A Generalization of Brémond-Valatin Method for Four-Body Correlation
and Its Application to a Charge Independent Pairing Interaction

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The new method presented by B. Brémond and J. G. Valatin is generalized which attached four body correlation to the Bogoliubov transformation. The generalization carried in this paper is similar to that from the Bogoliubov transformation to the generalized Bogoliubov one. We find a criterion of whether a trial ground state vector has an essential feature of four body correlation or it is like that of the generalized Bogoliubov transformation.

This method is applied to a system with a charge independent pairing interaction. There appear two different kinds of single quasiparticle excitation energy due to proton-neutron interaction.

§ 1. Introduction

First, we shall briefly review and compare several methods of particle number non-conserving canonical transformations.

It is well known that the Bogoliubov transformation is very useful in order to deal with the many fermion system with a certain type of pairing interactions. It is a unitary transformation which diagonalizes the approximate Hamiltonian linearized with respect to the states of the Bardeen, Cooper and Schrieffer's type.

It is expressed by

\[ U_B = \exp \sum_i \theta_i (a_i^* a_{-i} + a_i a_{-i}) \]  

where the coefficients \( \theta_i \) are real numbers and the single particle states \( i \) and \( (-i) \) are specially correlated (e.g. the states with momentum \( k \) and \( (-k) \) in the theory of superconductivity).

One of the defects of the Bogoliubov transformation (B.T.) is that we should know beforehand which states are strongly paired, and another is that it cannot deal with more than one interfering pairs on the same footing simultaneously, e.g. the system with not only proton-proton but also proton-neutron

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*) Unitarity of the Bogoliubov-like transformation is denied by H. Umezawa et al. and Y. Nambu et al. for different reasons. It can, however, be treated as unitary in its application to finite nuclear problems on which we shall concentrate our attention, since the number of states is denumerable and high energy states are usually cut off.
Attempts have been made to correct these defects through the invention of the generalized Bogoliubov transformation \(^{(3,6,7)}\) (G.B.T.)

\[
U_{G.B} = \exp\left[\left(\frac{1}{2}\right) \sum_{i,j} \left\{ \theta_{ij} a_i^* a_j^* - \text{h.c.} \right\} + \varphi_{ij} a_i^* a_j \right],
\]

where the matrix \(\theta_{ij}\) is antisymmetric and \(\varphi_{ij}\) is antihermitic. The structure of \(U_{G.B}\) is determined up to the algebra which commutes with the linearized Hamiltonian.\(^7\)

It is proved\(^{(7,8)}\) that there exist unitary transformations

\[
T_1(a) = \exp\left[\left(\frac{1}{2}\right) \sum_{i,j} \xi_{ij} a_i^* a_j \right]
\]

and

\[
T_1(\xi) = \exp\left[\left(\frac{1}{2}\right) \sum_{i,j} \xi_{ij}^* \xi_{ij} \right],
\]

which brings \(U_{G.B}\) to the canonical form

\[
U_{G.B} = T_1(\xi) U_n(b) T_1(a),
\]

where

\[
b_i = T_1(a) a_i T_1^{-1}(a),
\]

\[
\xi_i = U_n(b) b_i U_n^{-1}(b).
\]

Thus the above-mentioned defects of the Bogoliubov transformation are formally corrected. But even when there exist interfering interactions (e.g. \(p\)-\(p\) and \(n\)-\(p\)), those states constructed by the G.B.T. have not the additional binding energy attributed to four body correlation. This point has been noticed by many authors.\(^{(9,10)}\)

In this situation Brémond and Valatin\(^{(11)}\) proposed recently a different extension of the B.T. through inclusion of four body correlation. Their transformation (B.V.T.) reads\(^*\) as

\[
U_{B.V} = \exp \sum_i \left\{ \sum_{i=1}^{\nu} \theta_i^* (a_i^* \frac{a_i}{a_i^*} - \text{h.c.}) 
\right.
\]

\[
+ \theta^* (a_{i1}^* a_{i2}^* a_{i3}^* a_{i4}^* - \text{h.c.})
\]

+ (other elements of algebra associated with those explicitly written),\(^{(1.7)}\)

where \(\theta_i^*\) and \(\theta^*\) are real and it is assumed that single particle states are specified by the quantum number \([\sigma, i] i = 1, -1, 2, -2\).

