Resonant tidal excitations of rotating neutron stars in coalescing binaries

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Abstract
In a coalescing neutron star–neutron star or neutron star–black hole binary, oscillation modes of the neutron star can be resonantly excited by the companion during the final minutes of the binary inspiral, when the orbital frequency sweeps up from a few Hz to a few thousand Hz. The resulting resonant energy transfer between the orbit and the neutron star speeds up or slows down the inspiral, depending on whether the resonant mode has positive or negative energy, and induces a phase change in the emitted gravitational waves from the binary. While only g-modes can be excited for a non-rotating neutron star, f-modes and r-modes can also be excited when the neutron star is spinning. A tidal resonance, designated by the index \((jk,m)\) \((\{jk\}\) specifies the angular order of the mode, as in the spherical harmonic \(Y_{jk}\)), occurs when the mode frequency equals \(m\) times the orbital frequency. For the f-mode resonance to occur before coalescence, the neutron star must have rapid rotation, with spin frequency \(n_s \approx 710\) Hz for the \((22,2)\)-resonance and \(n_s \approx 570\) Hz for the \((33,3)\)-resonance (assuming canonical neutron star mass, \(M = 1.4\), and radius, \(R = 10\) km; however, for \(R = 15\) km, these critical spin frequencies become 330 and 260 Hz, respectively). Although current observations suggest that such high rotation rates may be unlikely for coalescing binary neutron stars, these rates are physically allowed. Because of their strong tidal coupling, the f-mode resonances induce a large change in the number of orbital cycles of coalescence, \(\Delta N_{orb}\), with the maximum \(\Delta N_{orb} \approx 10\)–1000 for the \((22,2)\)-resonance and \(\Delta N_{orb} \approx 1\) for the \((33,3)\)-resonance. Such f-mode resonant effects, if present, must be included in constructing the templates of waveforms used in searching for gravitational wave signals. Higher order f-mode resonances can occur at slower rotation rates, but the induced orbital change is much smaller (\(\Delta N_{orb} \approx 0.1\)). For the dominant g-mode \((22,2)\)-resonance, even modest rotation (\(n_s \lesssim 100\) Hz) can enhance the resonant effect on the orbit by shifting the resonance to a smaller orbital frequency. However, because of the weak coupling between the g-mode and the tidal potential, \(\Delta N_{orb}\) lies in the range \(10^{-3}\)–\(10^{-2}\) (depending strongly on the neutron star equation of state) and is probably negligible for the purpose of detecting gravitational waves. Resonant excitations of r-modes require misaligned spin–orbit inclinations, and the dominant resonances correspond to octopolar excitations of the \(j = k = 2\) mode, with \((jk,m) = (22,3)\) and \((22,1)\). Since the tidal coupling of the r-mode depends strongly on rotation rate, \(\Delta N_{orb} \approx 10^{-2}(R/10\,\text{km})^{10}(M/1.4\,M_\odot)^{-20/3}\) is negligible for canonical neutron star parameters, but can be appreciable if the neutron star radius is larger.

Key words: gravitation – hydrodynamics – binaries: close – stars: neutron – stars: oscillations – stars: rotation.

1 Introduction
The new generation of gravitational wave detectors such as the Laser Interferometer Gravitational Wave Observatory (LIGO) (Abramovici 1992) and its French-Italian counterpart VIRGO (Bradaschia et al. 1990) are designed to observe gravitational radiation in the frequency range from \(\sim 10\)–1000 Hz. Coalescing neutron star–neutron star (NS–NS) and neutron star–black hole (NS–BH) binaries are promising sources for such radiation, with the last few minutes of the binary inspiral producing gravitational waves which sweep upward through this frequency range (e.g. Thorne 1987; Cutler et al. 1993; Thorne 1998). Due to the low signal-to-noise ratios, a good understanding of the mechanisms...
that generate and affect the gravitational waveform is required to create accurate theoretical templates that can be used to extract the signals from the noise.

The binary inspiral and the resulting waveform can be described, to leading order, by Newtonian dynamics of two point masses, together with the lowest order dissipative effect corresponding to the emission of gravitational radiation via the quadrupole formula. To construct more accurately the theoretical templates, general relativistic effects must be taken into account by using post-Newtonian expansions (e.g. Blanchet et al. 1995) or techniques of numerical relativity (e.g. Brady, Creighton & Thorne 1998; Teukolsky 1999). Other corrections come from hydrodynamical effects due to the finite size of neutron stars (Bildsten & Cutler 1992; Kochanek 1992; Lai, Shapiro & Shapiro 1994). In particular, tidal interactions can have significant effects on the final stage of binary inspiral by destabilizing the orbit and accelerating the coalescence at small orbital radii (e.g. Lai et al. 1994; Lai & Shapiro 1995; Rasio & Shapiro 1995; Lai & Wiseman 1996; Baumgarte et al. 1997; New & Tohline 1997; Uryu & Eriguchi 1998; see also Shibata, Nakamura & Oohara 1993, Davies et al. 1994, Zhuge, Centrella & McMillan 1996 and Ruffert et al. 1997 for numerical simulations on binary coalescence).

The early stage of the inspiral, with the gravitational wave frequencies in the range of 10 Hz to a few hundred Hz, is particularly important for detecting the wave signal and for extracting binary parameters from the waveform using the matched filtering technique (e.g. Cutler et al. 1993; Cutler & Flanagan 1994). It is usually thought that tidal effects are completely negligible for this ‘low-frequency’ regime. This is indeed the case for the ‘quasi-equilibrium’ tides, as the tidal interaction potential scales as $D^{-6}$ (where $D$ is the orbital separation; see Lai et al. 1994 for analytic expressions of the orbital phase error induced by static tides). The situation is more complicated for the resonant tides: as two compact objects inspiral, the orbit can momentarily come in resonance with the normal oscillation modes of the NS. By drawing energy from the orbital motion and resonantly exciting the modes, the rate of inspiral is modified, giving rise to a phase error in the gravitational waveform. This situation has been studied (Lai 1994; Reisegger & Goldreich 1994) in the case of non-rotating neutron stars, where the only modes that can be resonantly excited are g-modes (with typical mode frequencies $\lesssim 100$ Hz). It was found that the effect is negligible, because the coupling between the g-mode and the tidal potential is weak. However, the NS can be rotating, and thus these results need to be re-examined. As we show in this paper, even modest rotation of the NS can enhance the resonant g-mode excitation by shifting the resonance to a lower frequency (where the orbital decay is slower and resonance is stronger). In addition, f-modes, which possess strong tidal coupling, can have their normal high frequencies in the non-rotating case ‘dropped’ or reduced by retrograde rotation to sufficiently low frequencies in the inertial frame, such that resonant excitations can occur well before binary merger. The NS spin also gives rise to a new class of modes which are absent in the non-rotating case: r-modes, which can have low frequencies and may also undergo resonant excitations.

The purpose of this paper is to study the resonant excitations of various modes of spinning NSs in inspiraling binary systems and to examine their effects on the orbital decay rate (particularly the phase error in the gravitational waveform). Section 2 develops the general formalism of resonant tides in coalescing NS binaries. In Section 3 we apply the analysis to f-modes, g-modes, and r-modes when the spin axis is aligned or anti-aligned with the orbital angular momentum axis. Section 4 describes the modification to the results of the aligned (anti-aligned) case when the spin and orbital axes are misaligned by an inclination angle $\beta$. In Section 5 we summarize our results and discuss their implications for gravitational wave detection.

2 Resonant Mode Excitation

2.1 Basic equations

Consider a NS with mass $M$, radius $R$, and spin $\Omega$, in orbit with a companion $M'$, which can be another NS or a BH. We treat $M'$ as a point mass. The orbital frequency vector is $\Omega_{\text{orb}}$. We define a coordinate system $(x, y, z)$ centred on $M$ with the $z$-axis along $\Omega_{\text{orb}}$, such that $\Omega_{\text{orb}} = \Omega_{\text{orb}}^z$ (with $\Omega_{\text{orb}}^z = 0$). Define another coordinate system $(x', y', z')$ with the $z'$-axis along $\Omega_\kappa$, such that $\Omega_\kappa = \Omega_\kappa^z$ (with $\Omega_\kappa^z = 0$) and the $y'$-axis is in the orbital plane; these are the body axes. Let the angle between the $z$-axis and $z'$-axis be $\beta$ (the spin–orbit inclination angle) and the angle between the $y$-axis and $y'$-axis be $\alpha$. Thus the $(x', y', z')$ frame is related to the $(x, y, z)$ frame by Euler angle $(\alpha, \beta, \gamma = 0)$.

The gravitational potential produced by $M'$ can be expanded in terms of spherical harmonics:

$$
U(r, t) = -\frac{GM'}{|r - D(t)|} = -GM' \sum_{lm} W_{lm}(D(t)) Y_{lm}(\theta, \phi),
$$

(2.1)

where $r = (r, \theta, \phi)$ is the position vector of a fluid element in star $M$ ($r, \theta, \phi$ are the spherical coordinates with respect to the $z$-axis), $D(t) = |D(t)\pi/2, \Phi(t)|$ is the position vector of the point mass $M'$, and $W_{lm}$ is defined in Press & Teukolsky (1977) as

$$
W_{lm} = (-)^{(l+m)/2} \left[ \frac{4\pi}{2l+1} \left( \frac{l+m}{2} \right) \right]^{1/2} \times \left[ l^2 \left( \frac{(l+m)^2}{2} \right) - \left( \frac{l-m}{2} \right)^2 \right]^{1/2}.
$$

(2.2)

Here the symbol $(-)^p$ is zero if $p$ is not an integer. For example, $W_{2,2} = (3\pi/10)^{1/2}$ and $W_{2,3} = (5\pi/28)^{1/2}$. In equation (2.1), the $l = 0$ and $l = 1$ terms can be dropped, since they are not relevant for tidal deformation. The tidal potential is specified by the index $|lm|$ (relative to the orbital angular momentum axis $\hat{z}$) and is non-zero only when $(l+m)$ is even, reflecting the symmetry of the tidal potential.

