Test-particle motion in superposed Weyl fields

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ABSTRACT
In a previous paper (Paper I), several superposed Weyl solutions of Einstein equations were investigated that might have some astrophysical relevance. We drew the gravitational field lines in the superposed space–times and the distortions of the Schwarzschild event horizon induced by the additional matter. Here we are concerned with the motion of free test particles in the same fields. In particular, the influence of the parameters (mass and location) of the additional sources on the positions of important circular equatorial geodesics is studied.

Key words: black hole physics – gravitation – relativity.

1 INTRODUCTION
One of the goals of relativistic astrophysics is to find the solution of Einstein’s equations describing the stationary system of a (rotating) compact body surrounded by an axially symmetric formation of (realistic) material. The results in this direction include perturbations of space–times of the central bodies (Will 1974: Schwarzschild + rotating ring; Chrzanowski 1976; Demianski 1976: Kerr + ring – distortion of the horizon), numerically constructed space–times (Lanza 1992: Kerr + thin disc; Nishida & Eriguchi 1994: Kerr + thick toroid), and also exact solutions. The latter have so far been explicitly found only in the static axisymmetric (Weyl) case (Chakrabarti 1988: Schwarzschild + Bach–Weyl ring; Lemos & Letelier 1994: Schwarzschild + inverted Morgan–Morgan thin disc; Klein 1997: Schwarzschild + isochrone thin disc), when one of the relevant Einstein equations can be written as a Laplace equation for the metric function interpreted as gravitational potential, and the fields of multiple sources can be obtained by linear superposition.

In a previous paper [Semeráč, Zellerin & Žáček 1999, hereafter Paper I (this issue)] we investigated several superposed Weyl space–times that might have some astrophysical relevance – those of a Schwarzschild black hole with a Bach–Weyl ring (Chakrabarti 1988) and with an inverted first Morgan–Morgan disc (Lemos & Letelier 1994), and those of an Appell ring (Gleiser & Pullin 1989), generating a field somewhat similar to the Kerr one, with the same two external sources. We drew the gravitational field lines in the superposed space–times and distortions of the Schwarzschild horizon induced by the additional matter. The results confirmed what one expected. We refer to Paper I for a more thorough introduction, references and notation. Let us only repeat here that the metric is considered in standard Weyl coordinates \( x^\mu = (t, \rho, \phi, z) \) in the form

\[
ds^2 = -e^{2\nu} dt^2 + e^{-2\lambda} d\rho^2 + e^{2(\lambda - \nu)} (d\phi^2 + dz^2),
\]

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where \( \nu \) and \( \lambda \) are functions of \( \rho \) and \( z \) only (see Paper I for \( \nu \) and \( \lambda \) corresponding to particular space–times considered here), \( \lambda \) being zero at the vacuum parts of the axis \( \rho = 0 \). Parameters of the central sources are \( M \) and \( a \), while \( M \) and \( b \) specify the external sources. The signature of the metric is \( -+++ \), Greek indices run from \( 0 \)–\( 3 \) and Latin indices from \( 1 \)–\( 3 \); geometrized units are used in which \( c = G = 1 \), \( c \) being the speed of light in a vacuum and \( G \) the gravitational constant.

In the present paper we study the motion of free test particles in the same superposed fields as in Paper I. Section 2 shows major types of motion in space–times obtained with different parameters (mass and location) of the additional sources. In Section 3, the influence of these parameters on the positions of important circular equatorial geodesics is studied. The results can be compared with those obtained by Abramowicz et al. (1984) (within Newtonian theory), Chakrabarti (1988) (within the exact superposition of a Schwarzschild hole with a Bach–Weyl ring) and Khanna & Chakrabarti (1992) (in the field of a velocity-dependent pseudo-Newtonian potential, simulating a Kerr black hole, superposed with a Bach–Weyl ring exact solution with an ad hoc dragging term added to mimic the rotation of the ring\(^1\)).

