
= [4n^2 - (n + \Delta n)^2] a,
\]  

(A16)

say, where we are now redefining \( \Delta n \) consistently with the modified prescription for assigning initial conditions and the new counting of the orbits and of the stars populating each orbit. We must accordingly replace equation (A11) with

\[
\frac{d^2 a}{dt^2} + (n + \Delta n)^2 a = 0.
\]  

(A17)

Thus the modified initial conditions suppress parametric resonance in the solution of equations (22) and (A2), but there remains a frequency shift \( \Delta n \) resulting from fluctuations in the N-body realization of the system.

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