Hydrodynamics of accretion discs of variable thickness

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ABSTRACT

We present a new model for the hydrodynamics of thin accretion discs. In the $(r, \phi)$ plane full hydrodynamics is taken into account, while the time evolution of the disc thickness $H$ is followed by assuming uniform expansion and contraction in the vertical direction. This amounts to a ‘one-zone’ approximation for the vertical equation of motion. We show that the model correctly incorporates the fundamental mode of vertical oscillation of a disc for waves long compared with the disc thickness. A numerical scheme to solve these equations on an Eulerian grid is introduced and tested for validity. As an application we compute the excitation of disc thickness oscillations by the tidal field of a companion. Strong excitation occurs at the 2:1 resonance, at $r \approx 0.32 a$, where $a$ is the binary separation.

Key words: accretion, accretion discs – hydrodynamics – waves.

1 INTRODUCTION

Accretion on to central stars like young stellar objects, white dwarfs or neutron stars is a common phenomenon in astrophysical situations. Whenever the specific angular momentum of the accreting gas is high, direct accretion is impossible and the mass is forced to form an accretion disc (see Frank, King & Raine 1992 for a general introduction). Accretion is connected with redistribution of angular momentum in the disc by some currently unknown process. A common property of most of these discs is their hypersonic azimuthal motion with Mach numbers $M = v_\phi/c_s \gg 1$, where $v_\phi$ is the azimuthal velocity, and $c_s$ the isothermal sound speed. Due to efficient cooling the disc thickness $H \sim c_s/\Omega_K$, with $\Omega_K$ the Keplerian angular velocity, is correspondingly small ($H/r \ll 1$, where $r$ is the distance to the central star). To study these geometrically thin discs, the gas properties are usually vertically averaged (Lynden-Bell & Pringle 1974), and the vertical stratification of the disc is assumed to be in thermal and pressure equilibrium.

The tidal response of discs to the gravitational interaction of a secondary (i.e., companion) star in a close binary system has been studied with two-dimensional numerical hydrodynamics in the $(r, \phi)$ plane by, among others, Matsuda et al. (1987), Różycka & Spruit (1993) and Savonije, Papaloizou & Lin (1994). These simulations show inward-travelling shock waves; their dissipation causes outward transport of angular momentum. The properties of these shocks were studied analytically by Spruit (1987) and Spruit et al. (1987). These calculations suffer from the restriction of being strictly two-dimensional, i.e., they are vertically confined between two ‘rigid walls’. Any vertical response of the disc (changes of the disc thickness) to tidal and thermal disturbances is suppressed.

We expect, however, that vertical oscillations are excited by the sudden change in the disc temperature, i.e., during the passage of heating and cooling waves in thermally and viscously unstable accretion discs, where the disc gas undergoes a thermal and viscous phase transition (see Cannizzo 1993 and Osaki, Hirose & Ichikawa 1993). We expect this to be accompanied by vertical disc oscillations, which might leave fingerprints on the light curves of cataclysmic variables. Surface waves can also be excited by the ram pressure of the bulk momentum of an overflowing accretion stream (Lubow 1989) hitting the disc near the Lubow–Shu radius. The accretion disc will be out of thermal and pressure equilibrium at the accretion stream impact at the outer accretion disc rim, known as the hotspot. Disc gas heating by the spiral shocks, excited by the tidal force of the secondary, will again result in a response of the disc normal to the orbital plane of the disc. Finally, Lubow (1981) pointed out that the eigenfrequency of the fundamental mode of vertical disc oscillation is close to the Keplerian frequency, and thus resonances are expected where this mode commensurates with the synodical period with which a disc particle ‘sees’ the tidal field of the secondary. With any of the examples mentioned above, energy can be deposited into the mode of vertical disc oscillation, which will travel along the disc as surface waves, possibly interacting nonlinearly with the motions in the disc plane. This might result to mass accretion on to the central star if the disc gas moves faster than the pattern speed of the excited wave (cf. Lynden-Bell & Kalnajs 1972; Lin & Papaloizou 1979).

It is the aim of this and forthcoming papers to study the influence of the fundamental vertical disc mode on the global hydrodynamics of accretion discs. In this paper we present the basis of a model for the hydrodynamics of geometrically thin accretion discs which is
still two-dimensional but includes the additional freedom of the fundamental vertical disc oscillation mode.

The basic assumption made is the usual thin-disc approximation. Thus the horizontal component of gravity is independent of the vertical height $z$ above the disc mid-plane, and the vertical component increases linearly with $z$. Similarly, the horizontal component of the velocity is independent of $z$, and the vertical velocity linear with $z$. This eliminates all modes with structure in the vertical direction but, as we show below, it is consistent with the properties of the fundamental mode, in the thin-disc limit.

With these assumptions we show in Section 2 how the vertical equation of motion can be reduced to two first order (in time) equations, one for the disc thickness and one for its rate of change. In Section 3 we reformulate the equations in a form suitable for numerical treatment on an Eulerian grid, using cylindrical coordinates centred on the primary star and corotating with the binary. The equations are investigated in Sections 4 and 5 in the linear regime, and in Section 6 we describe our numerical method. The validity of the numerical approach is tested in Section 7 and, finally, in Section 8 we show the results of axisymmetric disc calculations.

In a subsequent paper (Stehle 1999) we solve our equations in the potential field of a binary, and compare these computations with observations, done with Doppler tomography (Steeghs & Stehle 1999) and with eclipse mapping techniques (Stehle & Steeghs, in preparation).

