It should be clear both from the foregoing treatment and from the original paper, that in practical application to nonlinear systems, the **δ**-method will involve to a certain extent successive approximation interval by interval, to obtain reasonably good values of δ_i. This is particulary true in regions where the "direction" of change in δ (positive or negative) is not evident at the outset of the increment. For a specific example, if $\delta(x, v, \tau) = (x - a)^2f(t)/(v - b)$, it would be difficult to obtain satisfactory results for x near a and v near b. Certainly, the author's statement "... A subprofessional assistant can ... use the method ..." could not be applied literally to this case. As a matter of fact, the difficulties encountered by all curvature methods under such conditions as this, at branch points and other singularities, constitutes a serious limitation on their general applicability. It is evident by comparing Figs. 3 and 4 of the paper, that precision in the neighborhood of the origin for the case depicted is not practically possible.

Thus if $\delta(x, v, \tau)$ varies in certain critical and sensitive ways with the variables it will be necessary to use great care in the application of this technique. For the general case, unless the oscillatory character of the equation is clearly marked

$$\frac{\partial G}{\partial x} \gg \frac{\partial G}{\partial v}$$

there is no particular reason for preferring the curvature formulation over the more general form

$$\frac{dv}{d\tau} = \gamma(x, v, \tau) \qquad [17]$$

for which implicit approximate integrals may also be written as

$$v_{i+1} = v_i + \gamma_i \Delta\tau_i \qquad [18]$$

$$\gamma_i = \overline{[\gamma(x, v, \tau)]_i} \qquad [19]$$

$$x_{i+1} = x_i + \int_{\tau_i}^{\tau_{i+1}} v d\tau \qquad [20]$$

and corresponding to which, graphical procedures exist.

This consideration, in addition to the fact that the phase-plane curvature methods are not practical for nonlinear first-order, as well as third- and higher-order systems, led the writer[7] to the development of a graphical technique, patterned after the Bergeron method for wave-propagation problems,[8] which has come to be called the "slopeline method." This method has been further refined by himself, and collaborators,[9,10] and full details will shortly be available in book form,[11] together with applications to transient and vibration problems of all orders, both linear and nonlinear.

Author's Closure

The opening statement of Professor Paynter's discussion asserts that "curvature" methods of graphical phase-plane integration were brought to a high level of refinement long ago. In reviewing the discusser's references to the work of Lord Kelvin,[3] Léauté,[4] Meissner,[5] and Braun[6] the author finds that only the reference to Braun is relevant to the method of using "curvatures in the phase plane" unless it be maintained for the sake of completeness that many other names including those of Descartes and Euclid also should have been brought in as references.

The phase-plane delta method cannot be considered a special case of the Meissner[5] technique. However, Meissner's method is definitely more general than the phase-plane delta method since it is applicable to differential equations of any order, but this generality is not achieved by phase-plane considerations.

In calling attention to the work of Braun,[6] the discusser has contributed a welcome reference that the author regrets to have missed. Braun deals with the subnormal to the arc ds in Fig. 1(b) of the paper, but does not locate the center of the circular approximation on the x axis.

Professor Paynter's alternative derivation of the phase-plane δ method relationship is instructive as presenting a variant of those given by Professor Klotter and the author. The adjustment of the δ method to numerical calculations as shown in the discusser's Equations [11] to [16] is of interest to people who are numerically inclined, but it is of course not uniquely dependent on his way of deriving the δ relationship.

The author is grateful to the discusser for his attempt to appraise the inherent weaknesses and strengths of the δ method and agrees with him that no curvature procedure will give good accuracy near singular points. This fact, however, does not invalidate the author's statement that a subprofessional assistant can be instructed quickly in how to use the graphical δ method. Of course, the accuracy of a solution will suffer in a singular-point region, but the assistant will become forcefully aware of that by having to draw highly scalloped arcs there; consequently a finer subdivision or simple caution will automatically suggest itself to him.

It is interesting to note that Professor Paynter is planning to give full details of the "slope-line method" in the book[11] he is now preparing.