It is more convenient to define oscillation modes relative to the spin axis $\hat{z}'$, and we will need to express the tidal potential in terms of $Y_{lm}(\theta', \phi')$, the spherical harmonic function relative to $\hat{z}'$. The function $Y_{lm}(\theta, \phi)$ is related to $Y_{lm}(\theta', \phi')$ by

$$
Y_{lm}(\theta, \phi) = \sum_{m'} \mathcal{D}^{(l)}_{m'm}(\alpha, \beta) Y_{lm'}(\theta', \phi'),
$$

(2.3)

where the Wigner $\mathcal{D}$-function is given by (e.g. Wybourne 1974)

$$
\mathcal{D}^{(l)}_{m'm}(\alpha, \beta) = e^{im'a} [(l+m)!(l-m)!(l+m')!(l-m')!]^{1/2} \times \sum_p \frac{(-1)^{l+m-p}}{p(l+m-p)} \left( \frac{\beta}{2} \right)^{2l-2m+p+m'} \left( \frac{\sin\beta}{2} \right)^{2l-2m+p+m'}.
$$

(2.4)

Thus $|lm|$ is the tidal potential index relative to the spin axis.
The linear perturbation of the tidal potential on $M$ is specified by the Lagrangian displacement, $\xi(r,t)$, of a fluid element from its unperturbed position. In the inertial frame, the equation of motion takes the form

$$\frac{\partial^2 \xi}{\partial t^2} + 2(\mathbf{v} \cdot \nabla) \frac{\partial \xi}{\partial t} + C \xi = -\nabla U,$$

(2.5)

where $\mathbf{v} = \Omega \times \mathbf{r}$ is the unperturbed fluid velocity, and $C$ is a self-adjoint operator (a function of the pressure and gravity perturbations) acting on $\xi$ (Friedman & Schutz 1978a). Our analysis is accomplished through an eigenmode decomposition of the Lagrangian displacement: $\xi$ is taken to be the linear superposition of eigenmodes, $\xi_a(r,t)$, namely

$$\xi(r,t) = \sum \alpha a_a(t) \xi_a(r).$$

(2.6)

We shall specify the normal mode by the index $\alpha = \{njk\}$ with respect to the body axes; $n$ is the number of radial nodes, $j$ gives the number of nodes in the $\theta'$-eigenfunction, and $k$ is the azimuthal index (about the $z'$-axis). In the following we shall suppress the radial index $n$, so that $\alpha = \{jk\}$. In the absence of external perturbation, the eigenmode behaves as

$$\xi_a(r,t) = \xi_a(r) e^{i\omega_a t} \text{ or } e^{i\omega_a t + ik\phi},$$

(2.7)

where $\omega_a$ is the $\alpha$-mode frequency in the inertial frame. The mode eigenfunction satisfies the equation

$$-\omega_a^2 \xi_a + 2i\omega_a (\mathbf{v} \cdot \nabla) \xi_a + C \xi_a = 0,$$

(2.8)

and we shall adopt the normalization

$$\int d^3r \xi^*_a \xi_a = \delta_{\alpha\alpha'}.$$  

(2.9)

Substituting (2.6) into (2.5), using (2.7) and (2.9), and noting that (2.8) gives a relation for $C \xi$, the dynamical equation for the mode amplitude is

$$a_a - 2i T_a a_a + (\omega_a^2 - 2 T_a \omega_a) a_a = \sum_{l,m'} \frac{GM'}{D^{l+1}} W_{lm} D_{lm'}^{(j)} Q_{a,lm'} e^{-i\omega_m \phi},$$

(2.10)

where

$$T_a = i \int d^3r \rho \xi^*_a (\mathbf{v} \cdot \nabla) \xi_a = -k \Omega + i \int d^3r \rho \xi^*_a (\mathbf{\Omega \times \xi_a})$$

(2.11)

(this is called $B_a$ in Lai 1997), and $Q_{a,lm'}$ is the tidal coupling coefficient or overlap integral defined by

$$Q_{a,lm'} = \int d^3r' \rho \xi^*_a (\mathbf{v} \cdot \nabla) \xi_a = \int d^3x' \delta_{\alpha'\alpha} [r' Y_{lm'} (\theta', \phi')] = \int d^3x' \delta_{\alpha'\alpha} [r' Y_{lm'} (\theta', \phi')] .$$

(2.12)

Here $\delta_{\alpha'\alpha} = -\nabla_i (\rho \xi_i)$ is the Eulerian density perturbation. Clearly, $Q_{a,lm'}$ is non-zero only when $m' = k$; thus

$$Q_{a,lm'} = Q_{a,k,l} \delta_{m'k}.$$  

(2.13)

### 2.2 Resonant-mode energy

The right-hand side of equation (2.10) contains a sum of driving terms with time dependence proportional to $e^{-i\omega_m \phi(t)}$. For a given mode $\alpha = \{jk\}$, resonance occurs when

$$|\omega_a| = |m| \Omega_\alpha .$$  

(Resonance Condition)  

provided there exists at least one $l$ (obviously $l \geq |m|$, $l \geq |k|$) which satisfies

$$l + m = \text{even}$$  

(Symmetry of the Tidal Potential)

and

$$\mathcal{D}_{lm}^{(j)} Q_{a,k,l} \neq 0.$$  

(Non-Zero Tidal Coupling)

For $Q_{a,k,l}$ to be non-zero, the projected tidal potential $[\alpha Y_{\alpha} (\theta', \phi')]$ must have the same $\theta'$-symmetry, with respect to the equator, as the mode. This gives the requirement that $l + j = \text{odd}$ for spherical modes or $l + j = \text{even}$ (for $r$-modes) (see Section 3).

We consider now a specific resonance. Hereafter we shall adopt the convention $k > 0$ and $m > 0$. Thus a resonance is designated by the indices $(\alpha, m) = (jk, m)$, which indicates that the mode $\alpha = \{jk\}$ is excited by tidal potential of azimuthal order $m$ (with angular dependence $e^{i m \phi}$). In general, there can be many $l$s that satisfy the conditions (2.14)–(2.16), but clearly the smallest $l$ which satisfies these conditions will provide the dominant contribution. When appropriate, we shall also label a resonance using the notation $(jk, lm)$, with $l$ specifying the angular order of the dominant tidal field that contributes the resonance $(jk, m)$.

In the special case when the spin and orbit are aligned or anti-aligned, one must have $m = k$, i.e., each mode can experience resonance only once. However, in the general case of arbitrary spin–orbit inclination angle $\beta$, for a given mode $(jk)$, there may be an infinite number of $m$ which satisfies the conditions (2.14)–(2.16); in other words, the mode can be resonantly excited many times at different orbital frequencies. Thus, strictly speaking, the energy transfer from the orbit to the mode at a given resonance $(jk, m)$ depends on the amplitude of the mode induced by previous resonances $(jk, m + 2)$, $(jk, m + 4)$, …

However, as we shall see below, in almost all cases of interest, the energy transfer at the $(jk, m)$-resonance, $\Delta E_{jk,m}$, is much greater than the energy transfer at the earlier resonances, i.e., $\Delta E_{jk,m} \gg \Delta E_{jk,m+2}$. Therefore, throughout this paper we shall treat different resonances of the same mode as being independent of each other. This corresponds to assuming that before each resonance, the mode amplitude is essentially zero.

To calculate the energy transfer to a given mode at a particular $m$-resonance, $\Delta E_{jk,m}$, we only need to keep the resonant driving terms in (2.10), namely

$$\dot{a}_a - 2i T_a a_a + (\omega_a^2 - 2 T_a \omega_a) a_a = \sum_{l,m} \frac{GM'}{D^{l+1}} W_{lm} D_{lm}^{(j)} Q_{a,lm} e^{-i\omega_m \phi},$$

(2.10)

where

$$T_a = i \int d^3r \rho \xi^*_a (\mathbf{v} \cdot \nabla) \xi_a = -k \Omega + i \int d^3r \rho \xi^*_a (\mathbf{\Omega \times \xi_a})$$

(2.11)

Here $\delta_{\alpha'\alpha} = -\nabla_i (\rho \xi_i)$ is the Eulerian density perturbation. Clearly, $Q_{a,lm'}$ is non-zero only when $m' = k$; thus

$$Q_{a,lm'} = Q_{a,k,l} \delta_{m'k}.$$  

(2.13)

1 Clearly, in equations (2.7) and (2.10), $k$ and $m$ can be positive or negative.  
2 The exception is when the star is non-rotating, in which case $m = k$ and $l = j$.  
3 Consider a spherical $(l$ or $g)$ mode, $\alpha = \{jk\}$. For the $(jk,m)$-resonance to be in effect, it is necessary for $l + m = \text{even}$ (see equation 2.15) and $l + j = \text{odd}$ (otherwise $Q_{a,k,l} = 0$). Now, for the $(jk, m + 1)$ to be in effect, it is necessary for $l + m + 1 = \text{even}$, which is incompatible with $l + j = \text{even}$.  

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Since $a_\alpha \propto e^{i\omega n t}$ in the absence of external forcing (see equation (2.7), it is natural to define the mode amplitude $c_\alpha$ via

$$a_\alpha = c_\alpha e^{i\omega n t},$$  

(2.18)
and we expect $c_\alpha$ to be independent of time after the resonance.\footnote{Before and after a resonance, there is a small, non-resonant contribution to $a_\alpha$, which is proportional to $e^{-i\omega_n t}$. This contribution can be neglected compared to the resonance-induced amplitude.}

Equation (2.17) then becomes

$$\dot{c}_\alpha + 2i(\omega_n - T_n)c_\alpha = \left[ \sum_{l} \frac{GM'}{D^{l+1}} W_{lm}^D D_{lm}^0 \bar{Q}_{n,lr} \right] e^{-i\omega - i\omega_n t}$$

$$+ \left[ \sum_{l} \frac{GM'}{D^{l+1}} W_{-m} D_{lm}^0 \bar{Q}_{n,lr} \right] e^{i\omega - i\omega_n t}.$$  

(2.19)

Only one of the terms on the right-hand side of (2.19) can induce the resonance, which occurs when the argument of the exponential of either the first term or the second term goes through zero. The term that induces resonance depends on the sign of $\omega_n$.