\(^1\) When treating a similar problem with a Schwarzschild black hole, Fukue & Yamamoto (1986) used a similar pseudo-Newtonian approach involving the Paczyński–Wiita potential.
2 MOTION IN THE \((\rho, z)\) PLANE

In the four types of superposed space–times (Schwarzschild + ring, Schwarzschild + disc, Appell + ring, Appell + disc), considered with three different masses of the external sources \((M = 0.3M, M = 5M, M = 9M)\) being the mass of the central source), we follow three types of test motion: faster and slower particles ejected radially outwards from the vicinity of the centre, and those falling from rest at a larger distance. By faster/slower we mean those which can/cannot escape, having hyperbolic/elliptic energy with respect to infinity \(E > 1/\mathcal{E} < 1\). Note, however, that even ‘faster’ particles have \(\mathcal{E} < 1\) in the ‘supermassive’, \(M = 5M\) cases (this unrealistic additional mass was considered purely for interest and to demonstrate how strong a source the black hole is – see the figures). The resulting trajectories are depicted in the natural coordinates of the background source, i.e. in (prolate) Schwarzschild coordinates for a Schwarzschild centre, and on Weyl axes with a mesh of oblate coordinates for an Appell centre; relations to the Weyl system are, respectively:

\[
\rho = \sqrt{r(\rho - 2M)} \sin \theta, \quad z = (\rho - M) \cos \theta, \tag{2}
\]

and

\[
\rho = \sqrt{r^2 + a^2} \sin \theta, \quad z = r \cos \theta; \tag{3}
\]

\(r\) and \(\phi\) remain unchanged. In the second transformation, \(a\) is the Weyl radius of the Appell ring.

The trajectories are solutions of the geodesic equation,

\[\dot{x}^\mu + \Gamma_{\alpha\beta}^\mu x^\alpha \dot{x}^\beta = 0,\]

where \(\Gamma_{\alpha\beta}^\mu\) are Christoffel symbols of the space–time metric and the dot denotes differentiation with respect to the proper time of the particles, \(\dot{x}^\mu = u^\mu\) being their four-velocity. In a given space–time, each particle starts from a chosen initial point \(x^\mu_{in}\) with a chosen initial four-velocity \(u^\mu_{in}\). We parametrize the four-velocity in terms of speed \(\hat{v}\) and direction \((\hat{\alpha}, \hat{\beta})\) measured in a static \((x^\prime = \text{constant})\) local orthonormal tetrad made of unit vectors (no summation over \(\kappa\)),

\[e^\mu_k = |g_{\kappa\lambda}|^{-1/2} \delta_{\kappa}^k, \tag{5}\]

each pointing alone one (the \(k\)th) of the global (i.e. Schwarzschild or oblate) coordinates:

\[u^\mu = e^\mu_k u^k, \tag{6}\]

where the tetrad components of four-velocity read

\[u^k = \gamma (1, \hat{v} \cos \hat{\alpha}, \hat{v} \sin \hat{\alpha} \cos \hat{\beta}, \hat{v} \sin \hat{\alpha} \sin \hat{\beta}), \tag{7}\]

\(\gamma = \frac{1}{1 - u^\mu u_\mu} = (1 - \hat{v}^2)^{-1/2}\) being the relative boost factor (the relation to the conserved specific energy of a particle at infinity, \(E = -u^\mu u_\mu\), is \(E = E^\gamma\)).

We take a Schwarzschild black hole with mass \(M\) or an Appell ring with mass \(M\) and Weyl radius \(a = 3M\), and place an additional source with mass \(M = 0.3M, M = 5M\) (namely the ring or the inner edge of the disc) near the marginally stable circular geodesic of the background space–time (namely at \(r = 6M\) for the Schwarzschild case and \(r = 9M\) for the Appell case; see Section 3). The geodesic equation is solved numerically using the controlled sixth-order Runge–Kutta method for the chosen superposed Weyl solution (see Paper I for the metrics and details).

We follow the ejected particles in the meridional section within the interesting, strong-field central regions of the superposed space–times. As the arrangement of Figs 1–7 is very similar, we describe it here rather than repeat it in each caption. Eight trajectories are shown (solid lines) in each of the plots, together with the axes (which are \(r \sin \theta\), \(r \cos \theta\) in the case of a Schwarzschild centre, and \(\rho, z\) in the case of an Appell centre), the central source (empty half-circle for the Schwarzschild horizon at \(r = 2M\); solid absissa with a bullet at the end for the Appell disc at \(r = 0\) with singular rim at \(\theta = 90^\circ\); see the appendix of Paper I for the structure of Appell space–time) and the external source (bullet for the Bach–Weyl ring at \(r = 6M\) and \(\rho = 9M\); very thick solid half-line for the Lemos–Letelier annular disc with non-singular inner edge at \(r = 6M\) and \(\rho = 9M\)). The Schwarzschild coordinate mesh (in plots with a black hole) and oblate spheroidal mesh (in those with an Appell ring) are dotted. The values written along the axes are in the units of \(M\). Starting points, velocities and energies of the particles are given in figure captions.