## 2 Basic Assumptions and Model Equations

We formulate here the assumptions made and derive the equations of our model. We introduce coordinates $(r, z)$, where $r$ is the position vector parallel to the disc mid-plane and $z$ the distance normal to it. A subscript $\hat{\;}$ on a vector denotes the component of the vector parallel to the 2–dimensional disc mid-plane (i.e., $z =$ constant) and the subscript $\hat{\;}$ denotes the component normal to it (i.e., $r =$ constant). The velocity vector is thus given by $v(r, z) = [v_\hat{r}(r, z), v_\hat{z}(r, z)]$, where $v_\hat{r}$ is the component parallel to the disc plane, and $v_\hat{z}$ normal to it. The Lagrangian time derivative reads in this notation

$$d_t = \partial_t + v_\hat{r} \partial_{\hat{r}} + v_\hat{z} \partial_z. \quad (1)$$

As usual for geometrically thin discs, we neglect any terms of $O(H/r^2)$, so that the gravitational acceleration (produced by the primary and the secondary star) in the $z$-direction is given by

$$g_z(r, z) = g_z^0(r) z, \quad (2)$$

where $g_z^0(r)$ is independent of $z$, while the vertical dependence of $g_z^0$ is of higher order in $H/r$ and thus we assume

$$\partial_z g_z^0 = 0. \quad (3)$$

Assumptions are now needed for the vertical structure of the velocity field. We assume that we are dealing only with symmetric compressions and expansions of the disc, that is, we ignore ‘bending modes’. Because of the symmetry of the gravitational field, it excites only these symmetric modes. Tilting and warping can in principle be excited by a number of effects such as a disc wind (Schandl & Meyer 1994) or radiation pressure (Maloney Begelman & Pringle 1996); our model is not applicable to such discs. Sufficiently close to the mid-plane of the disc, the flow field in any symmetric mode of oscillation satisfies, by Taylor expansion,

$$\partial_z v_\hat{r} = 0, \quad (4)$$

and

$$\partial_z v_\hat{z}(r, z) = 0, \quad (5)$$

or

$$v_\hat{z}(r, z) = z v'_z(r), \quad (6)$$

where $v'_z(r)$ is independent of $z$. We now use this expansion near the mid-plane as an approximation for the entire disc height. This evidently ignores the vertical structure in the velocity field in modes with nodes in their vertical direction, but is a good approximation for the fundamental mode. Lubow (1981) has shown that the approximation is exact for the fundamental mode, in the limit of a disc which is thin compared with the horizontal wavelength. As a consequence of these assumptions, a vertical column of mass may be compressed, or may move horizontally as a whole, but it remains vertical at all times. Thus any overturning motions like convection are suppressed. However, the detailed vertical velocity structure will be of importance whenever the amplitude of the vertical disc oscillation becomes too large. This might, for example, be the case close to resonances. We might then underestimate the efficiency of the horizontal coupling of the wave with its vertical disc oscillation (Lubow 1981). A full three-dimensional study in the non-linear regime is then requested, which is beyond our current model assumptions.

Finally, we have to make an assumption about the vertical entropy distribution $s(z)$, which determines the average vertical density and temperature structure of the disc. For this we will use a polytropic or isothermal assumption, but for the moment $s(z)$ can be left unspecified.

Using only equation (4), integration of the continuity equation over the full disc thickness yields

$$\partial_t \Sigma + \nabla_{\hat{r}} (\Sigma v_\hat{r}) = 0, \quad (7)$$

where

$$\Sigma = \int_{-H}^{+H} \rho dz. \quad (8)$$

The horizontal component of Euler’s equation,

$$\rho (\partial_t v + v \nabla v) = \nabla P + \rho g, \quad (9)$$

can be integrated in the same way across the disc to yield

$$\Sigma (\partial_t v_\hat{r} + v_\hat{r} \nabla v_\hat{r}) = \nabla_{\hat{r}} \Pi + \Sigma g_\hat{r}, \quad (10)$$

where again equation (4) is used, and where the pressure integral $\Pi$ is given by

$$\Pi = \int_{-H}^{+H} P dz. \quad (11)$$

The continuity equation and the horizontal component of Euler’s equation are identical to the equations given previously in the literature (Pringle 1981). The vertical disc equations, however, are different. Instead of setting $H = H_c = c/\Omega$, we get two more equations, as follows.

With $v_\hat{z}(z) \sim z$ (i.e., equation 6), our fluid parcels move in the vertical direction by a uniform expansion or contraction. We describe this by defining for each mass element a normalized vertical position $\hat{z} = z/H$. Here $H = H(r, t)$ is a vertical scaling factor which is independent of $z$ but a time-dependent function of $r$. $H$ is chosen such that $\hat{z}$ is constant along a fluid path,

$$\frac{dz}{dt} = 0, \quad (12)$$

which defines $H$ uniquely except for a constant factor. Equation (6)
then yields
\[ v_z' = \frac{\partial v_z'}{\partial r} = \frac{dH}{H\delta t} \]
and, since \( \partial_t H = 0 \), we find
\[ \frac{dH}{\delta t} = \partial_t H + v_h \nabla_H H. \]
Equation (13) then becomes
\[ \partial_t H + v_h \nabla_H H = v_z' H. \]
This is to be read as an equation for the scaling parameter \( H \), and must be solved together with the vertical component of Euler’s equation (9). Integration of this component from the mid-plane to \( z = \infty \) yields, with our assumed velocity structure,
\[ \partial_t v_z' + v_h \nabla v_z' + (v_z')^2 = -\frac{1}{R} P_c + g'_c, \]
where \( R = \int_0^{\infty} z \text{d}z \), and \( P_c \) is the pressure at the mid-plane of the disc.

By our assumed velocity structure, the continuity equation yields
\[ \partial_t \left( \frac{\text{dln} \rho}{\text{dr}} \right) = -\partial_t \nabla \cdot v = \partial_t (\nabla v + \frac{\partial}{\partial r} v_z) = 0, \]
which is equivalent to saying that compression is uniform in \( z \). The most general form of \( \rho = \rho(r, \xi, t) \) that satisfies equation (17) is
\[ \rho = f(\xi)G(r, t), \]
where \( f \) is a constant of the motion, \( \partial f/\partial t = 0 \). Without further restrictions, \( f \) can be normalized, i.e., \( \int_{-\infty}^{\infty} f d\xi = 1 \). This yields \( G = \Sigma H \), so that \( \rho \) can be written as
\[ \rho = f(\xi)H(\Sigma)H. \]

To complete the equations, we need to express \( R \) and \( P_c \) in terms of \( \Sigma \). These quantities depend on the mass distribution with height, and therefore on the vertical entropy structure of the disc. This is again determined by the vertical transport of energy, which cannot be represented in our model. This missing physics is replaced in our model by adopting an isothermal or polytropic approximation to relate \( R \) and \( P_c \).

