Magnetic shear-flow instability in thin accretion discs

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ABSTRACT

The possibility that the magnetic shear-flow instability (also known as the ‘Balbus–Hawley’ instability) might give rise to turbulence in a thin accretion disc is investigated through numerical simulations. The study is linear and the fluid disc is supposed to be incompressible and differentially rotating with a simple velocity profile with \( \Omega \propto R^{-q} \). The simplicity of the model is counterbalanced by the fact that the study is fully global in all three spatial directions with boundaries on each side; finite diffusivities are also allowed. The investigation is also carried out for several values of the azimuthal wavenumber of the perturbations in order to analyse whether non-axisymmetric modes might be preferred, which may produce, in a non-linear extension of the study, a self-sustained magnetic field.

We find the final pattern steady, with similar kinetic and magnetic energies and the angular momentum always transported outwards. Despite the differential rotation, there are only small differences for the eigenvalues for various non-axisymmetric eigensolutions. Axisymmetric instabilities are by no means preferred; in fact for Prandtl numbers between 0.1 and 1, the azimuthal wavenumbers \( m = 0, 1, 2 \) appear to be equally readily excited. The equatorial symmetry is quadrupolar for the magnetic field and dipolar for the flow field system. The maximal magnetic field strength required to cause the instability is almost independent of the magnetic Prandtl number. With typical white dwarf values, a magnetic amplitude of \( 10^5 \) G is estimated.

Key words: accretion, accretion discs – instabilities – MHD.

1 INTRODUCTION

The long-standing problem of the generation of turbulence in various astrophysical situations, in which ordinary microscopic viscosities seem to be unable to produce the observed instabilities, has perhaps found a solution in recent years in the so-called ‘Balbus–Hawley instability’, in which the presence of a magnetic field has a destabilizing effect on a differentially rotating flow, provided that the angular velocity decreases outwards with the radius (Velikhov 1959; Chandrasekhar 1961). After the first work by Balbus & Hawley (1991), several other studies have been carried out to study the non-linear evolution of the instability (Hawley & Balbus 1991; Brandenburg et al. 1995; Hawley, Gammie & Balbus 1995; Matsumoto & Tajima 1995), but because of the complexity of non-linearities, all of these studies were aimed at understanding the ‘local’ properties of the instability. Recent investigations, however, have pointed out the importance of boundaries on the instability growth rates (Curry, Pudritz & Sutherland 1994; Curry & Pudritz 1995, 1996; Kitchatinov & Rüdiger 1997; Kitchatinov & Mazur 1997). All these studies were linear, a fully global non-linear approach being in its infancy (Drecker et al. 1997).

We investigate here the effect of the instability in a simple numerical model for a slim accretion disc in which a magnetized fluid is contained between two finite (in the axial direction) cylinders and is differentially rotating with a velocity profile \( \Omega \propto R^{-q} \), \( R \) being the radius in cylindric coordinates \( (R, \phi, Z) \). We deal with three cases: \( q = 1, q = 1.5 \) and \( q = 2 \). All these flows are stable (only marginally in the case \( q = 2 \)) with respect to the standard hydrodynamic Rayleigh criterion for Taylor–Couette flow, in such a way that an eventual instability is caused by the presence of the magnetic field. The choice \( q = 2 \) corresponds to a constant specific angular momentum along the radius between the two cylindrical surfaces. Such an angular velocity profile seems to fit the differential rotation for thin discs well (Papaloizou & Pringle 1984) while a Keplerian law (\( q = 1.5 \)) should be more suitable for thin discs. It is generally accepted that the index \( q \) in the rotation law depends in turn on the position and that in some zones of the disc, e.g. in the inner part, a profile with \( q = 2 \) is not unrealistic at all (Abramowicz, Brandenburg & Lasota 1996). In addition, the fluid is supposed to be incompressible, to get rid of the numerical problems related to the presence of sound waves. The system is embedded in an external vertical, uniform magnetic field that should drive the instability.