Errata:

In the author's Equation [4] change v to $-v$, and in connection with the solution of Bessel's equation change $-\delta_0$ to $+\delta_0$.

[7] "Methods and Results From M.I.T. Studies in Unsteady Flow," by H. M. Paynter, *Journal of the BSCE*, vol. 39, April, 1952, pp. 120–165.

[8] "Du coup de belier en hydraulique au coup de foudre en électricité," by L. Bergeron, Dunod, Paris, France, 1950.

[9] "Regulation of Hydroelectric Plants," by A. T. Gifford and H. M. Paynter, M.I.T. 1952 (mimeographed notes).

[10] "Slopeline-Graphical Solutions of Ordinary Nonlinear Differential Equations, With Engineering Applications," by A. C. Hendrickson, M.I.T. SM thesis, 1952 (unpublished).

[11] "Graphical Solution of Engineering Transients," by H. M. Paynter and A. C. Hendrickson (book in preparation).

Bending of a Uniformly Loaded Rectangular Plate With Two Adjacent Edges Clamped and Others Either Simply Supported or Free[1]

A. S. Veletsos.[2] The authors are to be complimented for their valuable paper. It might be of interest to note, however, that the problem of the bending of a uniformly loaded rectangular plate having two adjacent edges clamped and the others simply supported has been treated also by C. P. Siess and N. M. Newmark,[3] who used a modification of the method developed by Timoshenko. Included in this reference are the general solution of the problem and also numerical values for the moments on several sections of a square plate and of a rectangular plate having a ratio of sides equal to 0.5.

[1] By M. K. Huang and H. D. Conway, published in the December, 1952, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 74, pp. 451–460.

[2] Research Associate in Civil Engineering, University of Illinois, Urbana, Ill.

[3] "Moments in Two-Way Concrete Floor Slabs," by Chester P. Siess and Nathan M. Newmark, University of Illinois Engineering Experiment Station Bulletin 385, February, 1950, pp. 90–96.

For the purpose of comparison, the values of the moment on the fixed edge of a square plate which were presented in Table 3 of the paper are listed in the following, together with the corresponding values calculated from the solution obtained by Siess and Newmark:

Location		Clamping moments × $(1/qa^2)$		
x	y	Huang and Conway	Stiles	Siess and Newmark
0.2a	0	—0.0272	—0.0257	—0.0275
0.4a	0	—0.0594	—0.0608	—0.0598
0.6a	0	—0.0692	—0.0684	—0.0691
0.8a	0	—0.0510	—0.0515	—0.0507

Authors' Closure

The authors are grateful to Mr. Veletsos for bringing the work of Siess and Newmark to their attention. The agreement in the values of the clamping moments is most satisfactory.

The Elastic Sphere Under Concentrated Loads[1]

M. M. Frocht.[2] The paper raises several questions regarding the fundamentals in the theory of elasticity. Of particular significance is the conclusion that to the three conditions generally accepted as sufficient for a unique solution a fourth "limit condition" must be added and that failure to include such a fourth condition will, in the case of concentrated loads, lead to pseudosolutions, that is, in essence, to false solutions. This conclusion carries with it the need for a re-examination of the meaning of Saint Venant's principle.

To an experimentalist this paper is significant for an additional reason. It provides another illustration that mathematicians are not infallible and that the ultimate test of a theoretical solution lies in experimental verification.

The agreement between the theoretical results of the authors and the photoelastic results of Frocht and Guernsey are particularly noteworthy because the photoelastic solution is the first complete solution of its kind. The degree of agreement between the two sets of results is indeed remarkable, particularly at the center of the sphere where the difference in the value of (σ_z/σ_0) is about 2 per cent.

On the surface of the sphere, around the equator, where the stresses are relatively small, the agreement is less satisfactory. There the photoelastic value of (σ_z/σ_0) is zero and the theoretical value is 0.30, for $\nu = 0.48$. This aspect of the problem can be investigated by means of a strain-gage test. At the suggestion of the discusser, Messrs. K. P. Milbradt,[3] D. Landsberg,[4] and P. D. Flynn[5] made such a test using an aluminum sphere of 4.87 in. diam for that purpose, Fig. 1, herewith.