3 IMPORTANT EQUATORIAL ORBITS

As discussed in Paper I, we chose, from the exact superpositions available at present, those that are physically realistic, and which can approximate the existing astronomical objects modelled by accreting black holes. One of the essential characteristics of such systems is the position of the important circular geodesics in the equatorial plane \(z = 0\) with respect to which all the superposed fields are reflectionally symmetric) of the marginally stable, marginally bound and photon orbits. Accretion flow is strongly dependent on this position: the marginally stable orbit is assumed to roughly represent the inner edge of the accretion disc, the marginally bound orbit bounds the region from where the particles perturbed away from free equatorial circular motion can escape to infinity, and the circular photon trajectory is the limiting orbit below which circular motion is impossible. The positions of these three orbits in the fields of \textit{isolated} rotating black holes were studied by Bardeen, Press & Teukolsky (1972).

We note that the presumed position of the inner rim of an accretion disc was also our desideratum for where to place the additional sources (Paper I) – we put the Bach–Weyl ring and the inner edge of the inverted Morgan–Morgan disc near the marginally stable orbit of the background space–time. As the additional sources are self-gravitating, one should, however, consider the marginally stable (or other in some other respect) orbit of the \textit{total} space–time to be even more realistic. Of course, this is a more complicated, coupled problem. We plan to return to it elsewhere. [In any case, the self-gravitating ring with zero cross-section is singular, so it cannot lie at the (last stable) geodesic.]

We are interested in the dependence of the positions of important circular orbits on parameters of the additional sources. Using the formulae valid in stationary axisymmetric fields (e.g. Semerák 1998), one finds that in the static (Weyl) case the four-velocity and four-acceleration of steady circular motion (\(\rho = \text{constant}, z = \text{constant}, \) and azimuthal angular velocity with
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respect to infinity $\Omega = \frac{d\phi}{dt = \text{constant)}$ are

$$u^\mu = u'(1, 0, \Omega, 0),$$

$$a_\mu = -\frac{1}{2}(u')^2(g_{\phi\phi} + g_{\phi\mu}\Omega^2),$$

where $(u')^{-2} = -g_{tt} - g_{\phi\phi}\Omega^2$. The orbit is time-like for $(u')^{-2} > 0$, thus for $|\Omega| < \Omega_{\lim}$, where

$$\Omega_{\lim} = \sqrt{-g_{tt}/g_{\phi\phi}} = \frac{c^2}{\rho}$$

denotes the limiting physical magnitude of $\Omega$, corresponding to luminal motion. Owing to the symmetries, equatorial circular orbits have $a_r = a_\phi = a_z = 0$.

Equatorial circular geodesics also have $a_\rho = 0$, which implies

$$\Omega_z^2 = \Omega_\phi^2 = \frac{\rho v_\phi}{1 - \rho v_\rho}\Omega_{\lim}^2.$$  

The photon circular geodesics correspond to $\Omega_z^2 = \Omega_{\lim}^2$, i.e.

$$2\rho v_\rho = 1.$$  

Figure 1. Eight particles ejected radially outwards from a static local frame at $r = 3M$ and $\theta = 10^\circ, 20^\circ, \ldots, 80^\circ$ in the field of a Schwarzschild hole with mass $M$. The additional sources have mass $M = 0.3M$, $M$ and $5M$, as taken from top to bottom. In the left three plots, the additional source is a Bach–Weyl ring placed at $\theta = 90^\circ$, $r = 6M$; the particles have initial speed $\dot{\beta} = 0.885$, which gives them specific energy at infinity $E$ ranging from (1.172, 1.026, 0.481) at the axis to (1.164, 1.005, 0.434) at the equatorial plane (respectively from top to bottom plots). In the right three plots, an equatorial Lemos–Letelier annular disc has its inner edge at $r = 6M$; the initial speed is $\dot{\beta} = 0.850$ and $E$ ranges from (1.069, 1.008, 0.720) to (1.068, 1.004, 0.707) in the same sense.
The angular velocity of a marginally bound orbit is given by the condition that the specific energy with respect to infinity equals unity, \( \mathcal{E} = \mathcal{E}_\infty = 1 \), thus
\[
\Omega^2 = \Omega_{mb}^2 = (1 - e^{2\nu}) \Omega_{lim}^2,
\]
which yields, when equating \( \Omega^2 \) to \( \Omega_{mb}^2 \), the explicit relation
\[
2p\nu_\rho - 1 = e^{2\nu}(p\nu_\rho - 1). \quad (14)
\]
For stable circular motion, the square of the specific angular momentum with respect to infinity,
\[
\mathcal{L}^2 = u_\phi^2 = (g_{\phi\phi}u_\phi)\Omega^2, \quad (15)
\]
must increase with \( \rho \). The marginally stable circular equatorial geodesic is then obtained from equation \( d\mathcal{L}^2/d\rho = 0 \) with \( \mathcal{L}^2 \) evaluated at \( \Omega = \Omega_k \), i.e. with
\[
\mathcal{L}^2 = \frac{\rho^3 e^{-2\nu} \nu_\rho}{1 - 2p\nu_\rho}. \quad (16)
\]