### 2.2.1 General properties of the mode frequencies

Before proceeding with our calculation, let us briefly discuss various possibilities for the mode frequency. In general, for a given mode index $\alpha = (jk)$ (recall our sign convention $\kappa > 0$, and note that we have suppressed the radial mode index $n$), there are two possible solutions of opposite signs for $\omega^0_\alpha$, the mode frequency in the frame corotating with the star, which correspond to two different directions of propagation for the wave pattern. Recall from equation (2.7) that our convention for free oscillations is $\hat{e}^{i\omega_{nj} + i\phi_j}$ relative to the spin axis, where $\Omega_j = \Omega_0 \zeta_j$ and $\Omega_0 \neq 0$. The mode frequency in the inertial frame, $\omega self$, is related to $\omega^0_\alpha$ by

$$\omega self = \omega^0_\alpha - k\Omega_0.$$  

(2.20)

The first solution, with $\omega^0_\alpha > 0$, corresponds to the mode which propagates retrograde with respect to the spin (in the rotating frame). Rotation drags the mode frequency in the inertial frame to small values; thus the resonance is shifted to smaller orbital frequency. With increasing $\Omega_0$, the mode frequency $\omega self$ may become negative, corresponding to wave propagation in the same direction as the spin (in the inertial frame). This change in sign of $\omega self$ signals the onset of the so-called Chandrasekhar–Friedman–Schutz (CFS) instability (Chandrasekhar 1970; Friedman & Schutz 1978b) and can occur for both $f$-modes (e.g. Lindblom 1995) and $g$-modes (Lai 1999). At this point the mode energy in the inertial frame changes from positive (stable mode) to negative (unstable mode).\footnote{Since the growth time of the instability can be very long, and since viscosity can suppress the CFS instability for some modes (e.g. Lindblom 1995), we shall still consider such unstable modes in this paper.} Note that when the resonant excitation of a positive-energy mode occurs, the orbital energy is transferred to the mode and the orbit decays faster; when a negative-energy mode is excited, the orbit actually gains energy and the orbital decay is slowed.

The second solution, with $\omega^0_\alpha < 0$, propagates prograde with respect to the spin (in the rotating frame). Rotation causes $\omega self$ to become even more negative. This mode always has positive energy and is stable. The resonance is shifted to even higher orbital frequency. Thus for $f$-modes, the $\omega^0_\alpha < 0$ modes do not experience any resonance. For $g$-modes, $|\omega self|$ can still lie in the appropriate range for resonance.

The $r$-modes are a special case for which $\omega^0_\alpha > 0$ but $\omega self < 0$, meaning that in the rotating frame, the $r$-mode wave pattern always propagates opposite to the spin, while in the inertial frame it propagates in the same direction as the spin. This implies that the $r$-modes are always CFS unstable in the absence of fluid viscosity (see Section 3.3 below).

#### 2.2.2 Mode with $\omega self > 0$ (spin-retrograde mode)

For a mode with $\omega self > 0$, which corresponds to wave propagation opposite to the rotation, resonance occurs at $m\Phi = m\Omega_{orb} - \omega self$, where $\Omega_{orb}$ is given by

$$\Omega_{orb}^2 = \frac{GM(M + M')}{D^3},$$  

(2.21)

neglecting tidal and post-Newtonian corrections. In this case, only the second term on the right-hand side of (2.19) can induce resonance. Thus we have

$$\dot{c}_\alpha + 2i(\omega_n - T_n)c_\alpha = \left[ \sum_{l} \frac{GM'}{D^{l+1}} W_{-m} D_{lm}^0 \bar{Q}_{n,lr} \right] e^{i\omega - i\omega_n t}.$$  

(2.22)

Integrating over time around the resonance and noting that $c_\alpha$ is almost constant after the resonance (and $c_\alpha = 0$ before the resonance, since the mode amplitude induced by previous resonances is negligible; see the discussion at the beginning of Section 2), we find

$$c_\alpha(\text{after resonance}) = \frac{1}{2i(\omega_n - T_n)}$$

$$\times \int \left[ \sum_{l} \frac{GM'}{D^{l+1}} W_{-m} D_{lm}^0 \bar{Q}_{n,lr} \right] e^{(i\omega - i\omega_n t)}.$$  

(2.23)

Resonance occurs when the argument of the exponential is equal to zero. Doing a Taylor expansion of the argument of the exponential around the resonance epoch and using $m\Phi = \omega self$, we have

$$m\Phi - \omega self = m\Phi t + \frac{1}{2} \Phi^2 t^2 - \omega self t = \frac{m}{2} \Omega_{orb} t^2.$$  

(2.24)

$D$ varies slowly with time, so that

$$\left[ \int \frac{1}{D^{l+1}} e^{-i\omega self t} \right] = \left[ \frac{2\pi}{m\Omega_{orb}} \right]^{1/2}.$$  

(2.25)

Therefore the mode amplitude after the $(\alpha, m)$-resonance is

$$a_\alpha(t) = \frac{e^{i\omega self t}}{2i(\omega_n - T_n)} \left[ \sum_{l} \frac{GM'}{D^{l+1}} W_{-m} D_{lm}^0 \bar{Q}_{n,lr} \right] \left[ \frac{2\pi}{m\Omega_{orb}} \right]^{1/2},$$  

(2.26)

where $D_{\alpha}$ is the orbital radius where resonance occurs, and $\Omega_{\alpha}$ is evaluated at $D_{\alpha}$. The energy of the mode in the inertial frame is given by Friedman & Schutz (1978a) as

$$E = \frac{1}{2} \int d^3 \rho \left( \frac{\partial \hat{E}}{\partial t} + \hat{E} \cdot \hat{\nabla} \hat{E} \right).$$  

(2.27)

Using equations (2.6)–(2.9), this gives the mode energy associated with...
with the \((\alpha, m)\)-resonance as

\[
\Delta E_{a,m} = 2 \left[ \frac{1}{2} |\dot{a}_a|^2 + \frac{1}{2} (\dot{\omega}_a - 2T_a \omega_a) |a_a|^2 \right] \\
= 2 \omega_a (\omega_a - T_a) |a_a|^2 \\
= \left( \frac{\pi}{m \Omega_{orb}} \right) \left( \frac{\omega_a}{\omega_a - T_a} \right) \left[ \sum_k GM' \frac{w_{l-m} D_k^{(0)} W_{l-m} Q_{a,k}}{D_k} \right]^2 .
\]

(2.28)

The factor of 2 in the first line of (2.28) arises because, for each \(k > 0\) and \(\omega_a > 0\) resonant mode, there is also the identical \(k < 0\) and \(\omega_a < 0\) resonant mode (recall our sign convention \(m > 0\) and \(k > 0\)). Hereafter we shall define \(Q_{a,k}\) in units such that \(G = M = R = 1\). Equation (2.28) then becomes

\[
\Delta E_{a,m} = \left( \frac{GM}{R^2} \right) \left( \frac{GM'}{R} \right) \left( \frac{\pi}{m \Omega_{orb}} \right) \frac{\omega_a}{\omega_a - T_a} \left[ \sum_k GM' \frac{w_{l-m} D_k^{(0)} W_{l-m} Q_{a,k}}{D_k} \right]^2 .
\]

(2.29)

Here we have defined

\[
\varepsilon_a = \omega_a - T_a = \omega_a - i \int d^3 x \mu \xi_a(\Omega, \xi_a).
\]

(2.30)

Since the effect of the tidal interaction on the orbit is small, we use the quadrupole formula for the gravitational radiation of two point masses to obtain the orbital decay rate (e.g. Shapiro & Teukolsky 1983)

\[
\dot{\Omega}_{orb} = - \frac{3}{2} D \Omega_{orb}, \quad D = - \frac{64 G^3 M^l q (1 + q)}{5 \varepsilon_a} 
\]

(2.31)

where \(q = M'/M\) is the mass ratio. Equation (2.29) can be expressed in terms of index \((\alpha, m) = (jk, m)\) and quantities associated with the resonant mode, giving

\[
\Delta E_{a,m} = \frac{5\pi}{96} \left( \frac{GM}{R} \right) \left( \frac{RC}{GM} \right)^{5/2} \frac{q}{\varepsilon_a} \left[ \sum_k \frac{1}{(1 + q)^{2l+1/6}} \left( \frac{\omega_a}{m} \right)^{2l-1/3} W_{l-m} D_k^{(0)} W_{l-m} Q_{a,k} \right]^2 .
\]

(2.32)

where we have defined dimensionless quantities

\[
\dot{\omega}_a = \omega_a \left( \frac{R^3}{GM} \right)^{1/2} ,
\]

(2.33)

\[
\dot{\varepsilon}_a = \varepsilon_a \left( \frac{R^3}{GM} \right)^{1/2} ,
\]

(2.34)

\[
\dot{T}_a = T_a \left( \frac{R^3}{GM} \right)^{1/2} .
\]

(2.35)

In this case, the mode energy is always positive (see Section 2.2.1).

