Global vortex systems on standard accretion disc surfaces

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ABSTRACT
The hydrodynamical global stability of turbulent density-stratified and differentially rotating discs is studied. In the context of the mean-field approach, the small-scale turbulence produces the anisotropic kinetic alpha (AKA) effect, which is shown here to generate global vortices. Different rotation laws and aspect ratios, in particular those of standard accretion Kepler discs and galaxies, are considered. Linear stability analysis yields the AKA effect leading to the self-excitation of large-scale flow patterns. The preferred excitations are non-axisymmetric with an azimuthal wavenumber \( m = 1 \) and antisymmetric with respect to the disc mid-plane. Therefore, there are both hot and cold spots on each surface of the disc. The vertical flow always crosses the equator so that there is no concentration of dense material.

Key words: accretion, accretion discs – instabilities – planetary systems.

1 MOTIVATION
The present study is motivated by a paper of Abramowicz et al.'s (1992) reporting the existence of X-ray-active coherent structures at the surface of an accretion disc. In particular, large-scale vortices have been proposed as the source of short-term flickering of cataclysmic variables and the short-term variability of X-ray binaries (Bath, Evans & Papaloizou 1974).

Vortices, indeed, would produce time-dependent effects via rotational modulation if the observation were neither edge-on nor pole-on. The reason for this is that they are basically non-axisymmetric patterns. It is known that non-axisymmetric phenomena always need a particular excitation mechanism in order to be stationary, e.g. a non-axisymmetric magnetic field cannot survive against the smoothing action of differential rotation (Krause & Rädler 1980). Also, the arms of spiral galaxies would quickly be sheared by the differential rotation if there were no physical excitation like density waves.

The same could hold for maintaining large-scale vortices. Abramowicz et al. (1992) have suggested that non-axisymmetric structures can exist in discs in spite of their strong differential rotation.

The long-living vortices may also be important for planet formation. The balance between the centrifugal force and the pressure force in the vortical flow does not apply to planetesimals because of their relatively large density compared with the surrounding gas. The resulting planetesimal concentration in the central regions of vortices accelerates planet formation (Barge & Sommeria 1995; Tanga et al. 1996; Klahr & Henning 1997).

The generation of self-gravitating vortices in circumstellar discs in a local approximation has already been studied by Adams & Watkins (1995) considering an infinitely thin disc in a laminar regime. The present paper studies the alternative possibility, i.e. that a coherent global vortex system in an accretion disc is supported by turbulence. As known, the turbulent anisotropic kinetic alpha (AKA) effect is suspected to cause an instability of rotating, stratified turbulent fluids to global seed flows (cf. Frisch, She & Sulem 1987; Kitchatinov, Rüdiger & Khomenko 1994), similar to the hydromagnetic dynamo instability for large-scale magnetic seed fields. The expectation of general vortex generation by the AKA effect is suggested by recent simulations of v. Rekowski & Kitchatinov (1998), who considered spheres in uniform rotation. However, these results are not directly applicable to accretion discs with Kepler rotation. We have to check whether shearing by the differential rotation will preclude the excitation of non-axisymmetric vortices in the discs.

We shall see in the following that in spite of the shearing effect, non-axisymmetric flow patterns are always preferred in excitation. This conclusion follows from a series of accretion disc models for various values of the viscosity alpha. All of them favour patterns with the zonal wavenumber \( m = 1 \), which are antisymmetric with respect to the disc mid-plane. Each of the patterns will produce one hot and one cool spot on the disc surface, owing to the vertical mixing by their meridional flow.