**Figure 2.** The less energetic counterpart of Fig. 1. In the left three plots with Bach–Weyl ring, the particles have initial local speed \( \dot{v} = 0.835 \) and specific energies at infinity \( \mathcal{E} \) ranging from (0.991, 0.868, 0.407) at the axis to (0.985, 0.850, 0.367) at the equatorial plane (respectively from top to bottom). In the right three plots with Lemos–Letelier annular disc, the particles have initial speed \( \dot{v} = 0.805 \) and \( \mathcal{E} \) ranging from (0.949, 0.895, 0.639) to (0.948, 0.892, 0.628) in the same sense. Other parameters are the same as in Fig. 1.
The equation implies\(^3\)

\[
\rho v_{\rho\rho} + 2\rho v^2_{\rho}(2\rho v_{\rho} - 3) + 3v_{\rho} = 0. \quad (17)
\]

\(^3\)Bardeen (1970) gave a result valid for a general steady circular motion in a stationary axisymmetric field. The crucial expression, given on the right-hand side of his equation (24), should, however, read

\[
-(1 + v^2)v_{\rho\rho} + v^2\beta_{\rho\rho} - v^2\frac{\omega\rho}{\Omega - \omega} + 4v^2\rho v_{\rho} + v^4\frac{\omega^2}{(\Omega - \omega)^2}
\]

\[
-2v^2(\beta^2 - 2\beta_{\rho}v_{\rho}) - \frac{4}{\rho}(\beta_{\rho} - v_{\rho})v^2 - \frac{3}{\rho^2}v^2 + 2(1 - v^2)v_{\rho}^2
\]

(written exactly in Bardeen’s way), or (in a shorter form)

\[
-\nu_{\rho\rho} - 2\nu_{\rho} + \left(2\nu_{\rho} + \frac{v^2\omega_{\rho}}{\Omega - \omega}\right)^2
\]

\[
-\nu^2\left[\nu_{\rho\rho} - \beta_{\rho\rho} + \frac{\omega_{\rho}}{\Omega - \omega} + 2\left(\beta_{\rho} - v_{\rho} + \frac{1}{\rho}\right)^2 + \frac{1}{\rho^2}\right].
\]

where \(v = e^{i\beta - 2\beta(\Omega - \omega)}\) is a relative velocity with respect to the local zero-angular-momentum observer (who orbits with \(\Omega = \omega\)). Note that Bardeen’s \(\mu\), our \(\lambda - \nu\) and \(\omega = -\gamma_{0\phi}/\gamma_{00}\) is the dragging angular velocity; \(\beta\) is zero in our static case.
satisfied (see Paper I), in particular not for ring sources. Therefore we instead give the most explicit forms of the equations — those that will actually be solved (numerically) below to find the positions of the important orbits.

For a pure Schwarzschild solution, the positions of the important circular geodesics are familiar as $r_{ph} = 3M$, $r_{nb} = 4M$ and $r_{ms} = 6M$. For an Appell field, one finds them as solutions, respectively, of the equations

\[ (p^2 - a^2)^{3/2} - 2Mp^2 = 0, \]
\[ (1 - e^{-2M/\sqrt{p^2 - a^2}})(p^2 - a^2)^{3/2} - Mp^2 = 0, \]
\[ (p^2 - a^2)(p^2 - 4a^2) + 2Mp^2[2Mp^2 - 3(p^2 - a^2)^{3/2}] = 0. \]

or, as written in terms of the oblate radius $r$,

\[ r^3 - 2M(r^2 + a^2) = 0, \]
\[ (2 - e^{-2M/r})[r^3 - M(r^2 + a^2)] - r^3 = 0, \]
\[ r^3 - 3a^2r^2 + 2M(r^2 + a^2)[2M(r^2 + a^2) - 3r^3] = 0. \]

In the space–time of the Appell ring of radius $a = 3M$, considered in our figures, one finds the roots $p_{ph} = 4.60M$, $p_{nb} = 6.32M$ and $p_{ms} = 9.57M$, or $r_{ph} = 3.48M$, $r_{nb} = 5.57M$ and $r_{ms} = 9.08M$. Positions (both Weyl and oblate radii) of the important orbits in the Appell field are plotted in Fig. 8 against the radius of the ring $a$.