In the state vectors constructed by the B.V.T., the four body wave functions are included, which are independent of the two body ones, while the four body correlated terms in \(U_B|0\rangle\) and \(U_{G.B}|0\rangle\) come out only as the products of two body correlations. Thus, the four body correlation of a limited type can certainly

\(^*\) The complete algebraic form of \(U_{B.V}\) is found in the Appendix of the reference 11).
be included, but because of the special form of the transformation (1.7), there still remain the same defects as mentioned above in the Bogoliubov transformation, i.e. we do not know how to select the states \((i, -i)\) (see also §2.2). If we, for example, assign 1 to proton and 2 to neutron, most of the effects of \(n-p\) interaction disappear (e.g. no contribution to pairing potential).

Therefore we here attempt to generalize the Bogoliubov transformation in two ways as above-mentioned simultaneously.\(^{*}\) A natural extension in this direction would be

\[
U' = \exp\left[\frac{1}{2} \sum_{i,j} (A_{ij} a_i^* a_j^* - \text{h.c.}) + B_{ij} a_i^* a_j^* \right] + \frac{1}{4!} \sum_{i,j,k,l} (C_{ijkl} a_i^* a_j^* a_k^* a_l^* - \text{h.c.}) + \cdots ,
\]

but too much complexities are brought in if \(i\) runs over many states. Thus we are led to a simpler one,

\[
U = \exp\left[\sum_{i,j} \left\{ \frac{1}{2} \sum_{i,j} \theta_{ij}^r (a_i^* a_j - \text{h.c.}) + \frac{1}{2} \sum_{i,j} \phi_{ij}^r a_i^* a_j \right. \right. \\
+ \left. \left. \theta^* (a_i^* a_j^* a_i^* a_j^* - \text{h.c.)} \right) \sum_{i,j,k,l} \theta_{ijkl}^r (a_i^* a_j^* a_k^* a_l^* - \text{h.c.}) \right. \right. \\
+ \left. \left. \sum_{i,j,k,l} \phi_{ijkl}^r (a_i^* a_j^* a_k^* a_l^* - \text{h.c.}) \right. \right. \\
+ \right. \left. (\text{other elements of algebra associated with those explicitly written}) \right] ,
\]  

(1.8)

where we restrict \(\theta_{ij}, \phi_{ij}, \theta^*\) and other coefficients to real numbers, of which restriction corresponds to the assumption of reality of the matrix elements of the Hamiltonian (see §2). In comparison with the transformations (1.7) and (1.8), it should be emphasized that the introduction of the Hartree-Fock transformation represented by \(\phi_{ij}^r\) in Eq. (1.8) makes the transformation \(U\) free to mix the vectors in \((1, 2, 3, 4)\) space (e.g. spin-isospin space). From now on we shall call this transformation generalized Brémond-Valatin transformation (G.B.V.T.).

The relation between \(U\) and \(U_{b,v}\) is not so simple as that between \(U_{c,b}\) and \(U_b\) given in Eq. (1.5). We shall discuss it in §2.

Two methods have been developed to calculate (except the above-mentioned arbitrariness) the parameters in these transformations. One of them is the variational method (such as of Bardeen, Cooper and Schrieffer\(^{3}\)) for state vector coefficients or density matrix, pairing tensor and others. The other is to introduce the quasiparticle by means of elimination of so-called dangerous terms.\(^{3}\)

\(^{*}\) Possibility of such a generalization has already been suggested by Brémond and Valatin in reference 11).
For the B.V.T. and the G.B.V.T., the variational method is preferable which, however, may become a little tedious since we should find not only the ground state but also the excited state wave functions.