Assume an ideal gas of constant adiabatic index \( \gamma \). Then the entropy is given by \( s = s_0 \ln(P/\rho^\gamma) \). For adiabatic motions (no viscous heating or energy loss by radiation) we get \( \partial_s s = 0 \) or
\[ \frac{\text{dln} P}{\text{dt}} = \gamma \frac{\text{dln} \rho}{\text{dt}}, \]
and thus, by equation (17),
\[ \partial_t \left( \frac{\text{dln} P}{\text{dr}} \right) = 0. \]
This shows that \( P \) has the same general form as \( \rho \). Similarly, as in equation (19), we find a normalized function \( k(\xi) \) such that
\[ P = k(\xi)\Pi H, \quad \int_{-\infty}^{\infty} k(\xi) d\xi = 1, \]
where \( k \) is again a constant of motion. Note, however, that the assumption of adiabatic motion is essential here. If this is relaxed, \( \xi \) and \( f \) are still constants of motion, but not \( k \). The disc temperature \( T \) is given by
\[ T = \frac{\mu P}{R \rho K} = \frac{\Pi k \mu}{\Sigma f R}, \]
where \( R \) is the molar gas constant, and \( \mu \) the mean molecular weight. The constants of motion \( f \) and \( k \) still depend on the initial position \( r_0 \). In the spirit of polytropic approximations, we relate these constants by
\[ k = f^\Gamma, \]
where \( \Gamma = 1 \) for the isothermal case.

Now we concentrate particularly on the isothermal case, i.e., when \( \partial_t T = 0 \). In this case we know that the density follows a Gaussian distribution with scaleheight \( H_c \) when the disc is in vertical equilibrium. The arbitrary factor in the definition of \( H \) is fixed by considering a disc in hydrostatic equilibrium. Setting \( H \) equal to the pressure scaleheight
\[ H_c = c_s/\sqrt{g_c}, \]
in such a disc then completes the definition of \( H \). Here \( c_s \) is the isothermal sound velocity, given by \( c_s^2 = P/\rho \). By setting \( \partial_t v_z' = 0 \) in the vertical equation of motion (16), we find
\[ f = \sqrt{\frac{1}{2\pi}} \exp(-\xi^2/2). \]
This is independent of \( r \) and, since \( f \) is a constant of motion, \( f \) is constant in space and time for adiabatic motions. For the isothermal case we have \( \Gamma = 1 \), and thus \( k = f \). In that case we find for the mid-plane pressure
\[ P_c = \rho_c \frac{\Pi}{\Sigma}, \]
where the mid-plane density \( \rho_c \) is connected to the surface density, and the disc height by \( \Sigma = \sqrt{2\pi \rho_c H} \). \( k \) is a constant only for adiabatic motions. If dissipation or energy loss by radiation becomes important, \( k \) can change with time. As long as \( \partial_t \ln k \ll \partial_t \ln H \), the vertical disc movements can still be regarded as quasi-adiabatic, which is in fact true for most accretion discs. Otherwise the change of \( k \) has to be included to follow the time evolution of \( H \) accurately.

For adiabatic accretion discs (no dissipation or radiative cooling), equations (7), (10), (15) and (16), combined with an equation of state, determine the time evolution of the variables \( (\Sigma, \Pi, v_h, v_z', H) \) as a function of \( (r) \).

The vertical equation of motion is coupled to the other equations only by the internal energy. Vertical oscillations can be excited directly by the tidal force, but, in the absence of tidal forcing, the excitation can be thought of as resulting from changes in disc temperature. A change in the disc temperature (causing a change in the equilibrium disc thickness \( H_e \)) can be due to a sound wave, shock wave, or a thermal transition wave passing through the disc.

The assumption that the vertical velocity is proportional to \( z \), on which the derivation is based, is valid only for low-amplitude vertical displacements. It breaks down when the vertical velocity amplitude \( v_z'(H) \) becomes comparable to the sound speed \( c_s \) or, equivalently, when \( v_z' = \Omega_K \).

### 3 THE EQUATIONS IN THE POTENTIAL FIELD OF A BINARY

We investigate accretion discs in the potential field of a binary. We introduce a cylindrical coordinate system \( (r, \phi, z) \) centred on the primary, rotating with the binary angular velocity \( \Omega_0 = \Omega_0 e \). We determine the equations in these coordinates to solve them later numerically. We assume the disc to be isothermal in \( z \) (i.e., \( \partial_t T = 0 \)) which yields a Gaussian density distribution
\[ \rho(z) = \rho_c \exp\left( -\frac{z^2}{2H^2} \right). \]
The continuity equation \( \dot{\Sigma} \) in the binary potential field is

\[
g_{12}(z) = -g_{12}z = -\left( \frac{GM_{1}}{r} + \frac{GM_{2}}{\bar{r}} \right) z,
\]

where \( r \) and \( \bar{r} \) are the distances to the central star and the secondary respectively.

For discs in the potential field of a single star \((M_{2} = 0)\), equation (29) reduces to \( g_{12} = \Omega_{K} \), where \( \Omega_{K} = (GM_{1}/r)^{1/2} \) is the Keplerian angular velocity. In the case of a binary, the secondary acts like a non-axisymmetric perturber. The disc is vertically compressed most strongly by the companion’s gravity forces when the gas passes the line \( M_{1} - M_{2} = 0^{\circ}\). The secondary’s gravitational interaction is weakest at \( \phi = 180^{\circ}\).

The vertical acceleration \( \ddot{z} \) of motion we do this by using as dependent variables the vertical component of Euler’s equation, equation (10), has the radial and azimuthal components

\[
\partial_{r} \frac{\rho v_{r} \dot{v}_{r}}{r} + \partial_{\phi} \left( \rho \dot{v}_{\phi} \right) =
\]

\[
-\partial_{t} P + \frac{1}{r} \partial_{r} \left( rv_{r} \right) + \frac{1}{r} \partial_{\phi} \left( \rho v_{\phi} \right) =
\]

\[
\partial_{r} \pi - \frac{1}{r} \partial_{r} \left( rv_{r} \right) + \frac{1}{r} \partial_{\phi} \left( \rho v_{\phi} \right) =
\]

\[
-\partial_{t} P + \frac{1}{r} \partial_{r} \left( rv_{r} \right) + \frac{1}{r} \partial_{\phi} \left( \rho v_{\phi} \right) =
\]

\[
= -\partial_{t} P + \partial_{r} \pi - \partial_{\phi} \left( \rho v_{\phi} \right)
\]

and

\[
\partial_{r} \pi + \partial_{r} \left( rv_{r} \right) + \partial_{\phi} \left( \rho v_{\phi} \right) =
\]

\[
= -\partial_{t} P + \partial_{r} \pi - \partial_{\phi} \left( \rho v_{\phi} \right)
\]

The Roche potential \( \Phi \) includes the effects of both the gravitational and centrifugal forces, and reads

\[
\Phi(r, \phi, z = 0) = -\frac{GM_{1}}{r} - \frac{GM_{2}}{\bar{r}} - \frac{1}{2} \left( \Omega_{K} \right)^{2} \]

where \( r_{CM} \) is the distance to the centre of mass. The vertical structure of the disc is given by equations (15) and (16), where the mid-plane pressure \( P_{c} \) is given by equation (27).