2 BASIC EQUATIONS
We study the turbulent flow perturbations of a differentially rotating disc. For the turbulent flow we use the mean-field approach, \( \mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}' \), where \( \bar{\mathbf{u}} \) is the mean velocity and \( \mathbf{u}' \) describes the fluctuations. We then write the equation for the mean flow, which is assumed to be small compared with the unperturbed background differential rotation. Thus, we start from the linearized mean-field
momentum equation for a turbulent stratified rotating fluid,
\[
\frac{\partial \hat{m}_i}{\partial t} + \nabla (\hat{V} \hat{m}_i + \hat{m} \hat{V}) + \frac{\partial \bar{p}}{\partial x_i} = -\frac{\partial}{\partial x_j} \left[ \nu_{ij} \nabla \hat{m}_j + \hat{m}_j \nabla \hat{m}_i - G \hat{m}_j \hat{m}_i \right] - \frac{\partial}{\partial x_j} \left( \Gamma_{ij} \hat{m}_j \right)
\]
(1)

(Pipin, Rüdiger & Kitatinov 1996), where \( \hat{V} = \Omega \times r \) is the mean velocity for the undisturbed state that represents non-uniform rotation, \( \hat{m} = \rho \hat{u} \) is the mean momentum density, \( \bar{p} \) is the pressure, \( \nu_{ij} \) is the eddy viscosity, \( \bar{G} = \nabla \log \rho \) is the stratification vector, and the right-hand side of the equation involves the AKA effect with the \( \Gamma \)-tensor,
\[
\Gamma_{jk} = \Gamma_1 \left( \Omega^*_i \epsilon_{ijk} + \Omega^*_i \epsilon_{ijk} + \delta_{ij} \epsilon_{ijk} \Omega^*_i + \delta_{ij} \epsilon_{ijk} \Omega^*_i \right) G_p,
\]
(2)
with \( \Omega^* = \Omega / \Omega_0 \) as the vertical unit vector. The \( \Gamma \)-term of equation (1) comes from the symmetric correlation tensor for fluctuating velocities, i.e.,
\[
\langle \nu_i'(x,t) \nu_j'(x,t) \rangle = -\nu_{ij} \left( \frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i} \right) - \mu \nabla \hat{u} \hat{u} + \Gamma_{ij} \hat{m}_j
\]
(3)
(Krause & Rüdiger 1974; Roberts & Soward 1975; Moffatt & Tsinober 1992). It is linear in the mean velocity similarly to the hydromagnetic alpha effect, which is also linear but in the mean magnetic field. With the AKA effect included, the fluid may be unstable to generate large-scale flow patterns (Schüssler 1984; Sagdeev et al. 1984; Frisch, She & Sulem 1987; Khomenko, Moiseev & Tur 1991). Several models have been developed to study the vortex structure of the large-scale instability. While analytical Ansätze have been made for the velocity field to find the critical values for the onset of instability and the growth rates of the generated vortices in the linear regime (Moiseev et al. 1988; Gvaramadze, Khomenko & Tur 1989), non-linear 1D problems have been solved numerically to analyse the energy spectrum (Frisch, She & Sulem 1987; Sulem et al. 1989; Druzhinin & Khomenko 1991).

We perform the (linear) stability analysis by solving the eigenvalue problem for equation (1) for 3D flow perturbations in astrophysical discs. The flow is assumed to be inelastic, \( \nabla \times \hat{m} = 0 \), hence it can be described in terms of two scalar potentials,
\[
\hat{m} = \Omega^* \times \hat{U} + \nabla \times \left( \Omega^* \times \nabla A \right),
\]
(4)
where \( \hat{U} \) defines the toroidal part of the flow with its velocity vector normal to \( \Omega^* \), and \( A \) is the stream function for the poloidal flow.