### 2.2.3 Mode with \(\omega_a < 0\) (spin-prograde mode)

For a mode with \(\omega_a < 0\), corresponding to wave propagation in the same direction as the spin, the resonance occurs at \(m \Omega_{orb} = -\omega_a\), and only the first term on the right-hand side of equation (2.19) can induce resonance. Carrying out a similar calculation as in the \(\omega_a > 0\) case, we find

\[
\Delta E_{a,m} = \frac{5\pi}{96} \left( \frac{GM}{R} \right) \left( \frac{RC}{GM} \right)^{5/2} \frac{q}{\varepsilon_a} \left[ \sum_k \frac{1}{(1 + q)^{2l+1/6}} \left( \frac{\omega_a}{m} \right)^{2l-1/3} W_{l-m} D_k^{(0)} W_{l-m} Q_{a,k} \right]^2 .
\]

(2.36)

Note that the mode energy can be either positive or negative, depending on the sign of \(\omega_a \varepsilon_a\) (see Section 2.2.1 and equation 2.28).

Apart from the sign in \(\omega_a\), the only difference between (2.32) and (2.36) is the factor \(D_k^{(0)}\) and \(D_k^{(0)}\). This difference reflects the ‘preferred’ spin–orbit orientation for resonant excitation. In order to have resonance, the wave pattern of the mode in the inertial frame must propagate in the same direction as the orbital motion of the external mass, \(M'\). For \(\omega_a > 0\), the wave propagates opposite to the spin, and so the resonance will be strongest for retrograde spin–orbit inclination (\(\beta > 90^\circ\)). For \(\omega_a < 0\), the wave propagates in the same sense as the spin, and so we expect the resonance to be strongest for prograde spin–orbit inclination (\(\beta < 90^\circ\)).

### 2.3 Change in orbital phase

We now calculate the orbital phase error due to the resonant energy transfer between the binary orbit and the NS oscillation mode. The number of orbital cycles, \(N_{orb}\), is obtained from

\[
dN_{orb} = \frac{\Omega_{orb} \, dt}{2\pi} = \frac{\Omega_{orb} \, dE_{tot}}{2\pi} = \frac{\Omega_{orb} \, dE_{int}}{2\pi} ,
\]

(2.37)

where \(E_{tot} = E_{orb} + E_{int}\) is the total energy of the system, including both the orbit and the stellar mode. Comparing to the fiducial situation in which the mode energy is neglected and \(dE_{int} = (\Omega_{orb}/2\pi) dE_{orb}/E_{orb}\), we find that the change in the number of orbital cycles due to the \((\alpha, m)\)-resonance is (Lai 1994)

\[
(\Delta N_{orb})_{\alpha,m} = - \left( \frac{t_D}{t_{orb}} \frac{\Delta E_{a,m}}{|E_{orb}|} \right) ,
\]

(2.38)

where \(t_D = |E_{orb}/E_{tot}| = D/|D|\) is the orbital decay time, \(t_{orb} = 2\pi/\Omega_{orb}\) is the orbital period, and \(|E_{tot}| = GM'/(|2D|)\) is the orbital energy. The subscript \(\alpha\) implies that \(t_D, t_{orb}\), and \(|E_{orb}|\) are evaluated at the resonance radius \(D_\alpha\). Parametrizing by the resonance index \((\alpha, m) = (jk, m)\) and substituting in the expressions for \(\Delta E_{a,m}\), we obtain, for the mode with \(\omega_a > 0\),

\[
(\Delta N_{orb})_{\alpha,m} = - \frac{5}{64\pi} \left( \frac{R^3}{GM} \right) \left( \frac{RC}{GM} \right)^{5/2} \frac{q^{3/2}}{\varepsilon_a} \Delta E_{a,m} 
\]

(2.39)

\[
= - \frac{25}{644} \left( \frac{RC}{GM} \right)^5 \frac{1}{q^{3/2} \varepsilon_a} \left[ \sum_k \frac{1}{(1 + q)^{2l+1/6}} \left( \frac{\omega_a}{m} \right)^{2l-1/3} W_{l-m} D_k^{(0)} W_{l-m} Q_{a,k} \right]^2 .
\]

(2.40)

If we keep only the dominant \(l\)-term and label the resonance by
In the following (Section 3), we shall express our results in terms of \( \Omega \) and \( \Omega_{\text{orb}} \) instead of \( \omega_{\text{orb}} \) and \( \omega_{\text{orb}} \), and our procedure to calculate them are summarized. The case of general \( \beta \) will be discussed in Section 4.

For \( \beta = 0^\circ \) or \( 180^\circ \), we clearly have \( m = k \), i.e., a given mode \( \alpha = [jk] \) can only be excited once, at the orbital frequency given by \( k\Omega_{\text{orb}} = |\omega_{\text{orb}}| \).

### 3.1 F-modes

For non-rotating NSs, the f-modes have frequencies of order \( (GM/R^3)^{1/2} \), which are too large for any resonance to occur during binary inspiral. Rotation can reduce the mode frequencies, making resonant excitations possible. In this paper we shall consider only the f-modes with \( j = k \). Since \( \omega_{\text{orb}} = \omega_{\text{rot}} - k\Omega \), these \( j = k \) modes are more strongly affected by rotation. Only the modes with \( \omega_{\text{rot}}^j > 0 \) need to be considered, since rotation actually increases \( |\omega_{\text{rot}}| \) for the \( \omega_{\text{rot}}^j < 0 \) modes, making their resonances impossible (see Section 2.2.1).

To calculate the f-mode properties relevant for tidal resonance, we shall model the NS as an incompressible Maclaurin spheroid, whose equatorial radius and polar radius are \( a_1 \) and \( a_3 \), respectively. The spin frequency \( \Omega \), is related to the eccentricity \( e = (1 - a_2^2/a_1^2)^{1/2} \) of the spheroid via the expression

\[
\Omega^2 = \frac{2\pi G\rho}{e^2} \left[ \left( 1 - e^2 \right)^{1/2} - 3 e \left( 1 - e^2 \right)^{1/2} \right],
\]

where \( \rho = 3M/(4\pi R^3) \) is the (uniform) density of the spheroid, and \( R = (a_1^2a_3)^{1/3} \) is the mean radius. The frequency of the \( j = k \) mode in the rotating frame has been obtained by Bryan (1989) and Comins (1979):

\[
\omega_a^{(j)} = 1 \pm \left( 1 - \frac{2ke^2 R_k}{[\left( 3 - 2e^2 \right)^{1/2} - 3e(1 - e^2)]^{1/2}} \right)^{1/2},
\]

where the \( \pm \) sign refers to modes which propagate opposite to \( \Omega_a \) and in the same direction as \( \Omega_a \), respectively, and

\[
R_k = \frac{(1 - e^2)^{1/2}}{e} \frac{(2k - 1)!}{(2k)!} \sum_{p=k+1}^{\infty} \frac{(2p - 2)!}{(2p - 1)!} e^{2(p-k)}
\]

For \( j = k = 2 \), a more compact expression for the mode frequency is given by Chandrasekhar (1969) as

\[
\omega_a^{(2)} = \Omega_0 \pm 4\pi G\rho B_{11} - \Omega^2_a,
\]

where \( B_{11} \) is the index symbol (a dimensionless function of \( e \)) as defined in Chandrasekhar (1969). Also note that for \( \Omega_a = 0 \), the mode frequency becomes

\[
\omega_a^{(0)} = \pm \frac{2GM}{R^3} \frac{j(j-1)}{2j+1}.
\]
Clearly, for resonance to occur before merger would require
\[ \omega_a^{(r)} > 0 \] and \( \omega_a < 0 \) (spin-prograde) unstable modes. Note that the (22,4)-resonance is possible only for misaligned spin±orbit configurations (\( \beta \neq 0^\circ, 180^\circ \)). It is also clear that the unstable modes occur at extremely high spin rates. The vertical dashed lines indicate the frequency range of gravitational wave detectors, \( \nu_{gw} = 10-1000 \text{Hz} \).

This is essentially the Kelvin mode. As discussed above, we only consider the mode with + sign in (3.2).

Fig. 1 shows the normalized spin frequency,
\[ \nu_s = \nu M_1^{1/2} R_{10}^{3/2} = 2170 \hat{\Omega}_1 \text{Hz}, \]
where \( R_{10} \) is the NS radius in units of 10 km and \( \hat{\Omega}_1 = \Omega_1 (R_1^3/GM)^{1/2} \), plotted against the normalized gravitational wave frequency,
\[ \nu_{gw} = \nu_{gw} M_1^{1/2} R_{10}^{3/2} = 2170 \left( \frac{2}{M} \right) |\omega_a| \text{Hz}, \]
at which resonance occurs (see equation 2.14) for different f-modes (j = k = m = 2, 3, 4, 5). Higher order modes (k > 5) are not considered since, as shown below, their contribution to the orbital change is negligible. The spin frequency at which the star becomes secularly unstable (in the absence of viscosity) is where \( \omega_a \) becomes negative; for larger k-values, secular instability occurs at lower \( \hat{\Omega}_1 \). Resonance can also occur at these rotation frequencies. For completeness, we shall include these unstable modes, even though they would require very high spin rates.

The two vertical dashed lines in Fig. 1 indicate the frequency range of gravitational wave detectors, \( \nu_{gw} = 10 - 1000 \text{Hz} \). Note that the binary merger occurs at about \( r_{\text{min}} = 3R \) (for equal-mass NSs), corresponding to the maximum gravitational wave frequency
\[ \nu_{gw}^{(\text{max})} = 1181 M_1^{1/2} R_{10}^{3/2} \left( \frac{3R}{r_{\text{min}}} \right)^{3/2} \text{Hz}. \]
Near this frequency, the distortion to the NS(s) is significant. Clearly, for resonance to occur before merger would require \( \nu_{gw} < \nu_{gw}^{(\text{max})} \).