3.1 The disc thickness equations

For the numerical treatment of the advection terms it is important to write the equations in conservative form. For the vertical equation of motion we do this by using as dependent variables the vertical momentum and energy.

Because of the antisymmetry of \( v_{z}(z) \) the \( z \)-momentum in the upper half of the accretion disc \( p^{+}_{z} \) is opposite to that in the lower half \( p^{-}_{z} \). Instead of the vertical velocity we use \( p^{+}_{z} \), which is found to be

\[
p^{+}_{z} = \int_{0}^{z} \rho v_{z} dz = \sqrt{2\pi \Sigma H v_{z}}.
\]

The kinetic and potential energy \( E_{kin,z}, E_{pot,z} \) in \( z \)-direction are given by:

\[
E_{kin,z} = \int_{-\infty}^{+\infty} \frac{1}{2} \rho v_{z}^{2} dz = \frac{1}{2} \Sigma (v_{z} H)^{2} = \pi \frac{P^{+}_{z}}{\Sigma},
\]

\[
E_{pot,z} = \int_{-\infty}^{+\infty} \frac{1}{2} \dot{\Sigma} g_{z}^{2} dz = \frac{1}{2} \dot{g}_{z} \Sigma H^{2}.
\]

By using \( p^{+}_{z} \) as variable instead of \( v_{z} \), the kinetic energy (36), \( E_{kin,z} \sim p^{+}_{z} \) is also advected accurately. As second variable we use \( E_{pot,z} \) instead of \( H \). For \( E_{pot,z} \), we find, using equations (37) and (35),

\[
\dot{E}_{pot,z} + \frac{1}{r} \partial_{r} \left( rv_{r} E_{pot,z} \right) + \frac{1}{r} \partial_{\phi} \left( \rho v_{\phi} E_{pot,z} \right) = \frac{1}{\sqrt{2\pi}} \dot{g}_{z} \left( \frac{H^{2}_{z}}{\Sigma} - 1 \right).
\]

For the \( z \)-component of Euler’s equation we get

\[
\partial_{t} p^{+}_{z} + \frac{1}{r} \partial_{r} \left( rv_{r} p^{+}_{z} \right) + \frac{1}{r} \partial_{\phi} \left( \rho v_{\phi} p^{+}_{z} \right) = \frac{1}{\sqrt{2\pi}} \dot{g}_{z} \left( \frac{H^{2}_{z}}{\Sigma} - 1 \right).
\]

The last right-hand side of equation (39) results from the imbalance of gravitational and pressure forces. These forces accelerate the disc vertically, increasing \( H \) if \( H < H_{c} \), and decreasing \( H \) if \( H > H_{c} \).

3.2 The internal energy

Finally, we derive an equation for the internal energy which includes compressional heating by vertical disc oscillations. We do this assuming the equation of state of an ideal gas with constant ratio of specific heats \( \gamma \). We start by writing the energy equation in the form (e.g. Müller 1995)

\[
\partial_{t} \rho v + \nabla \rho v = -P \nabla \rho v,
\]

where \( \rho \) denotes the internal energy per unit mass. For an ideal gas with constant \( \gamma \), \( \rho \) is a function of the disc temperature only. It is then constant with \( z \) in our isothermal model. Integrating equation (40) over the full disc thickness then yields, for \( e = \rho \Sigma
\]

\[
\partial_{t} e + \frac{1}{r} \partial_{r} \left( rv_{r} e \right) + \frac{1}{r} \partial_{\phi} \left( \rho v_{\phi} e \right) = -P \nabla_{z} \rho_{t} - \frac{\sqrt{2\pi \Sigma H^{2} \rho^{+}_{z}}}{\Sigma H}
\]

because \( \lim_{z \to z_{c}} \rho_{c} = 0 \).

The integrated internal energy \( e \) is related to \( \Pi \) by

\[
\Pi = (\gamma - 1) e.
\]

4 VERTICAL DISC OSCILLATIONS

4.1 Unforced oscillations

In order to see what the consequences of adding of equations (15) and (16) are, without going to a full numerical simulation, we analyse their behaviour separately first, by ignoring the horizontal velocities. Equations (35), (39) and (41) then become

\[
\partial_{t} H = \sqrt{2\pi} \frac{E^{+}_{z}}{\Sigma},
\]

\[
\partial_{t} p^{+}_{z} = \frac{1}{\sqrt{2\pi}} \left[ (\gamma - 1)e - \frac{1}{H} H^{2} \right].
\]

\[
\partial_{t} \ln e = -\frac{1}{\sqrt{2\pi}} \left( (\gamma - 1) \rho^{+}_{z} + \frac{1}{H} \right)
\]

\[
\frac{1}{\Sigma H}
\]

Inserting equations (43) and (45) into equation (44) yields

\[
\frac{\partial^{2}}{\partial t^{2}} p^{+}_{z} + g^{+}_{z} \left( \frac{H^{2}_{z} H^{2} - 1}{H^{2}} \right) p^{+}_{z} = 0.
\]

In the linear regime \((H^{2}/H_{c}^{2} - 1 \ll 1)\), equation (46) is solved by \( p^{+}_{z} = e^{\omega t} \), where

\[
\omega = \sqrt{\left( 1 + \gamma \right) g^{+}_{z} - \frac{1}{H^{2}} \gamma \Omega_{K}}.
\]

The last equality is strictly true only for discs around single stars, but it is also a good approximation for a disc in a binary system. This
expression for $\omega$ agrees with the fundamental-mode frequency found by Lubow (1981).

In Fig. 1 we show several solutions obtained by numerical integration of equations (43), (44) and (45). The initial disc thickness is given by its equilibrium value ($H_i = H_e$), while the initial momentum $p_{z,i}^+$ is varied to show the effect of the non-linearity of the equations.