The procedure to derive the dynamical equations for given flow potentials is very similar to derivations for spheres (cf. Krause & Rädler 1980). We substitute equation (4) into equation (1), calculate the curl of the equation to exclude pressure and take the scalar product with the vector \( \Omega^* \). This yields the dynamical equation for the quantity \( \hat{U} \), where \( \hat{U} \) is the ‘horizontal part’ of the Laplacian delta,
\[
\hat{U} = \frac{1}{\Omega^*} \frac{\partial}{\partial \Omega^*} \frac{\partial^2}{\partial \Omega^* \partial \Phi} + \frac{1}{\Omega^*} \frac{\partial^2}{\partial \Omega^* \partial \Phi}.
\]
(5)

Then, we perform the curl of equation (5) twice and again take the scalar product with \( \Omega^* \). This yields the dynamical equation for \( \hat{W} \), where \( \hat{W} \) is the potential for the vorticity \( \hat{w} \),
\[
\hat{w} = \nabla \times \hat{m} = \Omega^* \times \nabla W + \nabla \times \left( \Omega^* \times \nabla U \right).
\]
(6)

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The Poisson equation resulting from equations (4) and (6),
\[
\Delta \Phi = -W,
\]
(7)
closes the equation system. The complete equation system is too bulky to reproduce here.

We apply the equations to discs bounded by parallel planes crossing the z-axis of rotation at the positions \( z = \pm H \) with \( H \) being the half-thickness of the disc. No boundaries in the radial directions are imposed. A new variable,
\[
\mu = \cos \theta = \left( \frac{\sigma}{\sigma_0} \right)^2 - 1 \left( \frac{\sigma}{\sigma_0} \right)^2 + 1,
\]
(8)
is introduced to compensate for the infinite range of \( r \) and the finite range of \( \mu \). Equation (5) in terms of the new variable is
\[
\hat{R} = \frac{1}{\sigma_0} (1 - \mu^2) \hat{L},
\]
(9)
\[
\hat{L} = \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial}{\partial \mu} + \frac{1}{1 - \mu^2} \frac{\partial^2}{\partial \mu^2},
\]
where \( \hat{L} \) has the Legendre polynomials, \( P_n^m \), as the eigenfunctions. Hence, the polynomial expansion is convenient. In fact, expansions in appropriate functions over all three dimensions were applied to solve the eigenvalue problem by the fully spectral method,
\[
A = \sum_{n,m,d} A_{n,m,d} \cos \left[ \pi \left( 1 - \frac{1}{2} \right) \frac{\sigma}{H} e^{im\Phi} P_n^m(\mu) \right]
+ \sum_{n,m,d} A_{n,m,d} \sin \left[ \pi \left( 1 - \frac{1}{2} \right) \frac{\sigma}{H} e^{im\Phi} P_n^m(\mu) \right],
\]
(10)
and the same expansion applies to \( W \). The representation for \( U \), however, is slightly different, i.e.,
\[
U = \sum_{n,m,d} U_{n,m,d} \cos \left[ \pi \left( 1 - \frac{1}{2} \right) \frac{\sigma}{H} e^{im\Phi} P_n^m(\mu) \right]
+ \sum_{n,m,d} U_{n,m,d} \sin \left[ \pi \left( 1 - \frac{1}{2} \right) \frac{\sigma}{H} e^{im\Phi} P_n^m(\mu) \right].
\]
(11)

The expansions (10) and (11) automatically satisfy the stress-free and no-vertical-flow boundary conditions, the symmetry conditions on the rotation axis and zero-mass-flow condition for \( r \rightarrow \infty \) (cf. Kitatinov & Mazur 1997). The first and second sums in equations (10) and (11) describe the flows with different types of symmetry about the disc mid-plane. The superscripts \( A \) and \( S \) denote the antisymmetric and symmetric flow patterns relative to the reflection about the mid-plane, respectively. It may also be shown that the complete equation system for the linear excitations splits into two independent subsystems describing the \( A \)- and \( S \)-modes, when the eddy viscosity and \( \Gamma_1 \)-coefficient of equation (2) are symmetric and the direction of the stratification vector \( G \) is antisymmetric about the mid-plane.

The assumption of the exponential time dependence, \( A, U, W \sim e^{i\Omega t} \), reduces the linear stability analysis to an eigenvalue problem. The eigenvalues were determined numerically for the models formulated in the following section.