To calculate the tidal coupling coefficient \( Q_{\alpha \beta} \), we need to determine the Lagrangian displacement, \( \xi_\alpha \). We begin with the Euler equation in the corotating frame for the inviscid fluid:
\[ \frac{\partial \nu}{\partial t} + (\nu \nabla) \nu + 2\hat{\Omega}_s \times \nu = -\nabla \Psi, \]
where
\[ \Psi = \frac{P}{\rho} + \Phi - \frac{1}{2} \hat{\Omega}_s^2 (x^2 + y^2). \]
\( \Phi \) is the self-gravitational potential. Linearizing and assuming small perturbations \( \nu, \Psi \propto e^{i\omega t} \), equation (3.9) reduces to
\[ i\omega_a^{(r)} \nu + 2\hat{\Omega}_s \times \nu = -\nabla \Psi. \]
Combining this with the continuity equation for an incompressible fluid, \( \nabla \cdot \nu = 0 \), we find
\[ \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \left( 1 - \frac{4\hat{\Omega}_s^2}{|\omega_a^{(r)}|^2} \right) \frac{\partial^2 \Psi}{\partial z^2} = 0. \]
A specific set of solutions to this equation, corresponding to the \( j = k \) mode, is
\[ \Psi = A_k (x + iy)^{k-1} e^{i\omega_a^{(r)} t}. \]
where \( A_k \) is a normalization constant. The Lagrangian displacement \( \xi = \nu/|\omega_a^{(r)}| \) can be obtained by substituting (3.13) into (3.11), giving
\[ \xi = \frac{k A_k}{|\omega_a^{(r)}|} (x + iy)^{k-1}(x + iy) e^{i\omega_a^{(r)} t}. \]
Adopting the normalization condition (2.9), we can write the displacement vector \( \xi \) as
\[ \xi(x, y, z) = \frac{(2k + 1)!!}{(2k)!! (2\pi k a_1^2 a_3)} \left( x + iy \right)^{k-1} e^{i\omega_a^{(r)} t}, \]
\[ \xi(r, \phi, z) = \frac{(2k + 1)!!}{(2k)!! (2\pi k a_1^2 a_3)} (x + iy)^{k-1} e^{i\omega_a^{(r)} t} \]
\[ \times Y_{k-1}^{(1)} (\phi, z) = \frac{(2k + 1)!!}{(2k)!! (2\pi k a_1^2 a_3)} \left( x + iy \right)^{k-1} e^{i\omega_a^{(r)} t}, \]
\[ \xi(r, \theta, \phi) = \left( \frac{k}{ar_{\text{min}}^2} \right)^{1/2} r^{k-1}, \xi_\theta = \frac{\xi}{k}, \]
and the spherical harmonic function \( Y_{kl} \) is given by
\[ Y_{kl}(\theta, \phi) = \frac{(-1)^k}{2k!} \sqrt{\frac{(2k + 1)!!}{4\pi}} \sin \theta^k e^{ik\phi}. \]
(2.12), can be written as

\[ Q_{kk} = \left( \frac{k}{2\pi r^2 a_3} \right)^{1/2} \int d^3 x \propto^{8k-2} \times \left( \frac{k|Y_{kk}|^2}{k} \left( \frac{|\partial Y_{kk}|}{\partial \theta} \right)^2 + \frac{1}{k} \frac{\|\partial Y_{kk}\|}{\|\partial \phi\|} \right). \] 

(3.20)

Note that for a given \( \theta \), the radial variable ranges from 0 to \( r_{\max}(\theta) \), with

\[ r_{\max}(\theta) = \left( \frac{\sin^2 \theta + \cos^2 \theta}{\alpha_1^2} \right)^{1/2} \alpha_1 \left( 1 + \mu \right)^{1/2}, \] 

(3.21)

where \( \mu = \cos \theta \) and \( \xi = (1 - c^2)^{1/2} \). Equation (3.20) can then be evaluated explicitly, giving

\[ Q_{kk} = \left( \frac{k}{4\pi} \right)^{1/2} \left( \frac{d}{\hat{R}} \right)^{k-1}, \] 

(3.22)

recalling the normalization of \( Q_{a,\ell} \) in units such that \( G = M = R = 1 \). Furthermore,

\[ \hat{\Omega}_a = -k - 1 \Omega_a, \] 

(3.23)

\[ \hat{\Omega}_a = \hat{\omega}_a + (k - 1) \Omega_a = \hat{\omega}_a^{(\ell)} - \Omega_a. \] 

(3.24)

Substituting equations (3.2), (3.22) and (3.24) into equation (2.41) or (2.42), the change in the number of orbital cycles due to the resonance can be readily calculated. Fig. 2 depicts \( \Delta \text{N}_{\text{orb}}(k,\ell) \) as a function of the normalized gravitational wave frequency \( \nu_{gw} \).

For \( m = k = 2 \), we have

\[ (\Delta \text{N}_{\text{orb}})_{2,22}(180^0) = \left( \frac{234M_1^{-4.5}R_1^{-3.5}}{\nu_g(1 + q)} \right) \times \left( \frac{0.895}{\hat{\xi}_a} \right) \left( \frac{Q_{2,22}}{0.69} \right)^2 \right)^{(100 \text{ Hz})}. \] 

(3.25)

Note that we have normalized \( \hat{\xi}_a \) and \( Q_{2,22} \) to their respective non-rotating values (rotation can change these by a factor of \( \sim 20 \) and \( \sim 1.5 \), respectively, for the range of \( \hat{\Omega}_a \approx 0.57 \) considered in this paper). The upper sign (negative \( \Delta \text{N}_{\text{orb}} \)) in (3.25) applies when \( \omega_a > 0 \) and \( \beta = 180^0 \), while the lower sign (positive \( \Delta \text{N}_{\text{orb}} \)) applies when \( \omega_a < 0 \) and \( \beta = 0^0 \). Note that \( \Delta \text{N}_{\text{orb}}(k,\ell) \) diverges for \( k = 2 \) as \( \nu_{gw} \) decreases. The reason for this is that at larger orbital radii (small resonant frequencies) the orbit decays more slowly, and the binary spends a longer time in resonance, giving rise to larger \( \Delta \text{N}_{\text{orb}} \). For higher \( k \)-resonances, the same effect is also operative, but the tidal potential is weaker so that the net effect is that \( \Delta \text{N}_{\text{orb}}(k,\ell) \propto \nu_{gw}^{3/2} \) decreases with decreasing \( \nu_{gw} \).

It is evident from Fig. 2 that the \((j,k,m) = (22,2)\) and (33,3)-resonances contribute significantly to the orbital phase evolution. However, to attain such resonances during the binary inspiral, the NS of canonical mass (1.4 M⊙) and radius (10 km) must be rotating very rapidly, i.e., \( \nu_s \gtrsim 710 \text{ Hz} \) for the \((22,2)\)-resonance, \( \nu_s \approx 570 \text{ Hz} \) for the (33,3)-resonance, \( \nu_s \approx 460 \text{ Hz} \) for the (44,4)-resonance, and \( \nu_s \approx 380 \text{ Hz} \) for the (55,5)-resonance (see Fig. 1). Such rapid rotation is physically possible but may be astrophysically unlikely. Observations of NS binary systems and the current theories regarding the formation of these systems indicate

- **Figure 2.** Orbital cycle change \( |\Delta \text{N}_{\text{orb}}|_{\ell,\ell} \), with \( M = 1.4 \text{ M}_\odot \), \( R = 10 \text{ km} \), \( q = 1 \), as a function of the normalized gravitational wave frequency, \( \nu_{gw} \). The solid lines are for the \( \omega_a^{(\ell)} > 0 \) and \( \omega_a < 0 \) (spin-retrograde) stable modes with \( |\Delta \text{N}_{\text{orb}}|_{\ell,\ell}(180^0) < 0 \). The dot-dashed lines are for the \( \omega_a^{(\ell)} > 0 \) and \( \omega_a < 0 \) (spin-prograde) unstable modes with \( |\Delta \text{N}_{\text{orb}}|_{\ell,\ell}(0^0) > 0 \). Note that both the (22,2) and (22,4) curves belong to the same (22,2)-resonance. The (22,42) curve serves as a baseline for the (22,44)-resonance (see Section 4 and Fig. 7). Also recall from Fig. 1 that the unstable modes require extremely high spin rates. The large dots with labels indicate the spin rate at that gravitational wave frequency. The vertical dashed lines indicate the frequency range of gravitational wave detectors, \( \nu_{gw} \sim 10 - 1000 \text{ Hz} \).

- **Figure 3.** Spin frequency, \( \nu_s \), of the \( R = 15 \text{ km} \) NS as a function of the gravitational wave frequency, \( \nu_{gw} \), for the f-mode \((j,k,m)\)-resonance (taking \( j = k = 2 - 5 \), where \( \nu_{gw} \) is related to the f-mode frequency by \( \nu_{gw} = (2/m)\nu_s \)). The labels in the figure give the values of \((j,k,m)\). The solid lines are for the \( \omega_a^{(\ell)} > 0 \) and \( \omega_a < 0 \) (spin-retrograde) stable modes. The dot-dashed lines are for the \( \omega_a^{(\ell)} > 0 \) and \( \omega_a < 0 \) (spin-prograde) unstable modes. Note that the (22,4)-resonance is possible only for misaligned spin–orbit configurations (\( q \neq 0, 180^0 \)). It is also clear that the unstable modes occur at extremely high spin rates. The vertical dotted line at \( \nu_{gw} = 640 \text{ Hz} \) denotes the maximum gravitational wave frequency allowed by equation (3.8), which sets a lower limit to the spin frequency. The vertical dashed lines indicate the frequency range of gravitational wave detectors, \( \nu_{gw} \sim 10 - 1000 \text{ Hz} \).
that such rapid rotation of the NS is not possible (see Section 5). If such resonances indeed occur, then their effect on the orbital phase of the binary inspiral must be taken into account when constructing theoretical wave templates in the search for gravitational waves from these systems.