The behaviour of the important orbits for very large masses of the external sources follows, when writing $v = v_{\text{centre}} + v_{\text{ext}}$ and fixing $\theta = 90^\circ$, from the fact that $v_{\text{ext}}$ is negative and directly proportional to $M$ for both the Bach–Weyl ring and the Lemos–Letelier annular disc (see Paper I, sections 3.3 and 3.4). Hence, when $M \to \infty$, equation (14) reduces to equation (12) because its right-hand side vanishes, i.e. the marginally stable orbit coincides with the photon orbit in that limit. Two or three different asymptotic ($M \to \infty$) solutions of equation (12) can be found.

(i) At small radii (below the external source), $v_{\text{ext},p} < 0$, so the asymptotic solution lies where $v_{\text{centre},p}$ goes to (plus) infinity. In the case of a Schwarzschild hole this happens on the horizon ($p = 0$), while in the case of an Appell ring it holds at its actual radius, $p = a$.

(ii) At very large (asymptotic) radii one can write $v_{\text{ext},p} \sim M/p^2(> 0)$ and $v_{\text{centre},p} \sim M/p^2$ which, for $M \to \infty$, yields $p = 2M$ as the solution of equation (12).

(iii) From the expressions for $v_{\text{ext}}$, calculated in section 4 of Paper I, one finds that, for the Lemos–Letelier disc, $v_{\text{ext}}(\theta = 90^\circ) = 0$ (so it does not diverge with $M$) at $p = \sqrt{3/2}b = 1.225b$. At this radius the third photon orbit can therefore be expected to be found for very large $M$ in both superpositions containing the Lemos–Letelier disc.

The marginally stable orbit(s) can only be found at very large radii for very large $M$. Using the asymptotic expressions $v_{\text{ext}} \sim M/p^2$, $v_{\text{pp}} \sim -2M/p^2$ in equation (17), we obtain $p = (3 \pm \sqrt{5})M$ for $M \to \infty$.

The above analysis is confirmed by Figs 9 and 10, which show the conditions of free circular motion in the equatorial planes of the four superposed space–times, with the additional sources lying respectively at $r = 9M$ (around a Schwarzschild hole) and $p = 12M$ (around the Appell ring with $a = 3M$). We use four grey levels to visualize the meaning of different sectors (and curves) in the plots. They indicate where circular motion is (i) possible and

**Figure 4.** Eight particles ejected from a static local frame at $\theta = 10^\circ$, $20^\circ$, ..., $80^\circ$ in the field of the Appell ring with mass $M$ and radius $a = 3M$. In the top two plots, they start radially outwards from $r = 2M$; in the first one, the initial speed is $v = 0.830$ and $\mathcal{E}$ ranges from 1.532 at the axis to 1.123 at the equatorial plane; in the second one, $v = 0.500$ and $\mathcal{E}$ ranges from 0.987 to 0.723. At the bottom, particles fall from rest ($v = 0$) from $r = 16M$; their $\mathcal{E}$ ranges from 0.941 to 0.939 in the same sense.
stable (dark grey), (ii) possible and bound but unstable (medium grey), (iii) possible but unbound and unstable (light grey), (iv) impossible (white). Note that the last region is bounded not only by photon orbits, given by equation (12), but also by the curve of $\nu_\rho = 0$, which is a limit for real $\Omega_\theta$ and $\xi$ – see equations (11) and (16).