3 THE MODELS

The density distribution in astrophysical discs is usually much more non-uniform over the vertical direction than the horizontal one. For this reason, we neglect the radial component of the stratification vector, \( G \). For the vertical stratification a parametrization is
applied, i.e.

$$G = \Omega^2 G(z), \quad G(z) = \frac{1}{\rho} \frac{\partial P}{\partial z} = \frac{G_0}{H} \sin \left( \frac{\pi z}{H} \right).$$ \hspace{1cm} (12)$$

Here $G$ is a negative quantity fixing the density stratification. The density decreases outwards as long as the sine function is positive, i.e. up to $z = H$. In contrast to the standard accretion disc models our density profiles are rather flat in the disc ‘atmosphere’ so that also the AKA effect is only small there.

The angular velocity distribution is much more variable with axial distance than with vertical position. The differential rotation profile is parametrized by a Brandt-type function,

$$\Omega(\sigma) = \frac{\Omega_0}{\left[ 1 + (\sigma/\sigma_0)^n \right]^{1/n}},$$ \hspace{1cm} (13)

with $n = 2$ (cf. Ruzmaikin, Shukurov & Sokoloff 1988). The rotation profile (13) approaches a power law, $\tilde{q}(\sigma) = \Omega_0 (\sigma/\sigma_0)^{-q}$, with $q > 1$. A value of $q = 3/2$ corresponds to Keplerian rotation in accretion discs, while $q = 1$ is appropriate for galaxies.

The eddy viscosity $\nu_1$ is assumed to be constant. $\Gamma_1(\sigma)$ is a function which has the dimension of a viscosity. It is known that the AKA effect comes from the rotation, therefore we assume the normalized $\tilde{\Gamma}_1$ to be proportional to the normalized rotation profile,

$$\tilde{\Gamma}_1(\sigma) = \Gamma_1 \frac{\Omega(\sigma)}{\Omega_0}.$$ \hspace{1cm} (14)

Apart from the $q$-parameter and the aspect ratio,

$$\tilde{\Gamma}_0 = \frac{\sigma_0}{H},$$ \hspace{1cm} (15)

the models have two key governing parameters. The first is the Taylor number

$$Ta = \frac{4 \Omega_0^2 H^3}{\nu_1}.$$ \hspace{1cm} (16)

With the viscosity Ansatz $\nu_1 = \alpha_{SS} H^2 \Omega_0$, one arrives at the estimate

$$Ta = \left( \frac{2}{\alpha_{SS}} \right)^2$$ \hspace{1cm} (17)

in terms of the Shakura–Sunyaev viscosity parameter $\alpha_{SS}$. The application of the above accretion disc viscosity parametrization to galaxies can be justified because the resulting values for $\nu_1$ for $\alpha_{SS}$ ranging between 0.001 and 1 cover the value of the viscosity in galaxies. The other parameter

$$C_T = \Gamma_1 |G_0|$$ \hspace{1cm} (18)

measures the power of the AKA effect.

Our problem is to define the critical values of $C_T$ for the onset of the mean-flow instability and the shape of the preferred, i.e. having the lowest critical $C_T$, flow patterns.

4 RESULTS AND DISCUSSION

Figs 1 and 2 present the neutral stability lines for a thin standard accretion disc and for a galaxy disc in the $Ta-C_T$-plane ($\alpha_{SS}-C_T$-plane). Above the lines, the real parts of the eigenvalues $\lambda$ are positive, implying that the system is unstable to the flow excitations.

The accretion disc is modelled by assuming a Keplerian profile at large distances from the rotation axis. The central object (black hole, neutron star or white dwarf) is considered as a rigidly rotating core with radius $\sigma_0$. Also close to the central object, where the rotation law stops its Keplerian behaviour, the aspect ratio of the disc half-thickness to the radius runs as $0.017 \alpha_{SS}^{1/10}$. For small viscosity alphas of order $10^{-3}$ it thus makes sense to use an aspect ratio of $\tilde{\Gamma}_0 = 30$. As known, galaxies possess a rigidly rotating core with a size of a few kpc so that a slightly smaller aspect ratio of $\tilde{\Gamma}_0 = 10$ results.