Flow and field perturbations are developed after the azimuthal Fourier modes \( \exp(i n \phi) \). Although the flow and field quantities...
2 EQUATIONS AND MODEL

We solve the incompressible, dissipative, linearized MHD equations completely numerically in cylindrical coordinates for a global model. The basic rotation law is

$$\Omega = \Omega_0 \left( \frac{R_0}{R} \right)^q e_\phi,$$

(1)

with $R_0$ as the inner radius of the disc, and the rotating fluid may be threaded by a strictly vertical magnetic field,

$$B_0 = B_0 e_z,$$

(2)

consistent (as a very special case) with Ferraro’s law (cf. Stone & Norman 1994). Extra radially dependent azimuthal fields are also possible in this approximation (Curry & Pudritz 1995; Ogilvie & Pringle 1996; Terquem & Papaloizou 1996; Papaloizou & Terquem 1997). By linearizing around this equilibrium state and using dimensionless quantities, our equations read

$$\frac{du}{dt} = C_\Omega (-A - \nabla p) + \left( \frac{\text{Ha}^2 \text{Pm}}{C_\Omega} \right) \mathcal{L} + \text{Pm} \Delta u,$$

(3)

$$\Delta p = \nabla \left[ -A + \left( \frac{\text{Ha}^2 \text{Pm}}{C_\Omega} \right) \mathcal{L} + \frac{\text{Pm}}{C_\Omega} \Delta u \right],$$

(4)

$$\frac{\partial B}{\partial t} = \text{curl}[C_\Omega (\mathcal{R} + \mathcal{E}) - \text{curl}B],$$

(5)

where $p$ denotes the pressure, $u$ the velocity perturbation and $B$ the magnetic perturbation.

In the equations above, $A$ and $\mathcal{L}$ are the linearized advection term and Lorentz force given by

$$A = \frac{\Omega}{\Omega_0} \left( \begin{array}{c} \text{imu}_r - 2u_\phi \\ \text{imu}_\phi + (2 - q)u_R \\ \text{imu}_z \end{array} \right)$$

(6)

and

$$\mathcal{L} = \left( \begin{array}{cc} \frac{\partial B_R}{\partial z} & \frac{\partial B_z}{\partial r} \\ - \frac{im}{r} B_z + \frac{\partial B_\phi}{\partial z} & 0 \end{array} \right),$$

(7)

while

$$\mathcal{R} = \frac{\Omega}{\Omega_0} \left( \begin{array}{c} B_z \\ 0 \\ -B_R \end{array} \right).$$

(8)


3 RESULTS

3.1 The neutral-stability lines

The neutral-stability lines for $m < 3$ in the $C_\Omega$–Ha plane for $q = 1$ and $q = 2$ and various Pm are given in Fig. 1 for rigid boundary...
conditions. As there is little qualitative change with \( q \), we only show results for \( q = 1.5 \) for the stress-free case in Fig. 2. Above the curves the rotation law is unstable while below the eigenvalues the stratification is stable. Instabilities in a disc with stress-free boundary conditions are excited more easily than in a rigid-boundary disc. A direct comparison with the plots of Kitchatinov & Mazur (1997) for \( m \) = 0 shows that they are similar to ours, in spite of the difference in the definitions. We checked that the curves have no intersection with the \( C_\Omega \)-axis, because we found the energy decreasing exponentially for \( \Ha = 0 \). This is because, in the absence of a magnetic field, the basic flow is linearly stable by the Rayleigh criterion.

We find for higher values of the Hartmann number (for stronger magnetic fields) either the mode \( m = 1 \) or \( m = 2 \) with slightly lower excitation eigenvalues. As a main result of the present study, very similar excitation conditions for both axisymmetric and non-axisymmetric modes are found. It is worth noting that in the case of the sphere (another case in which the fluid is bounded in all directions) Kitchatinov & Rüdiger (1997) found that the mode \( m = 1 \) was more easily excited for a strong magnetic field. Also the mode \( m = 2 \) was found to be preferred to the axisymmetric one, though only for a rather strong magnetic field.