Free disc oscillations conserve the total energy:

$$\delta E_{\text{tot}} = \delta E_{\text{pot}} + \delta E_{\text{kin,z}} + \epsilon = 0,$$

which is verified by inserting equations (43), (44) and (45) into equation (48). Another conserved quantity during the oscillation is the specific entropy $s(z = 0)$ at disc mid-plane $z = 0$. Up to a constant factor, it is given by

$$s(z = 0) = \epsilon H_{\text{e}}^{(r-1)}.$$

### 4.2 Disc thickness oscillations driven by the tidal force

Next, we consider the excitation of disc thickness variation by the tidal force. This force excites both horizontal and vertical motions, and the vertical motions are themselves also coupled to the horizontal equations of motion. In order to isolate the basic driving process, we simplify the physics by studying a single disc zone revolving around the primary in Keplerian orbits at a constant radius. We therefore assume $v_r = 0$ and $v_{z,i} = \Omega K_r(r)$ throughout this analysis. The disc gas thus ‘sees’ the change of the tidal force of the secondary star on its synodical revolution period $\Omega = \Omega_2 - \Omega_1$.

The excitation is then governed by equations (43), (44) and (45) due to the changing potential field of the secondary normal to the disc plane. First, we study the linearized problem.

Linearizing (46) and neglecting $d_g z$ along the particle’s trajectory, which is valid as soon as the vertical oscillation of the single zone has been excited, we derive with $g_i$ from the gravitational field (29)

$$\frac{\partial^2}{\partial t^2} p_{z,i}^+ + \Omega K^2 \left[ 1 + \hat{\gamma} + \frac{q \sigma^3}{1 + \sigma^2 - 2 \sigma \cos(\Omega t)} \right] p_{z,i}^+ = 0,$$

where $a$ is the binary separation, $\sigma = r/a$, $q = M_2/M_1$, and $\hat{\gamma} = c_s^2 \Omega K^2 H_{\text{e}}^{(r-2)} = \gamma$ near thermal equilibrium. Expanding the last part of the brackets in equation (50) into ultraspherical polynomials $C_n^{(3/2)}(\cos \Omega t)$ (Abramowitz & Stegun 1965) and rearranging the order of summation (see Appendix A), we derive

$$\frac{\partial^2}{\partial t^2} p_{z,i}^+ + \Omega K^2 \left[ 1 + \hat{\gamma} + q \sigma^3 \sum_{j=1}^{\infty} C_j^{(3/2)}(\sigma) \cos(j \Omega t) \right] p_{z,i}^+ = 0,$$

where the $L_j$ are now only functions of $\sigma = r/a$ (see Appendix A). If the summation is reduced to a single $j$-value, i.e., to $j = m$, then equation (51) is Mathieu’s equation, describing parametric instability. A similar equation had been found by Hirose & Osaki (1990) for tidal resonances within the plane of the accretion disc, and we further follow their line of argument.

Resonance can occur at different values of the azimuthal order $m$, but for most $m$ they occur outside the outer radius of typical discs in binaries. For $m = 2$, parametric instability occurs at a radius which can be within the actual size of an accretion disc. We therefore concentrate on the $m = 2$ component in the potential field of the secondary star. In that case, Mathieu’s equation reads (see Bender &
The resonance radius \( \hat{R}_{21} \) for tidally excited oscillations of the disc thickness (full line) as a function of \( q \), for \( \hat{\gamma} = 1.4 \). The Roche radius of the primary \( R_{\text{Ro}} \) (dot-dashed line), the distance to the Lagrangian point \( R_{\text{L}} \) (dashed line) and the tidal resonance radius \( R_{\text{t}} \) with the epicyclic frequency (dotted line) are shown for comparison (see text for references).

\[
\frac{d^2 p_+}{d\tau^2} + (a + 2\epsilon \cos \tau)p_+ = 0,
\]

(52)

where \( \tau = 2\Omega t \), \( \epsilon = q \pi L_z/4\Omega^2 \) and \( a = \bar{\omega}^2/4\Omega^2 \). \( \bar{\omega} \) is the frequency of oscillation in the absence of the driving term (i.e., setting \( L_{\pi} = 0 \) in equation 51):

\[
\bar{\omega}^2 = \Omega_K^2 [1 + \gamma + q \pi L_z^{(3/2)}(\pi)].
\]

This is the orbital frequency, and it differs from \( \Omega_K \) by a correction due to the azimuthal average of the secondary’s potential.

For small values of \( \epsilon \), Mathieu’s equation yields an unstable solution where \( a = n^2/4 \) with \( n = (0, 1, 2, \ldots) \). In particular, for \( n = 2 \) we find a 2:1 resonance at a disc radius where the condition \( \bar{\omega}(\pi) = 2(\Omega_K - \Omega_0) \)

is fulfilled. This equation is similar to that given by Hirose & Osaki (1990) or Whitehurst & King (1991) for tidal resonances which appear within the plane of the accretion disc, but with \( \bar{\omega} \) the epicycle frequency instead of the fundamental vertical disc oscillation frequency. This result is in full agreement with the linearized study of Lubow (1981) for local vertical disc oscillations, which again justifies the approach taken in our study (see equation 41 in Lubow 1981).

For \( \hat{\gamma} = 1.4 \), the radius \( R_{21}(\pi) \) can be fitted by the approximate expression

\[
R_{21}/a = 0.370 - 0.123q + 0.041q^2,
\]

(55)

which is accurate to within 1 per cent between \( q = 0 \) and 1.5. This shows, as mentioned earlier by Lubow (1981), that the 2:1 resonance is in the possible range of disc sizes in binaries. That is why we studied the \( m = 2 \) component.

In Fig. 2 we show the 2:1 resonance radius \( R_{21}(q) \) for \( \hat{\gamma} = 1.4 \) as a function of \( q \). The figure also shows the radius to the inner Lagrange point \( L_1 \) (Plavec & Kratochvil 1964), the average radius of the primary Roche lobe \( R_{\text{Ro}} \) following (Eggleton 1983), and the 3:2 tidal resonance radius \( R_{32} \) between the binary orbit and the epicyclic frequency for motions in the plane of the disc (cf. Warner 1995). We remark that the truncation radius of accretion discs in binary systems is at \( \approx 70 \) per cent of the primary Roche radius \( R_{\text{Ro}} \). This is also the maximum size to which accretion discs can grow in binary systems.

We may expect considerable variations in the disc thickness near the 2:1 resonance. To show this, we integrate equations (43), (44) and (45) numerically. Fig. 3 shows the maximal and minimal disc thickness, \( H_{\text{max}}, H_{\text{min}} \) versus \( \pi \) for \( q = 0.1, 0.5 \) and 1 after about 100 Keplerian orbits. Outside the regime where the resonance instability applies, the maximal amplitude of the disc oscillation is found within our time of integration. At the location of the resonance instability we expect the disc height to become infinite, however, which is in our numerical approach not resolved. The disc thickness remains finite in our calculation.