When adding mass to the external sources, the original photon orbit and the marginally bound orbit sink towards the centre, while the marginally stable orbit goes up to meet its counterpart that has come down from the external source (thus the region of stability, existing between the marginally stable orbit of the central source and the additional source, shrinks and eventually disappears). Above the external ring, an additional photon, marginally bound and marginally stable orbits exist with increasing radii; on the other hand, two additional orbits of each of the types (thus the whole new regions of forbidden, unbound and unstable circular motion) arise within the external disc (for very large masses, even the third marginally stable orbit is seen just above the rim of the

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**Figure 5.** Eight particles ejected radially outwards from a static frame at $r = 2M$ and $\theta = 10^\circ$, $20^\circ$, ..., $80^\circ$ at the Appell ring with mass $M$ and Weyl radius $a = 3M$. The additional sources have mass $M = 0.3M$, $M$ and $5M$ from top to bottom. In the left three plots, the additional source is a Bach–Weyl ring at $\theta = 90^\circ$, $\rho = 9M$; the particles start with $\nu = 0.830$ and their specific energy at infinity $\xi$ ranges from $(1.483, 1.374, 0.890)$ at the axis to $(1.084, 1.000, 0.629)$ at the equatorial plane (respectively from top to bottom plots). In the right three plots, an equatorial Lemos–Letelier annular disc has its rim at $\rho = 9M$; there $\nu = 0.805$ and $\xi$ ranges from $(1.420, 1.375, 1.140)$ to $(1.040, 1.006, 0.831)$ in the same sense.
disc, however in a forbidden region). Below the ring, a region of \( \nu_\rho < 0 \) (within dashed lines) grows where circular motion is impossible; in the case of the disc, this region only occurs at some minimal \( M \) and then widens gradually both below and above the inner radius of the disc. The ring (left plots) apparently divides the space–time into two parts in the equatorial plane, each of which has its own important orbits; the disc figures (right, see upper plots) would resemble that with a ring if extended to asymptotic values along the axes.

In Figs 9 and 10, the ring (left) is represented by a dot at given (\( M \), radius of the ring), while the disc (right) corresponds to a vertical line going upwards from (\( M \), radius of the inner rim of the disc). Shifting the additional source to the right from zero, one sees how the properties of circular motion in the resulting space–time change with increasing external mass \( M \). The picture obtained can be compared with several older results. Abramowicz et al. (1984) studied the effect of a self-gravitating thin torus on the Keplerian distribution of angular momentum in the central field in the Newtonian case. Our singular ring corresponds to their fig. 4(d), while the disc (with the major part of the matter concentrated

Figure 6. The less energetic counterpart of Fig. 5. In the left three plots with a Bach–Weyl ring, the particles have initial local speed \( \hat{\theta} = 0.500 \) and specific energies at infinity \( \mathcal{E} \) ranging from (0.955, 0.885, 0.573) at the axis to (0.698, 0.644, 0.405) at the equatorial plane (respectively from top to bottom). In the right three plots with a Lemos–Letelier annular disc, the particles have initial speed \( \hat{\theta} = 0.530 \) and \( \mathcal{E} \) ranging from (0.994, 0.962, 0.798) to (0.728, 0.704, 0.581) in the same sense. Other parameters are the same as in Fig. 5.

at the inner edge) could match their figs 4(a)–(c). The results are very similar, the main differences (with obvious reasons) being as follows.

(i) In the vicinity of the central body, there always remains a region of possible free circular motion and a smaller region of stability in the Newtonian case, whereas in relativity there always exist a region around the centre where circular motion is not possible and a larger region where it is not stable.

(ii) Above the external source, circular motion is everywhere possible in the Newtonian case, whereas in relativity a finite forbidden region occurs.

Notice that in both descriptions it is impossible for particles to circulate freely in a certain region below the external source (within dashed lines): there, attraction of the external source overwhelms that of the centre and ‘no $\Omega$ is small enough’ (below the photon orbit of relativity, in contrast, no $\Omega$ is large enough). As seen in ‘global’ plots with disc, the forbidden region grows (with $M$) not only below, but also above the inner rim of the disc,

\textbf{Figure 7.} Particles falling from rest ($\dot{v} = 0$) from a static local frame at $r = 16M$ and $\theta = 10^\circ, 20^\circ, \ldots, 80^\circ$ in the same fields as in Figs 5 and 6. In the left three plots with a Bach–Weyl ring at $\rho = 9M$, the specific energy of the particles at infinity $\mathcal{E}$ ranges from (0.926, 0.891, 0.716) at the axis to (0.921, 0.879, 0.674) at the equatorial plane (respectively from top to bottom). In the right three plots with Lemos–Letelier annular disc going from $\rho = 9M$, $\mathcal{E}$ ranges from (0.932, 0.911, 0.797) to (0.926 0.897, 0.744) in the same sense.
making the interpretation of the latter in terms of free orbital motion no longer possible. However, this happens only for $M > 1.9843M$ (in the case of a Schwarzschild centre; see upper right of Fig. 9) and $M > 2.2033M$ (in the case of an Appell centre; see upper right of Fig. 10) with our choices of $b$, with realistic estimates of $M$, all matter within the disc is in stable physical motion.