Fig. 1 yields linear relations for the Hartmann number as a function of the ‘dynamo number’ such as

\[
\Ha_{\text{max}} = \varepsilon C_\Omega. \tag{12}
\]

for the maximal possible Hartmann number. \( \varepsilon \) is of order unity (\( \varepsilon \approx 0.8 \)) for \( \Pm = 1 \). It is possible to see from the figure that the scaling of the quantity \( \varepsilon \) with \( \Pm \) is approximately

\[
\varepsilon \propto \Pm^{-0.5}. \tag{13}
\]

If the curves in Fig. 1 are taken for \( C_\Omega/\sqrt{\Pm} \) as a function of \( \Ha \), the neutral-stability lines are very close together for all values of both \( \Pm \) and \( m \), even for the (few) examples obtained for \( \Pm = 0.01 \). Note that in this formulation the critical number for the onset of instability proves to be

\[
C_\Omega / \sqrt{\Pm} = \frac{\Omega_0 H^2}{\sqrt{\mu_0}}. \tag{14}
\]

In terms of the Alfvén velocity \( V_A \) and the speed of sound \( c_{ac} \) is

\[
V_{A,\text{max}} \propto \Omega_0 H = c_{ac}. \tag{15}
\]

Figure 1. Neutral-stability lines for various azimuthal wavenumbers \((m = 0, 1, 2)\) and rigid boundary conditions. The parameters are \( q = 1 \) (left) and \( q = 2 \) (right). \( r_1 = 30, \Pm = 0.1 \) and \( \Pm = 1 \). The isolated symbols for \( q = 2 \) denote simulations for the small Prandtl number \( \Pm = 0.01 \).

Figure 2. Neutral-stability lines for various azimuthal wavenumbers \((m = 0, 1, 2)\) and stress-free boundary conditions. A Keplerian rotation law is used \((q = 1.5)\). Again, \( r_1 = 30, \Pm = 0.1 \) and \( \Pm = 1 \).

The sound velocity and also the radial density profile are known for the standard accretion disc model (cf. Frank, King & Raine 1985), hence

\[
B_{\text{max}} = \sqrt{\mu_0 \rho c_{ac}}. \tag{16}
\]

The sound velocity and also the radial density profile are known for the standard accretion disc model (cf. Frank, King & Raine 1985), hence

\[
B_{\text{max}} = 1.4 \times 10^4 \alpha_{\text{SS}}^{-9/20} R_6^{-21/16} M_{16}^{17/40} M_{16}^{7/16} G \tag{17}
\]

in terms of \( R_6 = R/(10^6 \text{ cm}) \), \( M_1 = M/M_\odot \) and \( M_{16} = M/(10^{16} \text{ g s}^{-1}) \). All three quantities are of the order of unity for discs around white dwarfs. The definition of the viscosity parameter \( \alpha_{\text{SS}} \) is given below in (19). With its standard value of 0.01, the maximal possible magnetic field for the Balbus–Hawley instability is

\[
B_{\text{max}} = 10^5 G, \tag{18}
\]

in accordance with the observed surface field strength for white dwarfs of 10$^{3.9}$ G. After (16), the ratio of the gas pressure to the maximally possible magnetic pressure is of order unity.
3.2 The angular momentum transport

Although our study is linear, it is worth analysing the efficacy of the instability for the angular momentum transport. The effective viscosity that should bring the angular momentum transport is generally parametrized through the quantity \( \alpha_{SS} \) (Shakura & Sunyaev 1973), defined in terms of the Reynolds and Maxwell stress tensors by

\[
\left< \frac{u_R u_\phi - B_R B_\phi}{\mu_0 \rho} \right> = - \nu_T R \frac{\partial \Omega}{\partial R} \tag{19}
\]

with the normalization \( \nu_T = \alpha_{SS} H^2 \Omega \),

where \( \alpha_{SS} \) is called the viscosity alpha. Hence

\[
\left< \frac{u_R u_\phi - B_R B_\phi}{\mu_0 \rho} \right> = q \alpha_{SS} H^2 \Omega^2, \tag{20}
\]

i.e. in dimensionless units

\[
\alpha_{SS} = \frac{1}{q} \left( \frac{\Omega_0}{\Omega} \right)^2 \left< \frac{u_R u_\phi - \frac{H_0^2 \rho m}{C_s} B_R B_\phi}{B_R B_\phi} \right> \tag{21}
\]

(Brandenburg et al. 1996). In our linear approach, the amplitudes of all the quantities are unknown as they are free of a common positive or negative factor. Hence, the sign of quadratic terms can be computed, as well as ratios of fluctuating quantities.