We note that for large values of \( q \) a second resonance of the vertical disc oscillation with its synodical period might appear within the possible sizes of accretion discs in outburst, i.e., a 3:1 resonance. For \( q = 1 \), this second resonance peaks at \( \pi = 0.5 \) and is also observed in Fig. 3. Apart from the response near the resonances, the oscillation amplitude increases with radius and with increasing mass ratio, as expected for tidal forcing.

## 5 Axisymmetric Linear Waves

The purpose of this section is to test the effect of our simplified vertical structure equations by analysing linear waves, and comparing them with the known behaviour of waves in the full axisymmetric problem, using the shearing sheet approximation (Goldreich & Lynden-Bell 1965).

We assume an axisymmetric disc of constant surface density in isothermal equilibrium \( (\partial_r T = \partial_\phi T = 0) \) around a single star. We set \( \psi_0 = 0 \) and \( p_0 = 0 \). The disc revolves around the central star in Keplerian orbits. We study the equations in a frame of reference which corotates with the equilibrium disc at \( r_0 \) [i.e., \( \Omega_K = \Omega_K(r_0) \)] and linearize the equations. A Cartesian coordinate system \((x, y)\) is introduced by the transformation

\[
x = r - r_0 \quad \text{and} \quad y = r_0(\phi - \Omega_K t),
\]

(56)

which yields for the partial derivatives

\[
1/r_0 \partial_\phi \rightarrow \partial_y \quad \text{and} \quad \partial_r \rightarrow \partial_x.
\]

(57)
The equilibrium rotation frequency in the corotating frame is approximated by
\[ \Omega - \Omega_{K,0} = 2Ax/r_0 = \frac{d\Phi}{dr_0} x, \] (58)
where the local shear rate \( 2A \) is taken as a constant.

Transformation to shearing sheet coordinates \( \hat{x} = x, \hat{y} = y - 2Ax \) and \( \hat{t} = t \) yields
\[ \partial_{\hat{x}} = \partial_x - 2A\partial_y, \]
\[ \partial_{\hat{y}} = \partial_y, \quad (59) \]
\[ \partial_{\hat{t}} = \partial_t - 2A\partial_x. \]

With a Fourier transform in the \( y \)-direction and a WKB approximation in \( x \), the linearized equations are (cf. Bender & Orszag 1978)
\[ \partial_t \Sigma_i = -i\Sigma_0 k_x v_{x,i} - i\Sigma_0 k_y v_{y,i,1}, \]
(60)
\[ \partial_t v_{x,i} = -i\left( \frac{\gamma - 1}{\Sigma_0} k_x e_1 + 2\Omega_{K,0} v_{x,i,1}, \right), \]
(61)
\[ \partial_t v_{y,i,1} = -2A k_x v_{x,i,1} - i\left( \frac{\gamma - 1}{\Sigma_0} k_y e_1 - 2\Omega_{K,0} v_{x,i,1}, \right), \]
(62)
\[ \partial_t p^{+}_{z,i,1} = \frac{(\gamma - 1)}{\sqrt{2\pi H_e}} e_1 - \frac{2\Omega_{K,0} \Sigma_0}{\sqrt{2\pi}} H_1 - \frac{\Omega_{K,0}^2 H_2}{\sqrt{2\pi}} \Sigma_1, \]
(63)
\[ \partial_t H_1 = -v_{x,i,1} \frac{dH_1}{dr_0} + \frac{\Omega_{K,0}^2}{\Sigma_0} p^{+}_{z,i,1}, \]
(64)
\[ \partial_t e_1 = -ir_0 e_0 k_z v_{x,i,1} - ir_0 e_0 k_y v_{y,i,1} - (\gamma - 1) \sqrt{2\pi} \left( \frac{r_0 e_0}{\Sigma_0} \right) \frac{d^2 \hat{H}}{dr_0^2} p^{+}_{z,i,1}, \]
(65)
where the ‘wavenumber’ \( k_z \) is time-dependent:
\[ k_z(t) = k_{z,0} - 24At. \]
(66)

The problem reduces to an algebraic one in the axisymmetric case \( (k_y = 0) \). Then \( k_y = k_{z,0} \) is independent of time. We derive the following dispersion relation in dimensionless form:
\[
\hat{\omega}^4 - \hat{\omega}^2 \left( \hat{k}^2 + (\gamma + 1) + \gamma \hat{k}_z^2 \right) + \left( \hat{k}^2(1 + \gamma) + (3\gamma - 1)\hat{k}_z^2 \right) = 0,
\]
(67)
where \( \hat{\omega} = \omega/\Omega_{K,0}, \hat{k} = k/\Omega_{K,0} \) and \( \hat{k}_z = k_z H_e \). Here \( \kappa \) denotes the epicyclic frequency
\[ \kappa = \sqrt{4\Omega_{K,0}(A + \Omega_{K,0}). \] (68)

The dispersion relation is plotted in Fig. 4.

We compare this dispersion relation with a full description of a local isothermal accretion disc. Lubow & Pringle (1993) investigated the propagation of waves in isothermal accretion discs in a shearing sheet approximation. They paid considerable attention to the axisymmetric case.

We integrate equations (33) and (34) in Lubow & Pringle (1993). We take the upper boundary at \( z = 5H_e \), and apply a free surface condition there:
\[ \frac{dP_1}{dt} + v_{z,1} \frac{dP_0}{dz} \bigg|_{z=5H_e} = 0. \]
(69)

The spectrum has two series of modes, one corresponding to the p-modes of stars, and one to the g-modes. In Lubow & Pringle (1993) the g-mode spectrum was eliminated by the regularity condition they imposed at \( z = \infty \). With our free surface, this part of the spectrum survives. For the lowest modes, the dispersion relation is only very slightly influenced by the choice of the boundary condition, since the density in the disc at heights of the order 5 scaleheights is already so low that these regions contribute very little of the mode inertia.

The numerical results are shown in Fig. 4, together with the two wave modes from dispersion relation (67), for \( \gamma = 1.4 \) and \( \kappa = 1. \) We identify the curve in big dots and the upper branch of the dispersion relation with a modified acoustic pressure wave. The spectrum of the higher harmonics which are at higher frequencies are, however, not shown in the plot.