It can also be seen from equations (3.1) and (3.2) that the NS spin frequency and mode frequencies depend on the radius of the NS and thus the NS equation of state (EOS). Current equations of state allow NSs with \( R = 15 \) km (e.g. Shapiro & Teukolsky 1983). This implies that the spin and mode frequencies are lowered, and resonance occurs at these lower frequencies as illustrated by Fig. 3. Fig. 4 shows the corresponding \( \Delta \nu_{\text{orb}}(k,m) \) as a function of the \( v_{gw} \) for the \( R = 15 \) km NS. The lower limit to the spin frequency required for resonance is set by the associated gravitational wave frequency given by equation (3.8). This limit is indicated by the dotted lines in Figs 3 and 4, and yields \( \nu_s \approx 330 \) Hz for the \((22,2)\)-resonance, \( \nu_s \approx 260 \) Hz for the \((33,3)\)-resonance, \( \nu_s \approx 200 \) Hz for the \((44,4)\)-resonance, and \( \nu_s \approx 150 \) Hz for the \((55,5)\)-resonance.

It is necessary to check that the higher-\( l \) contribution to the \((k, m)\)-resonance is negligible. Except for the \( \Omega_k = 0 \) case, the \( l = k + 2, k + 4, \ldots \) tides can also contribute to \( \Delta \nu_{\text{orb}}(k,m) \). The mode frequency is the same, but the new tidal coupling coefficient must be calculated. We shall consider just the \((22,2)\)- and \((33,3)\)-resonances here, since the higher order terms are much smaller. Using the mode eigenfunction (3.17), we find

\[
Q_{22,42} = \frac{45}{112} \left[ \frac{9}{2\pi(1 - e^2)} \right]^{1/2} \left( \frac{a_1}{R} \right)^4 \int_{-1}^{1} \frac{-5\mu_k^2 + 6\mu_m^2 - 1}{\left( 1 + \frac{\mu_k^2}{\frac{\mu_m^2}{\xi^2}} \right)^{1/2}} d\xi.
\]

(3.26)

**Figure 4.** Orbital cycle change \( |\Delta \nu_{\text{orb}}(k,m)\rangle\), with \( M = 1.4 \) M\(_{\odot}\), \( R = 15 \) km, \( q = 1 \), as a function of the gravitational wave frequency, \( \nu_{gw} = (2/m)\nu_{s}\), for the \( f \)-mode \((j,k)m\)-resonance. The labels in the figure give the values of \((j,k,lm)\). The solid lines are for the \( \omega_0^{(j)} > 0 \) and \( \omega_0^{(m)} > 0 \) (spin-retrograde) stable modes with \( \Delta \nu_{\text{orb}}(j,k,m) \) \((180°) < 0 \). The dot-dashed lines are for the \( \omega_0^{(j)} > 0 \) and \( \omega_0^{(m)} < 0 \) (spin-prograde) unstable modes with \( \Delta \nu_{\text{orb}}(j,k,m) \) \((0°) > 0 \). Note that both the \((22,2)\) and \((22,4)\) curves belong to the same \((22,2)\)-resonance. The \((22,4)\) curve serves as a baseline for the \((22,44)\)-resonance (see Section 4 and Fig. 7). Also recall from Fig. 3 that the unstable modes require extremely high spin rates. The vertical dotted line at \( \nu_{gw} = 640 \) Hz denotes the maximum gravitational wave frequency allowed by equation (3.8). The large dots with labels indicate the spin rate at that gravitational wave frequency. The vertical dashed lines indicate the frequency range of gravitational wave detectors, \( \nu_{gw} = 10-1000 \) Hz.

and

\[
Q_{33,33} = \frac{35}{96} \left[ \frac{11}{\pi(1 - e^2)} \right]^{1/2} \left( \frac{a_1}{R} \right)^4 \int_{-1}^{1} \frac{7\mu_k^6 - 15\mu_k^4 + 9\mu_k^2 - 1}{\left( 1 + \frac{\mu_k^2}{\frac{\mu_m^2}{\xi^2}} \right)^{1/2}} d\xi.
\]

(3.27)

Both expressions can be evaluated numerically. In Fig. 2 the results for the orbital change \( \Delta \nu_{\text{orb}}(22,42) \) is plotted, indicating that it is indeed much smaller than \( \Delta \nu_{\text{orb}}(33,33) \). The orbital change \( \Delta \nu_{\text{orb}}(33,33) \) is not shown, since it is below the limit of interest for this paper.

### 3.2 G-modes

Gravity modes in neutron stars arise from the composition (proton-to-neutron ratio) gradient in the stellar core (Reisenegger & Goldreich 1992; Lai 1994), density discontinuities in the crust (Finn 1987), as well as thermal buoyancy associated with finite temperatures, either due to internal heat (McDermott, Van Horn & Hansen 1988) or due to accretion (Bildsten & Cutler 1995). In this paper we shall focus on the core g-modes, since the gradient of proton-to-neutron ratio in the NS interior provides the largest buoyancy. For a non-rotating NS, the typical frequencies of low-order g-modes are around 100 Hz, and are smaller for higher order modes. Thus we expect that even modest stellar rotation rate (\( \approx 100 \) Hz) can change the mode properties dramatically.

The core g-modes of rotating NS have been calculated by Lai (1999) using the traditional approximation. Two NS models are considered based on different EOS. Model UU gives a non-rotating NS radius \( R = 13.47 \) km for a \( M = 1.4 \) M\(_{\odot}\), and its lowest order \((n = 1) j = 2 \) g-mode has frequency \( \nu_g = 148 \) Hz and \( Q_{22,22} = 0.62 \times 10^{-3} \). Model AU gives a non-rotating NS radius \( R = 12.37 \) km for a \( M = 1.4 \) M\(_{\odot}\), and its lowest order \((n = 1) j = 2 \) g-mode has frequency \( \nu_g = 72 \) Hz and \( Q_{22,22} = 0.11 \times 10^{-3} \).

Fig. 5 shows the spin frequency, \( \nu_s \), as a function of the observed gravitational wave frequency, \( \nu_{gw} = (2/m)\nu_s \), for the \( m = 2 \) resonance of the \(( j \; k \rangle = (22) \) g-modes \((n = 1)\). The \( \omega_0^{(j)} > 0 \) mode propagates in the opposite direction to the spin (see the discussion following equation 2.20). As \( \nu_s \) increases, the mode frequency, \( \nu_g \), in the inertial frame decreases; \( \nu_s \) can even become negative when \( \nu_s \) exceeds a certain critical value (95 Hz for model UU and 47 Hz for model AU). This corresponds to the onset of secular instability for the g-mode (Lai 1999). We include these unstable modes because, as shown by Lai, viscosity tends to stabilize these modes for typical parameters of the NS.

Fig. 5 also shows the results for the spin-prograde mode, which has \( \omega_0^{(j)} < 0 \) (see the discussion following equation 2.20). These are calculated in a similar manner as in Lai (1999). These modes always have negative \( \omega_0^{(j)} \), but they have positive energy and thus are secularly stable. Rotation clearly increases the gravitational wave frequency at which resonance occurs for these modes.

Note that to calculate the total orbital change due to the \(( j \; k \rangle = (22,2)\)-resonance, one needs to use equation (2.40), which includes contributions from all possible \( l \)-terms. However, the smallness of \( \Delta \nu_{\text{orb}}(22,2) \) implies \( \Delta \nu_{\text{orb}}(22,2) \approx 0 \) to a good approximation. The quantity \( \Delta \nu_{\text{orb}}(22,42) \) will also be used in Section 4 as a fiducial number for the \((22,44)\)-resonance.
(2.42) (depending on the sign of \( \omega_\alpha \)), we find
\[
(\Delta N_{\text{orb}})_{22,22} (180^\circ, 0^\circ) = \mp 0.0086 \frac{1}{q(1+q)} \times \left( \frac{0.107}{\tilde{\epsilon}_\alpha} \right) \left( \frac{Q_{22,22}}{6.2 \times 10^{-4}} \right)^2 \left( \frac{50\text{Hz}}{\nu_{GW}} \right)
\]
for modelUU and
\[
(\Delta N_{\text{orb}})_{22,22} (180^\circ, 0^\circ) = \mp 0.00045 \frac{1}{q(1+q)} \times \left( \frac{0.046}{\tilde{\epsilon}_\alpha} \right) \left( \frac{Q_{22,22}}{1.1 \times 10^{-4}} \right)^2 \left( \frac{50\text{Hz}}{\nu_{GW}} \right)
\]
for model AU. We have normalized \( \tilde{\epsilon}_\alpha \) and \( Q_{22,22} \) to their respective non-rotating values (rotation can change these by a factor of \( \sim 2 \) for the range of \( \tilde{\Omega}_e \leq 0.3 \) considered in this paper). The upper sign in (3.28) and (3.29) applies when \( \omega_\alpha > 0 \) and \( \beta = 180^\circ \), while the lower sign applies when \( \omega_\alpha < 0 \) and \( \beta = 0^\circ \). However, when \( \omega_\alpha < 0 \) and \( \tilde{\epsilon}_\alpha < 0 \); therefore \( \Delta N_{\text{orb}} < 0 \) (a decrease in the number of orbits).

The large dots in Fig. 6 correspond to the case when the NS is non-rotating. We see that retrograde rotation \( (\beta = 180^\circ) \) can increase \( |\Delta N_{\text{orb}}|_{22,22} \) for the spin-retrograde mode by shifting the resonance to smaller orbital frequency, where the orbital decay is slower. For the spin-prograde modes, rotation decreases \( |\Delta N_{\text{orb}}|_{22,22} \) from the non-rotating value. It is clear from Fig. 6 that the orbital change is small, especially in comparison to the orbital change generated by f-modes. However, the g-mode resonances can occur more easily than the f-modes, since the g-mode resonance does not require the rapid rotation of the NS.

The above results apply for particular neutron star models. As in equation (3.25), we have the scaling: \( (\Delta N_{\text{orb}})_{22,22} \propto M^{-1.3} R^{3.5} \); thus the orbital change is larger for low-mass NSs which typically have larger \( R/M \).