In Fig. 3, the calculated values of \( \alpha_{ss} \) are shown for a point on the marginally stable curves for the modes \( m = 0 \) and \( m = 2 \), respectively. It is normalized to unity. The averages are taken in the azimuthal and vertical directions, so that \( \alpha_{ss} \) is a function of the radius. In both cases \( \alpha_{ss} \) is positive, meaning that the instability is effective in transporting angular momentum outwards. Owing to the fact that the eigenfunctions are concentrated close to the inner boundary of the integration domain, the correlation between the \( R \) and \( \phi \) components of \( u \) and \( B \) is non-vanishing only in proximity to \( r_\Omega \).

It is also possible to give an estimate of the relative strength of the Reynolds stress in comparison with the Maxwell stress by measuring the quantity

\[
\gamma = \frac{C_s}{H_0 \rho m} \left< \frac{u_R u_\phi - \frac{H_0^2 \rho m}{C_s} B_R B_\phi}{B_R B_\phi} \right> \tag{22}
\]

plotted in Fig. 4. The sign is positive in both cases, meaning that both the Reynolds and Maxwell stresses are working to transport the angular momentum in the same direction. The maximum \( \gamma \) is of order 1.0 to 2.0 so that the contributions of the velocity field and magnetic stress to this transport are similar.

The same can be done with the total energy. The ratio of kinetic and magnetic energy (in our units) is

\[
\Gamma = \frac{C_s^2}{H_0^2 \rho m} \left< \frac{u^2}{B^2} \right>, \tag{23}
\]

taken over the complete cylinder. The result for the same model as that in Figs 3 and 4 is plotted in Fig. 5. The magnetic energy slightly dominates the kinetic energy.

3.3 The eigenfunctions

In Fig. 6, the induced toroidal magnetic field is shown. It is, obviously, an equatorially symmetric (‘S’) field. Correspondingly, the associated zonal kinetic flow has an antisymmetric structure.
with respect to the equator (Fig. 7). The induced fields and flows here are concentrated at the inner boundary.

The zonal flow with stress-free boundary conditions is very similar to the flow with rigid boundary conditions shown in Fig. 7. Those contour lines, which are strongly bent near the upper and lower boundaries of the disc in Fig. 7, are just open in the \( z \)-direction in the stress-free case. The similarity of flow patterns helps us to understand the similarity of the respective neutral-stability lines in Figs 1 and 2.

The most important question relevant to such eigenfunctions is the fate of electrically charged dust grains in the flow field system. Do they concentrate in the equatorial plane or not under the influence of the various electromagnetic forces? The ionization states of protoplanetary discs have been studied by Dolginov & Stepinski (1994) and Stepinski & Valageas (1996).

4 A DISC DYNAMO MODEL?

Our numerical simulations of thin accretion discs revealed the existence of stationary laminar flow and field patterns for a certain amplitude range of external magnetic fields. There are even indications that for stronger magnetic fields (increasing Hartmann number) non-axisymmetric modes are more easily excited. Owing to the Cowling theorem, only these modes could excite, in a non-linear regime, a self-maintained axisymmetric magnetic field, so that the results seem to be promising in view of a possible non-linear extension of the code for investigating the possibility of a dynamo effect. After our experiences with large-scale dynamo models, however, the existence of a vertical density stratification should be necessary for dynamo self-excitation (cf. Brandenburg et al. 1995; Hawley, Gammie & Balbus 1996; Stone et al. 1996).

Before moving to a non-linear study, however, there are several possible improvements that could be made. The velocity profile could be generalized and different choices for the structure of the external background magnetic field (e.g. a dipole) may be made. A forthcoming extension of the study concerns the choice of the parameters in a model that fits a more ‘realistic’ situation. Runs are in progress with different values of the \( r_{10} \) parameters to simulate thinner accretion disc structures and values of the Prandtl number of \( \text{Pm} \approx 10^{-2} \), closer to typically estimated values for accretion discs.

Let us finally stress that the dominance of the mode \( m = 2 \) for the higher Hartmann numbers shown in Figs 1 and 2 could be relevant in the case of a galactic dynamo, although the intrinsic limitations of our model prevent us from any direct conclusion.

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