Figs 9 and 10 may also be compared with an example of an angular-momentum curve for the hole ± Bach–Weyl ring system given in fig. 3 of Chakrabarti (1988) or fig. 1 of Khanna & Chakrabarti (1992) (fig. 2a of the latter shows how the radii of the marginally stable and marginally bound orbits behave for low masses of the ring). Cf. also fig. 3 of Lanza (1992) and fig. 5 of Nishida & Eriguchi (1994) for a similar curve obtained in numerical space–times of a rotating hole, respectively with a disc and a toroid.

### 4 CONCLUDING REMARKS

In Paper I we confirmed, for several astrophysically relevant superposed Weyl space–times, that self-gravitating external sources modify the shape of the field of a central body in a predictable way. The present paper shows that additional sources also influence the motion of test particles according to intuition. In Section 2 we saw that the trajectories of test particles are more and more attracted by external matter when its mass is increased. The positions of important equatorial circular geodesics turned out to be strongly dependent on position and mass of the external source (Section 3). The inverted first Morgan–Morgan disc around a Schwarzschild or Appell centre was shown to be interpretable in terms of counter-rotating streams of particles on stable time-like circular geodesics up to rather high mass.

When studying the behaviour of important equatorial orbits, we admitted masses $M$ of the external sources that were as large as several tens of times that of the ‘main’ source. A natural question would be whether a horizon does not occur that encloses the whole system (or whether it does not inflate to enclose the system – see Paper I, section 5) at some stage of increase of $M$. As pointed out in section 5 of Paper I, however, in Weyl coordinates the horizon may appear only at $\rho = 0$, so the answer is no. To understand this physically, we realize that by increasing the mass of the external source (placed at some $\rho = b$), one in fact makes the source greater and greater, namely its circumferential radius $R$ increases exponentially with $M$. Actually, in the equatorial plane ($z = 0$, or $\theta = 90^\circ$) the general formula

$$v_{\text{ext}} = -\frac{2M}{\pi(\rho + b)} K \left( k = \frac{2 \sqrt{\delta p}}{\rho + b} \right)$$

(24)

when the external source is a Bach–Weyl ring, while

$$v_{\text{ext}}(\rho > b) = -\frac{M}{\rho} \left( 1 - \frac{b^2}{2\rho^2} \right)$$

(25)

$$v_{\text{ext}}(\rho < b) = -\frac{M}{\rho^2} \left[ \left( 1 - \frac{b^2}{\rho^2} \right) \arccot \sqrt{\frac{\rho^2}{b^2} - 1} - 1 \right]$$

(26)

when it is a Lemos–Letelier annulus. At the actual Bach–Weyl ring ($\rho = b$), $v_{\text{ext}}$ (and thus $R$) is infinite; however, the circumferential radius of any circle at non-zero $\rho \neq b$ varies as $e^{\lambda A}$. For the inner edge (at $\rho = b$) of the Lemos–Letelier annular disc, one obtains

$$R = b \exp \left[ -\left( \sqrt{\frac{b^2 + M^2 - M}{b^2 + M} - \frac{M}{2b}} \right)^{1/2} \right] \exp \left( \frac{M}{2b} \right)$$

(27)

with the Schwarzschild centre (of mass $M$), while

$$R = b \exp \left( \frac{M}{\sqrt{b^2 - a^2}} \right) \exp \left( \frac{M}{2b} \right)$$

(28)

with the Appell centre (of mass $M$ and radius $a$).

With the exception of the horizon distortion, the results of both papers were presented in Weyl coordinates ($\rho$, $\varphi$) or in corresponding prolate (Schwarzschild) or oblate coordinates ($r$, $\theta$). It would instead be desirable to use something having geometrical meaning as a radial coordinate, so that the conclusions about the positions of additional sources and of important orbits have proper sense. The natural choice is either a ‘circumferential radius’ $R = \sqrt{\delta_{\phi\phi}}$ or a proper radial distance $P = \int \sqrt{\delta_{\rho\rho}} \, dp$. Unfortunately, neither of them is applicable for singular sources such as rings, when the map is usually not unique and the sources are typically shifted to infinity.