### 3.3 R-modes

For non-rotating stars, r-modes are ‘degenerate’ toroidal modes with zero frequency and zero density perturbation. Rotation breaks the degeneracy via the Coriolis force, giving rise to finite-frequency r-modes. For the analysis of the r-modes in this section and in Section 4.3, we shall consider the slowly rotating \( (\tilde{\Omega}_e = \tilde{\Omega}_{\text{e}}/(GM/R^3)^{1/2} \ll 1) \) incompressible NS models. As shown by Provost, Berthomieu & Rocca (1981), only the \( j = k \) r-modes exist for barotropic stars.

Following Saio (1982) and Kokkotas & Stergioulas (1999), it is convenient to introduce a new radial coordinate \( a \) which is related to the radial distance \( r \) through \( r = a(1 + e) \), where \( e(a, \theta) \propto \tilde{\Omega}_e \) represents the centrifugal distortion of the star. In the new coordinates, all unperturbed quantities are functions of \( a \) only. The displacement eigenfunction of the \( \alpha = \{ kk \} \) r-mode can be written as
\[
\frac{\xi_{kk}}{a} = \left( 0, K_{kk} \frac{1}{\sin \theta} \frac{\partial}{\partial \phi}, -K_{kk} \frac{\partial}{\partial \theta} \right) Y_{kk} + \left( S_{k+1k}, H_{k+1k} \frac{1}{\sin \theta} \frac{\partial}{\partial \phi}, H_{k+1k} \frac{\partial}{\partial \theta} \right) Y_{k+1k},
\]
where the first term represents the toroidal (axial) component and the second term represents the spheroidal (polar) component. The
r-mode is predominantly toroidal, and the radial eigenfunctions scale with the rotation rate as $K_{kk} \sim \tilde{\Omega}_r^2$, while $S_{k+1,k} = H_{k+1,k} \sim \tilde{\Omega}_r^2$. For a uniform density star, the mode frequency in the rotating frame is given to order $\tilde{\Omega}_r^3$, by (Saio 1982; Kokkotas & Stergioulas 1999)

$$\omega_{1}(\alpha) = \frac{2}{k+1} \tilde{\Omega}_r + \frac{5k(k+1)^2 - 8}{(k+1)^3} \tilde{\Omega}_a^3.$$  

(3.31)

In the inertial frame, this gives

$$\dot{\omega}_a = -\frac{(k+2)(k-1)}{k+1} \tilde{\Omega}_r + \frac{5k(k+1)^2 - 8}{(k+1)^3} \tilde{\Omega}_a^3.$$  

(3.32)

Note that the $r$-mode always has positive $\omega_{1}(\alpha)$ and negative $\omega_{1}(\alpha)$, i.e., the wave pattern propagates opposite to spin in the rotating frame while it becomes spin-prograde in the inertial frame, indicating that the $r$-mode is secularly unstable, although viscosity tends to stabilize the mode for small $\tilde{\Omega}_r$ (Andersson 1998; Andersson, Kokkotas & Schutz 1999; Friedman & Morsink 1998; Lindblom, Owen & Morsink 1998).

The eigenfunction (3.30) implies the density perturbation for the $r$-mode $\delta \rho_{kk} \propto Y_{k+1,k}$. Thus the tidal coupling coefficient, $Q_{kk,k}$, is non-zero only for $l = k+1$. To evaluate $Q_{kk,k}$, we write the pressure perturbation of the $r$-mode as

$$\delta p_{kk} = \rho g a \xi_{k+1,k} Y_{k+1,k},$$  

(3.33)

where $g$ is the gravitational acceleration. The dimensionless function $\xi_{k+1,k}(a)$ is given by (Kokkotas & Stergioulas 1999)

$$\xi_{k+1,k}(a) = \frac{\Omega^2}{a^2} e^{-\frac{k^2}{k^2+3}}.$$  

(3.34)

and the corresponding radial eigenfunction is

$$K_{kk}(a) = i \frac{(k+1)^2 \sqrt{k^2+3}}{4k} \left( \frac{a}{R} \right)^{k-1}.$$  

(3.35)

The Eulerian density perturbation is given by

$$\delta \rho_{kk} = -\xi_{kk} \nabla \rho = \rho \xi_{kk} Y_{k+1,k} \delta(a - R).$$  

(3.36)

However, the boundary condition at the stellar surface gives $S_{k+1,k}(R) = \xi_{k+1,k}(R)$, which is obtained by requiring the Lagrangian pressure perturbation to vanish at the surface. We then find

$$Q_{kk,k} = \rho \Omega^4 R^{k+4}.$$  

(3.37)

Note that this value corresponds to the normalization as defined by (3.34). To obtain the proper normalization (see equation 2.9), we use equation (3.30) and find, to leading order in $\tilde{\Omega}_r$,

$$\int d^3 x \rho \xi_{kk}^2 = k(k+1)\rho \int_0^R da a^d |K_{kk}(a)|^2 = \frac{(k+1)^5}{16k} \rho R^5.$$  

(3.38)

Thus the renormalized $Q_{kk,k}$, expressed in units where $G = M = R = 1$, is

$$Q_{kk,k} = \left[ \frac{12k}{\pi(k+1)^2} \right]^{1/2} \Omega^2.$$  

(3.39)

Thus tidal coupling will be weak for slow rotation. With the $\xi_{kk}$ given by (3.30), we can similarly calculate the function $T_{\alpha}$ and $\epsilon_{\alpha}$ (see equations 2.11 and 2.30), yielding

$$T_{\alpha} = -k \tilde{\Omega}_r,$$  

(3.40)

$$\epsilon_{\alpha} = \tilde{\omega}_a + k \tilde{\Omega}_r = \tilde{\omega}_{\alpha}^{(r)}.$$  

(3.41)

The next order correction to $\tilde{T}_a$ is of order $\tilde{\Omega}_a^5$ and can be neglected.

Now consider resonant excitation of the $\alpha = \{kk\}$-mode. For aligned spin–orbit we require $m = k$, while to have non-vanishing $Q_{kk,k}$ we require $l = k+1$. Since $W_{k+1,k} = 0$, we conclude that for $\beta = 0^\circ$ or $180^\circ$, there is no effective tidal excitation of the $r$-mode (at least for the barotropic stellar model considered here). Resonant excitations of $r$-modes can only occur for misaligned spin–orbit configurations, which we consider in Section 4.3, using the properties of the $r$-mode derived above.

4 NON-ALIGNED SPIN–ORBIT

In this section we consider tidal resonances for general spin–orbit inclination angle $\beta$.

4.1 F-modes

We again focus on the $j = k$ modes. For a given $\alpha = \{kk\}$ mode, in addition to the $(\alpha, m) = (kk, k)$ resonance considered in Section 3.1, many new resonances become possible for general $\beta$. This can be seen by inspecting the resonance conditions (2.14)–(2.16): for $Q_{kk,k}$ to be non-zero, we require $l = k, k + 2, k + 4, \ldots$ To have non-zero tidal potential ($W_{lm} \neq 0$) we must then have $m = k, k + 2, k + 4, \ldots (m > 0)$. For example, the $j = k + 3$ mode can be resonantly excited at orbital frequencies $\Omega_{orb} = |\omega_a|$, $\Omega_{orb} = |\omega_a|/3$, and $\Omega_{orb} = |\omega_a|/5$, etc.

The properties of the $f$-modes have already been calculated in Section 3.1. For $\omega_a > 0$ (the spin-prograde mode), the relevant Wigner $D$-functions are

$$D_{k,k}^{(k)} = e^{i\alpha} \left( \sin \frac{\beta}{2} \right)^{2k},$$  

(4.1)

$$D_{k,k+2}^{(k)} = e^{i\alpha} \sqrt{k(k-1)} \left( \cos \frac{\beta}{2} \right)^2 \left( \sin \frac{\beta}{2} \right)^{2k-2},$$  

(4.2)

$$D_{k,k+4}^{(k)} = e^{i\alpha} \sqrt{k(k-1)(k-2)(k-3)} \left( \cos \frac{\beta}{2} \right)^4 \left( \sin \frac{\beta}{2} \right)^{2k-4},$$  

(4.3)

$$D_{k,k-2}^{(k+2)} = e^{i\alpha} \left( \sin \frac{\beta}{2} \right)^{2k} \left( \cos \frac{\beta}{2} \right)^4 \left( \sin \frac{\beta}{2} \right)^2 + (k+1)(2k+1) \left( \cos \frac{\beta}{2} \right)^4.$$  

(4.4)

From equation (2.41), we find that the orbital change due to the $(kk, lm)$-resonance at inclination angle $\beta$ can be written as

$$(\Delta \Omega_{\text{orb}})_{k,k,lm}(\beta) = (\Delta \Omega_{\text{orb}})_{k,k,180^\circ}(180^\circ) \times f_{k,k,lm}(\beta),$$  

(4.5)

where $(\Delta \Omega_{\text{orb}})_{k,k,180^\circ}$ is the value for anti-aligned spin–orbit. 

7 This is because the eigenfunction of the $\alpha = \{kk\}$ mode is even (odd) with respect to the stellar equator for even (odd) $k$; the tidal potential must have the same symmetry in order to produce non-zero coupling.
and the \( f_{kk,lm} \) is a dimensionless function

\[
    f_{kk,lm}(\beta) = \left( \frac{W_{l-m}}{W_{l-k}} \right)^2 |D_{k-l/m}^{(4/11/3)}(k/m) |
\]

(4.6)

Fig. 7 shows \( f_{kk,lm}(\beta) \) as a function of the inclination angle for a number of different resonances. This figure should be read together with Fig. 2 or Fig. 4 to determine the actual orbital change at different \( n_{gw} \) and \( \beta \). The results for the \((33,53)-\) and \((33,55)-\)resonances are not shown, since their contributions to the orbital change are below the limits of interest. It is evident from Fig. 7 that even when the rotation of the NS is mostly prograde with respect to the orbital motion \( \beta < 90^\circ \), there still exist modes that can be resonantly excited and thus contribute to orbital energy dissipation. Also of note is the large orbital change caused by the \((55,51)-\)resonance relative to the \((55,55)-\) and \((55,53)-\)resonances. It is thus possible for this resonance to produce an orbital change which is comparable to those produced by the \((44,4m)-\)resonances.