The curve plotted in triangles and the lower branch of the dispersion relation is an axisymmetric acoustic wave, a mode previously described by, e.g., Goldreich & Tremaine (1979). For long wavelengths, it approaches the epicyclic frequency; at lowest order in \( k^2 \) it is given by
\[ \omega^2 = \kappa^2 + \frac{\gamma \Pi}{\Sigma} k_z^2. \]
(70)

In two-dimensional calculations with fixed disc thickness this is the only mode.

In addition to the study of Lubow & Pringle (1993), we find surface waves (Fig. 5 and small dots in Fig. 4, the number of nodes

Figure 4. Dispersion relation for axisymmetric waves in the one-zone model (solid lines). The modes of a full vertically stratified disc atmosphere are shown for comparison (dotted lines).

Figure 5. The momentum \( p_\perp(z) \) for surface waves and \( \hat{k} = 1 \) in modes 1–4 of an isothermal disc. The momentum peaks near 2 disc scaleheights. Towards the disc surface (here taken at 5 pressure scaleheights) \( p_\perp(z) \) decreases but does not vanish exactly as we apply a free boundary condition.
Hydrodynamics of accretion discs of variable thickness

6 NUMERICAL PROCEDURE

The partial differential equations (31), (32), (33), (38), (39) and (41) describe a two-dimensional, time-dependent hydrodynamic problem. Such problems are often within reach of accurate numerical simulations. In the case of accretion discs, however, the problem still presents a significant challenge. On the one hand, the flow in a disc is hypersonic, so that a method must be chosen that treats shock waves adequately without excessive computational overhead. On the other hand, the flow in a disc is hypersonic, so that a method must be chosen that treats shock waves adequately without excessive computational overhead.

The comparison shows that, for long wavelengths $\lambda \gg H$, our model represents the behaviour of linear wave modes well. This supports the applicability of the one-zone approximation for the study of global wave modes in thin discs.

6.1 The disc thickness step

The disc thickness, vertical momentum $p^+_{\phi}$ and the internal energy $e$ are updated in this step by integrating equations (43), (44) and (45) in time with a fourth-order Runge–Kutta method with adaptive step size control. The equations are integrated with an accuracy of $\Delta e = \Delta p^+/p^+ = \Delta H/H \leq 10^{-6}$. Usually the overall time-step is a factor of 2–3 larger than the time-step in the Runge–Kutta integration. Note that equations (44) and (45) imply equation (49). Thus the disc thickness step is adiabatic.

In some test cases we checked our integration also for the conservation of the total energy (48), which is shown to be fulfilled within the applied accuracy of the integration. We finally note that this part of the solution is about as computationally expensive as the advection or the source step.

7 TEST CALCULATIONS

For comparison with existing results we first show results for three standard tests. Following this, the integration of the disc thickness equations is tested in Section 7.4.

7.1 Advection test

We tested the transport step for different geometries. Fig. 6 shows a one-dimensional advection test, in which a square density enhancement with a width of 50 grid points is shown after travelling over $\sim 180$ grid cells. The initial discontinuity has widened to 50 grid points in 80 grid cells, which is comparable to other published codes based on a van Leer (1977)-type staggered grid.

7.2 Shock tube test

Sod’s (1978) shock tube experiment has become a standard test case for hydrodynamic algorithms. For the initial conditions and an analytic solution we refer to Courant & Friedrichs (1948)
Hawley et al. (1984). Fig. 7 shows the result for a grid with 100 points, for an artificial viscosity coefficient of 1. The variables $\Sigma$, $\Pi$, $\epsilon$ and $v$ are plotted and compared to the analytic solution. The accuracy of the numerical results is quite similar to that of other codes of this type.

7.3 Pressure-free collapse of a sphere

In this test we changed the metric coefficients (Stone & Norman 1992) to test our code in spherical symmetry with the pressure-free collapse of a self-gravitating sphere. Hunter (1962) derives an analytic solution for this problem. The integration is shown (Fig. 8) after a free-fall time $t = 0.543$. We resolve a linear velocity distribution (squares, labels at the right-hand axis) in the regime of the collapsing cloud. The density (filled dots, left-hand axis) is nearly constant. Only in the innermost two grid zones is the density somewhat increased.

7.4 Disc thickness oscillations

This is a test of the applicability of our one-zone approximation for vertical disc oscillations of moderate amplitudes.

We study the time evolution of an isothermal, non-

![Figure 6. One-dimensional advection test. A density enhancement 50 points wide is shown after it has travelled over ~180 grid points. The initial discontinuity has widened to ~12 grid cells, comparable to other staggered grid codes.](https://academic.oup.com/mnras/article-abstract/304/3/674/1023976)

![Figure 7. The numerical integration of Sod’s (1978) shock tube problem compared (dots) to the analytic solution for $\Sigma$, $\epsilon$, $v$ and $\epsilon$.](https://academic.oup.com/mnras/article-abstract/304/3/674/1023976)

![Figure 8. The pressure-free collapse of a self-gravitating sphere after a free-fall time $t = 0.543$. We resolve a linear velocity distribution (squares, labels at the right-hand axis) in the regime of the collapsing cloud. The density (filled dots, left-hand axis) is nearly constant. Only in the innermost two grid zones is the density somewhat increased.](https://academic.oup.com/mnras/article-abstract/304/3/674/1023976)
self-gravitating, Gaussian density distribution in an external gravity field \( g_z \sim z \). A one-dimensional integration of this problem is done in full hydrodynamics and compared with the one-zone approximation to the same problem. The computational domain is \( 0 < z < 6H_e \), where \( H_e \) is the equilibrium pressure scaleheight of the disc. The initial state is obtained from the equilibrium density distribution by a uniform adiabatic expansion by a factor 1.1. Fig. 9 shows the results after 2.18 oscillations. It is seen that the density distribution remains Gaussian, and that the velocity is linear with \( z \).

The internal energy per unit mass \( \epsilon \) varies with time during the oscillation, but remains independent of height.

In Fig. 10 we show the time evolution of the disc thickness \( H \) and the internal energy per unit mass \( \epsilon \) for the hydrodynamic calculation (dots) and the one-zone model (full line). The agreement shows that the one-zone approximation is an excellent representation of the fundamental mode of disc thickness oscillations in the long-wave \( k_h \to 0 \) limit, at moderate amplitudes.