There are also other properties peculiar to singular toroidal solutions: the ring singularity typically has a non-trivial structure rather than being a simple line source – it is ‘directional’, not locally cylindrical (Hoenesaers 1995 and references therein). In the figures, this was indicated by the strange behaviour of test particles in the closest vicinity of both the Appell and the Bach–Weyl rings: the test trajectories preferably hit the innermost regions around the rings along certain directions while being ‘repelled’ elsewhere. Of course, this is mainly caused by the coordinates employed. A natural check, for example, is to compute a proper distance to the ring, $\int \sqrt{\delta_{\rho\rho}} \, dp + dx^2$, along different directions. Its behaviour at the ring can easily be found in terms of the toroidal variables used in section 3.2 of Paper I, $\sqrt{\Sigma}$ (mean distance to the ring) and $\psi$ (angle going round the ring, with $\psi = 0^\circ/180^\circ$ pointing outwards/inwards along the
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The $c$-dependence of those terms in $l^2_n$, which can make the proper distance infinite, i.e. which can tend to $+\infty$ quickly enough at $\Sigma \to 0^+$, is important. For the Appell ring, we obtain

$$\lambda - \nu = -(M^2/4)[\Sigma^{-2} a^2 \cos 2\psi + \Sigma^{-1}(1 + \cos \psi \cos 2\psi)] + O(\Sigma^{-1/2}),$$

thus the ring is at finite distance when approached from the right or left quadrant ($|\psi| \leq 45^\circ$ or $|\psi - 180^\circ| \leq 45^\circ$), whereas it is infinitely distant when approached from the top or bottom quadrant ($|\psi - 90^\circ| < 45^\circ$ or $|\psi - 270^\circ| < 45^\circ$). For the Bach–Weyl ring, one first uses the asymptotics $K(k) \propto \ln(4/\sqrt{1 - k^2})$, valid at $k \to 1^-$ (i.e. at the ring), for the elliptic integral $K(k)$ involved in the Bach–Weyl solution for $\nu$ and $\lambda$ (see section 3.3 of Paper I). Then

$$\lambda - \nu = -(2M^2/\pi^2)\Sigma^{-1} \cos \psi + O(\ln^2 \Sigma),$$

thus the ring is at finite/infinite distance when approached from the right/left half-space ($|\psi| < 90^\circ$ or $|\psi - 180^\circ| < 90^\circ$). Now, a fixed physical distance from the ring means a greater/smaller coordinate distance in the directions where $\lim_{\Sigma \to 0^+}(\lambda - \nu)$ is

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**Figure 9.** Positions of important equatorial circular orbits in the fields of a Schwarzschild hole of mass $M$ with a Bach–Weyl ring at $r = 9M$ (left column) and with a Lemos–Letelier annular disc with its inner edge at $r = 9M$ (right column). The Schwarzschild radius $r$ of the orbits (vertical axes, in the units of $M$) is plotted against the mass $M$ of the additional sources (horizontal axes, in units of $M$). Three different scales are chosen to show, from top to bottom, the overall arrangement and asymptotics, details of the shifting of the ‘original’ marginally bound (upper) and photon (lower) orbits, and details of the behaviour of the marginally stable orbit(s).

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small/large. Hence, the particles should appear ‘repelled’/'attracted’ in the directions from which the ring is physically nearby/far away. This agrees with what can actually be seen in the figures.

Astrophysically, however, singular rings are, by far, not as important as discs. For disc solutions with reasonable density, the ‘physical’ radial measures are unique and are also usually applied (e.g. Bičák, Lynden-Bell & Pichon 1993). It is our plan now to concentrate on the field of a Schwarzschild black hole surrounded by one of the two realistic dust discs — the annular disc of Lemos & Letelier (1994) or the isochrone disc of Klein (1997). The density profile of the former resembles that of an accretion disc, while the latter is mainly used to model galaxies. For each of the two possibilities, we will generate, in a future paper, a sequence of superposed solutions by gradually increasing the mass of the disc, while keeping its inner edge just at the marginally stable circular orbit of the complete space–time. We will plot the positions of this orbit (and also of the other important circular orbits), the horizon and density and the Keplerian velocity profile of the disc in terms of the circumferential radius, proper distance from the horizon and Schwarzschild radial coordinate.

Other interesting problems concerning superposed space–times are the appearance of self-gravitating external sources compared with that of test sources of similar shape and temperature, and the
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induction of chaos in the motion of test particles by self-gravity of the external source. Work in these directions is also in preparation.

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