As one can see from Fig. 1, in accretion discs the non-axisymmetric modes with an azimuthal wavenumber $m = 1$ are the first that are excited by the action of the AKA effect. This is true over the whole range of studied values of the viscosity parameter $\alpha_{SS}$, which is commonly assumed to be smaller than unity. With respect to the reflection about the mid-plane, the antisymmetry is clearly preferred. The symmetric modes require much higher values for $C_T$ to be excited: they are therefore not plotted. Surprisingly, the $C_T$-value is nearly completely independent of $\alpha_{SS}$, assuming values of $C_T = 5.6$.

For galactic discs the situation is slightly more complicated. Fig. 2 shows that the type of mode that is excited first depends on the

![Figure 1](https://github.com/academic-oup/mnras/article-abstract/0034/1062983)

**Figure 1.** The diagram for marginal instability for a thin standard accretion disc with Keplerian rotation ($q = 1.5$) and $\tilde{\Gamma}_0 = 30$. Only the equatorially antisymmetric modes are plotted. The critical values of $C_T$ for the symmetric ones are always larger.

![Figure 2](https://github.com/academic-oup/mnras/article-abstract/0034/1062983)

**Figure 2.** The same as in Fig. 1 but for a galaxy disc with uniform azimuthal velocity ($q = 1$) and $\tilde{\Gamma}_0 = 10$. 

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For relatively high $\alpha_{SS}$, $A_1$ dominates and $C_T$ remains nearly constant, $C_T = 6$. With decreasing $\alpha_{SS}$, $C_T$ increases and the dominant mode changes from $A_1$ to $A_2$. For $\alpha_{SS} < 0.002$ ($Ta > 10^6$), $C_T$ remains constant at $C_T = 8$, with the dominant mode now being $S_1$. Therefore, non-axisymmetric flow structures, e.g. vortices, are always preferred. However, the boundary of the instability region is defined by different azimuthal wavenumbers and both symmetry types relative to the mid-plane.

Figs 3–7 show the surface flow patterns for the primary $A_1$ (accretion discs) or else $A_1$, $A_2$ and $S_1$ (galaxy discs) modes for different $\alpha_{SS}$ where they are dominant.

The radial and azimuthal surface velocities describe a vortex structure which is extended over the whole disc (Figs 3, 5 and 7). In the case of $m = 1$ (Figs 3 and 5), as well as $m = 2$ (Fig. 7), there exist outside the rigidly rotating core two (in Figs 3 and 5, four in Fig. 7) spots with almost no horizontal velocity and a vertical flow going up in one and down in the other. The vertical flow attains its maximum around the mid-plane for the $A_1$ modes and between the mid-plane and the surface for the $S_1$ and $A_2$ modes (Figs 4, 6 and 7). In the small range of $\alpha_{SS}$ between 0.002 and 0.003, where $A_2$ is dominating, the vertical flow, as well as the horizontal flow, is concentrated outside the core (Fig. 7).

All the eigenvalues have finite imaginary parts indicating the rotation of the flow pattern with time. The frequency of this rotation increases with an increasing basic rotation rate (or decreasing viscosity).
We note that the eigenvalue problem for a 1D galaxy model was solved by Kitchatinov, Rüdiger & Khomenko (1994). The critical value was found to be $C_G = 5.98$ which is comparable to $C_G = 6–8$, found here in the 3D model. The flow evolution of the 1D problem was studied in v. Rekowski, Kitchatinov & Rüdiger (1995). As for the present model, the flow patterns are antisymmetric about the mid-plane and rotating on the order of $10^6$ yr.

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