8 AXISYMMETRIC CALCULATIONS

We study waves in a disc with constant temperature and a surface density \( \Sigma \) varying as \( r^{-3/4} \). The Mach number of the orbital motion is of the order 100.

8.1 Standing waves

We follow a low-amplitude perturbation of a disc with inner and outer radii \( r_{in} = 0.9 \) and \( r_{out} = 1 \). The internal energy is perturbed at \( t = 0 \) with a wavelength \( \lambda = 0.02 \), i.e., \( 1/5 \) of the computational domain. The time evolution is shown in Fig. 11, where \( \epsilon/\Sigma, p_\perp, p_z \).
and $H/r^{3/2}$ are plotted as functions of $r$ and $t$ in units of the Kepler period at $r = r_{\text{out}}$.

Fig. 11 shows that two waves with different frequencies coexist. They correspond to the two branches of dispersion relation (67) (see Fig. 4). The high-frequency pressure wave is best seen in $p_\tau$, and the low-frequency surface wave in $H$ and $p_\tau^+$. Time in units of the orbital period at $r = 1$.

8.2 Broadening of wave packet

In a second calculation we follow a perturbation which initially is localized in radius at a single grid point. In this example the disc has inner and outer edges at $r_{\text{in}} = 0.1$ and $r_{\text{out}} = 1.0$. At $t = 0$ the disc thickness is increased by 22 per cent at $r = 0.55$. Fig. 12 shows the time evolution of the temperature $\epsilon/\Sigma$ of this perturbation. The wave is seen to spread outward. In Fig. 13 we plot the evolution of the disc thickness at the location of the initial perturbation $r = 0.55$.

The vertical oscillation excites waves which travel in radial direction. In this way energy is removed from the disc oscillation at $r = 0.55$, and the amplitude at this point decreases. The initial time-scale for this damping is of the order of 8.5 times the orbital period at that point. As the frequency of the fundamental mode $\omega(r) = \sqrt{1 + \gamma\Omega_K(r)}$ decreases with disc radius, the amplitude of the wave excited at $r = 0.55$ is damped as the wave travels inward. Note that there is no loss of total energy in the process, since dissipation by shock waves is negligible at these amplitudes and no radiative damping was included.

9 DISCUSSION

We have extended the numerical investigation of geometrically thin accretion discs in two dimensions by removing the restricting assumption of a fixed disc thickness or a fixed opening angle.

Instead, we derive, from the vertical equation of motion, two additional equations which allow us to follow the time dependence of the disc thickness at each point on the disc surface.

We consider our approach as an economical next step towards greater realism in numerical disc hydrodynamics, as long as full three-dimensional studies of accretion discs are unpractical with currently available CPU power.

We have tested our equations against the local description of a stratified disc by Lubow & Pringle (1993) for the axisymmetric case by comparing the dispersion relations. Our model reproduces the two lowest modes of vertical motion, a pressure wave and a surface wave. We have shown that, in the limit of small wavenumbers, the properties of these waves agree with those of a fully stratified disc.

In the limit $k_r \to 0$ the oscillation frequency is given by $\omega_z = \sqrt{1 + \gamma\Omega_K}$, where $\Omega_K$ is the Keplerian angular velocity. This agrees with the frequency of the fundamental mode of vertical disc oscillations as given by Lubow (1981).

We note that our treatment of the disc thickness and its time evolution with only two additional variables can reproduce only the fundamental reaction of the disc in the vertical direction. Modes of higher order in $z$ or vertical convective instabilities are not represented in the model. Similarly, we have to postpone the investigation of waves with more complicated vertical structure to full three-dimensional studies of the hydrodynamic equations.

The tidal perturbation by a companion star excites variations in disc thickness. As previously studied by Lubow (1981), we find by neglecting horizontal wave propagation a resonance at a radius between 0.25 and 0.35 in units of the binary separation, dependent on the mass ratio $q = M_1/M_2$. This is well within the size of observed discs in binary systems, e.g., in cataclysmic variables.

We have presented a method for two–dimensional time-dependent numerical simulations, in the spirit of existing astrophysical hydro codes, but extending them to include the time dependence of the
In the linear regime, our results are comparable with those of Lubow (1981) and Lubow & Pringle (1993). In the non-linear regime, test calculations include advection, shock tube and spherical collapse calculations for the conventional part of the hydro code. Two wave propagation tests for discs with time-dependent thickness demonstrate the behaviour of the two wave modes.

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REFERENCES

Figure 12. The reaction of an axisymmetric disc upon an initial increase of the disc thickness by 22 per cent at a radius of \( r = 0.55 \). The initially localized oscillation excites waves which travel to the outer disc edge.

Figure 13. The time evolution of the disc thickness at \( r = 0.55 \) in the calculation of Fig. 10.
APPENDIX: TRANSFORMATION OF THE ULTRASPHERICAL POLYNOMIALS

The ultraspherical polynomials $C_n^{(3/2)}$ for $|x| < 1$ have the generating function

$$
\frac{1}{(1 + x^2 - 2x \cos \phi)^{3/2}} = \sum_{n=0}^{\infty} C_n^{(3/2)}(\cos \phi)x^n. \quad (A1)
$$

The series expansion of $C_n^{(3/2)}$ is (Abramowitz & Stegun 1965):

$$
C_n^{(3/2)} = \sum_{m=0}^{n} \frac{\Gamma(3/2 + m)\Gamma(3/2 + n - m)}{m!(n-m)!\Gamma(3/2)^2} \cos(n - 2m)\phi, \quad (A2)
$$

where $\Gamma(x)$ denotes the gamma function. Inserting equation (A2) into equation (A1) and rearranging the order of summation results in

$$
\frac{1}{(1 + x^2 - 2x \cos \phi)^{3/2}} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\Gamma(3/2 + j)\Gamma(3/2 + l + j)}{j!(i+j)!\Gamma(3/2)^2} \cos(i\phi)x^{i+2j}. \quad (A3)
$$

A more elegant form is

$$
\frac{1}{(1 + x^2 - 2x \cos \phi)^{3/2}} = L_0^{(3/2)}(x) + 2 \sum_{m=1}^{\infty} L_m^{(3/2)}(x) \cos(m\phi), \quad (A4)
$$

with

$$
L_m^{(3/2)}(x) = \sum_{j=0}^{m} \frac{\Gamma(3/2 + m + j)\Gamma(3/2 + l)}{(n+j)!(\Gamma(3/2)^2)} x^{n+2j}. \quad (A5)
$$

These equations are used in Section 4.2.

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