For spin-prograde modes \( (\omega_0 < 0) \), the above results still apply except that one should replace \( \Delta N_{orb}(180^\circ) \) by \( \Delta N_{orb}(0^\circ) \) and, in the expression for \( f_{kk,lm}(\beta) \), replace \( \beta \) by \( (180^\circ - \beta) \).

### 4.2 G-modes

The g-mode resonance considered in Section 3.2 corresponds to the \((\alpha, lm) = (22,22)\). The function \( f_{kk,lm}(\beta) \) derived in Section 4.1 equally applies to the g-mode resonance and is plotted in Fig. 7. Although other resonances of g-modes are also induced, their effects are below the limits of interest.

### 4.3 R-modes

Recall that only the \( j = k \) \( r\)-modes exist for barotropic stars, and \( Q_{kk,l} \neq 0 \) only if \( l = k + 1 \), in which case \( Q_{kk,k+1} \) is given by (3.39). Resonances at \( m \) are restricted to \( m = k + 1, k - 1, k - 3, \ldots (m > 0) \), since \( W_{k+1m} = 0 \) otherwise. We consider only the \((22,3m)-\)resonances (for \( m = 3, 1 \), since the orbital effect decreases with increasing \( l \). The relevant Wigner \( D\)-functions are

\[
    D_{23}^{(3)} = e^{-2i\alpha} \sin \beta \left[ \cos \beta \right]^{5/2},
\]

(4.7)

\[
    D_{21}^{(3)} = e^{-2i\alpha} \sin 10 \sin \beta \left[ \cos \beta \right]^{3/2} \left[ 2 \sin \beta \right]^{2} - \left( \cos \beta \right)^{2}.
\]

(4.8)

For the \((22,33)-\)resonance, we find, to leading order in \( \hat{\Omega} \), that the gravitational wave frequency is given by

\[
    \nu_{gw} = \frac{2}{3} |\nu_{\alpha}| = \frac{8}{9} \nu_{s},
\]

(4.9)

where in the second equality, we have dropped the \( \hat{\Omega}^{3} \) term in
equation (3.32). The orbital change is given by

$$
(\Delta N_{\text{orb}})_{22,33} = +3.0 \times 10^{-4} \frac{Rc^2}{GM} \frac{1}{q(1+q)^{5/3}} \times \left[ \frac{\hat{\omega}_m^{1/3}}{\hat{\omega}_n^{1/3}} \Omega_s \right]^4 \left( \sin \beta \right)^2 \left( \cos \beta \right)^{10/3} 
$$

$$
= +4.6 \times 10^{-5} M_{1.4}^{2/3} R_{10}^{10} \frac{1}{q(1+q)^{5/3}} \times \left( \frac{v_s}{100 \text{ Hz}} \right)^{10/3} \left( \sin \beta \right)^2 \left( \cos \beta \right)^{10} . 
$$

(4.10)

Recall that the r-mode has negative energy, and as a result, \((\Delta N_{\text{orb}})_{22,33}\) is positive, i.e., the resonance slows down the inspiral. At the maximum \(v_{gw} = 1000 \text{ Hz}\) (or \(v_s = 1125 \text{ Hz}\)) and \(\beta = 48^\circ\), which would give the maximum orbital change, we have

$$
(\Delta N_{\text{orb}})_{22,33}^{\text{(max)}} = 0.010 M_{1.4}^{2/3} R_{10}^{10} \frac{1}{q(1+q)^{5/3}} . 
$$

(4.11)

The orbital change increases with increasing spin frequency. However, note that our analysis of r-mode is based on the assumption of slow rotation, and so (4.11) should be considered as an estimate.

For the \((22,31)\)-resonance, we have

$$
v_{gw} = 2 |r_a| = \frac{8}{3} v_s 
$$

(4.12)

and

$$
(\Delta N_{\text{orb}})_{22,31} = +4.3 \times 10^{-4} \frac{Rc^2}{GM} \frac{1}{q(1+q)^{5/3}} \left| \frac{\hat{\omega}_m^{1/3}}{\hat{\omega}_n^{1/3}} \Omega_s \right|^{4/3} \left( \sin \beta \right)^2 \left( \cos \beta \right) \left[ 2 \left( \sin \beta \right)^2 - \left( \cos \beta \right)^2 \right] \left( \frac{v_s}{100 \text{ Hz}} \right)^{10/3} 
$$

$$
= +6.6 \times 10^{-5} M_{1.4}^{2/3} R_{10}^{10} \frac{1}{q(1+q)^{5/3}} \left( \frac{v_s}{100 \text{ Hz}} \right)^{10/3} \times \left( \sin \beta \right)^2 \left( \cos \beta \right) \left[ 2 \left( \sin \beta \right)^2 - \left( \cos \beta \right)^2 \right] \left( \frac{v_s}{100 \text{ Hz}} \right)^{10/3} 
$$

(4.13)

Again, at the optimal parameters \(v_{gw} = 1000 \text{ Hz}\) (or \(v_s = 375 \text{ Hz}\)) and \(\beta = 34^\circ\), we have

$$
(\Delta N_{\text{orb}})_{22,31}^{\text{(max)}} = 2.0 \times 10^{-4} M_{1.4}^{2/3} R_{10}^{10} \frac{1}{q(1+q)^{5/3}} . 
$$

(4.14)

Thus we conclude that for canonical NS parameters, the \((22,33)\)- and \((22,31)\)-resonances (and indeed all r-mode resonances) produce orbital changes that are below the limits of interest. However, if we consider a larger NS radius, such as \(R = 15 \text{ km}\) for a \(M = 1.4 \text{ M}_\odot\) NS, which is allowed for some nuclear equations of state (e.g. Shapiro & Teukolsky 1983), then the orbital change for the \((22,33)\)-resonance may become significant \((\Delta N_{\text{orb}} \sim 0.6)\).

### 5 SUMMARY AND DISCUSSION

In this paper we have presented a systematic study on the resonant mode excitations of rotating neutron stars in coalescing binaries and their effects on the gravitational waveform. The essential effect of rotation is that it can modify the mode frequency in the inertial frame, thereby changing the strength of the g-mode resonance, which already exists for non-rotating neutron stars, and also making a variety of new resonances involving f-modes and r-modes possible.

We find that for the f-mode resonance to occur during the last few minutes of the binary inspiral, with the emitted gravitational wave frequency \(10 < v_{gw} < 1000 \text{ Hz}\) (the sensitivity band of the interferometric gravitational wave detector), the neutron star must have very rapid rotation (see Fig. 1), e.g., assuming canonical neutron star mass and radius, \(M = 1.4 \text{ M}_\odot\) and \(R = 10 \text{ km}\), respectively, \(v_s \geq 710 \text{ Hz}\) for the \((jk,m) = (22,2)\) resonance (see the beginning of Section 2.2 for notation) and \(v_s \geq 570 \text{ Hz}\) for the \((33,3)\)-resonance. Observations (e.g., PSR 1913 + 16 has spin period 59 ms, and PSR 1534 + 12 has 38 ms) and our current understanding of the formation of neutron star binaries seem to indicate that neutron stars in compact binaries (with another neutron star or a black hole as a companion) can never achieve such a rapid rotation, although such a rotation rate is physically allowed. For a \(M = 1.4 \text{ M}_\odot\) and \(R = 15 \text{ km}\) neutron star, the critical frequencies are lowered, e.g., \(v_s \approx 330 \text{ Hz}\) for the \((22,2)\)-resonance and \(v_s \approx 260 \text{ Hz}\) for the \((33,3)\)-resonance (see Fig. 3). If the neutron star in a coalescing binary indeed has the required rapid rotation, then the induced orbital change due to the f-mode resonance is significant (see Figs 2 and 4) and must be included in the templates of waveforms used for searching gravitational wave signals.

The resonance of g-modes is strongly affected by even a modest rotation (see Fig. 5). Although there are large uncertainties associated with the property of the g-mode (since it depends on the symmetry energy of nuclear matter), we show that for a canonical \(1.4 \text{ M}_\odot\) neutron star, the orbital change \(\Delta N_{\text{orb}}\) due to the g-mode resonance is below the \(10^{-2}\) level (see Fig. 6). Such a small phase error in the waveform is unimportant for the search of gravitational wave signals from the binary (E. Flanagan, private communication). However, we note that if we consider very low-mass \((M \leq 0.5 \text{ M}_\odot)\) neutron stars, which have large \(R/M\) ratio, the orbital change due to g-mode resonance may well be significant. Of course, although such low-mass neutron stars are physically allowed, there is no astrophysical evidence for their existence.

Resonant excitations of r-modes occur for misaligned spin–orbit configurations. Since the r-mode always has negative energy, the resonance actually transfers energy to the orbit, slowing down the inspiral. Since the tidal coupling of the r-mode depends strongly on the stellar rotation rate, we find that for canonical neutron star mass and radius \((1.4 \text{ M}_\odot\) and 10 km, respectively), the orbital change \(\Delta N_{\text{orb}}\) is negligible. However, if we consider a larger stellar radius, such as \(15 \text{ km}\) for a \(1.4 \text{ M}_\odot\) neutron star (as is allowed by some nuclear equations of state), then \(\Delta N_{\text{orb}}\) may become important, especially in the high-frequency band \((v_{gw} \approx 1000 \text{ Hz})\) (see equations 4.10 and 4.11).

Finally, we note that our calculations of the properties of the f-modes and r-modes are based on incompressible neutron star models. Our results for the r-modes should break down for very high rotation rates. Since neutron stars are not exactly barotropic, other r-modes, in addition to the \(j = k\) mode considered in this paper, can also exist. It is desirable to study the resonance effects using more realistic neutron star models.
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