Four $^1/_8$-in. SR-4 strain gages were mounted at points A-A, Fig. 2, on the equator. Two gages were set parallel to ϵ_z and the remaining two parallel to ϵ_γ. Tests were made at 28,000 lb, 56,000 lb, 84,000 lb, etc., up to 500,000 lb. Typical strain-load curves for several cycles of loading and unloading are shown in Fig. 3.

[1] By E. Sternberg and F. Rosenthal, published in the December, 1952, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 74, pp. 413–421.

[2] Research Professor of Mechanics, Illinois Institute of Technology, Chicago, Ill. Mem. ASME.

[3] Assistant Professor of Civil Engineering, Illinois Institute of Technology.

[4] Assistant Research Engineer in Experimental Stress Analysis, Illinois Institute of Technology.

[5] Research Corporation Fellow in Experimental Stress Analysis, Illinois Institute of Technology.

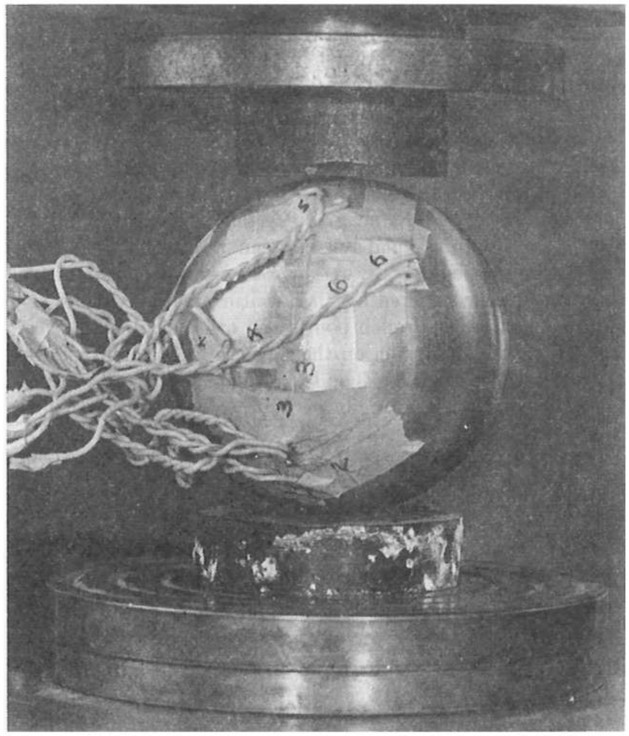

Fig. 1

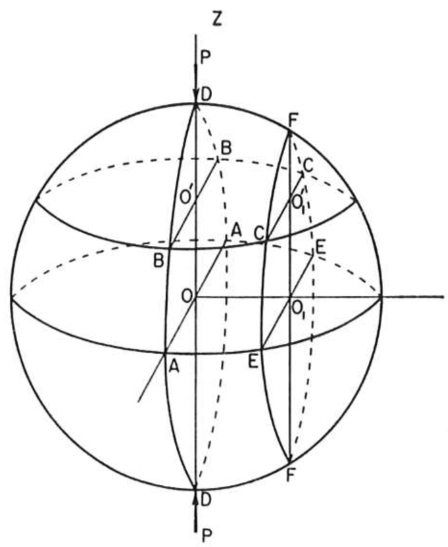

Fig. 2

It will be observed that the stress-strain relation is essentially linear. This linearity extended up to 280,000 lb, and the deviations from linearity at higher loads were small. It also will be noted that recovery was somewhat incomplete in the first stage of each new load. The meaning of this initial set is at present not clear. Perhaps it may be connected with the development of small plastic zones in the region of the applied load.

The stresses were based on the mean recoverable strain. The error resulting from neglecting the initial set is believed to be small.

The final results are summarized in Table 1 of this discussion. The gages thus show that in the equatorial region there exist compressive σ_z stresses of a magnitude of $0.25\sigma_0$ and transverse