We formulate the G.B.V.T. and study the difference between the trial state vector of the G.B.T. and that of the G.B.V.T. (especially as to four body correlation and volume dependence), and also the relation between $U_{b,v}$ and $U$ in § 2. Then we apply this method to the system with a charge independent pairing interaction and study the consequence of proton-neutron interaction to the usual pairing between the same kind of particles by this model in § 3.

§ 2. Formulation of the generalized Brémond-Valatin transformation

2.1 Formulation

As is described in § 1, the G.B.V.T. is expressed by

$$U = \exp S,$$

with

$$S = \sum \sigma S_{\sigma},$$

$$S_{\sigma} = \frac{1}{2} \sum_{i,j} \{ \theta_{ij}^{\sigma} (a_{\sigma i}^* a_{\sigma j} - a_{\sigma j} a_{\sigma i}) + \varphi_{ij}^{\sigma} a_{\sigma i}^* a_{\sigma j} \}$$

$$+ \sum_{i,j,k,l} \theta_{ijkl}^{\sigma} (a_{\sigma i}^* a_{\sigma j}^* a_{\sigma k} a_{\sigma l} - \text{h.c.})$$

$$+ (\text{other elements of algebra associated with those explicitly written}),$$

where $\theta_{ij}^{\sigma}$ and $\varphi_{ij}^{\sigma}$ are both real antisymmetric tensors and other coefficients are also real (see the Appendix). These restrictions will be discussed later.

Then the trial state vector of even particles is given by

$$\phi = \prod_{\sigma} \phi_{\sigma} \vert 0 \rangle,$$

with

$$\phi_{\sigma} = \alpha_{\sigma}^* + \frac{1}{2} \sum_{i,j} \alpha_{ij}^{\sigma} a_{\sigma i}^* a_{\sigma j} + \beta_{i}^{\sigma} a_{\sigma i}^* a_{\sigma 2} a_{\sigma 2}^* a_{\sigma 4}$$

where $\alpha_{ij}^{\sigma}$ is a real antisymmetric tensor and $\alpha_{\sigma}^*$ and $\beta_{i}^{\sigma}$ are real, and $\vert 0 \rangle$ denotes the vacuum state for the free Hamiltonian. From now on we drop the suffix $\sigma$ when no confusion occurs. The normalization relation is given by

$$\alpha_{\sigma}^* + \sum_{\sigma' \neq \sigma} \alpha_{ij}^{\sigma} \beta_{i}^{\sigma} = 1$$

for each $\sigma$.

Now we define the following quantities such as density matrix and pairing
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For calculating the expectation values of operators:

\[ h_{ij} = h_{ji} = \langle a_i^* a_j \rangle = \langle a_j^* a_i \rangle = \sum_k \alpha_{ik} \alpha_{kj} + \beta^2 \delta_{ij}, \]

\[ \chi_{ij} = -\chi_{ji} = \langle a_i^* a_j^* \rangle = \langle a_j a_i \rangle = a \alpha_{ij} + \frac{1}{2} \sum_k e_{ijkl} \beta \alpha_{kl}, \]

\[ \gamma_{ijkl} = \langle a_i^* a_j^* a_k a_l \rangle = \alpha_{ij} \alpha_{kl} + \beta^2 (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}), \]  

(2·6)

where the notation \( \langle \rangle \) denotes the expectation value with respect to the state \( \Phi \) and \( e_{ijkl} \) is the antisymmetrizing tensor.

The starting Hamiltonian is given by

\[ H = H_{s.p} + V, \]

\[ H_{s.p} = \sum_e \epsilon_e a_e^* a_e, \]

\[ V = \frac{1}{4} \sum_{\kappa_1 \kappa_2} \langle \kappa_1 \kappa_2 | V | \kappa_1' \kappa_2' \rangle a_{\kappa_1}^* a_{\kappa_2} a_{\kappa_1'} a_{\kappa_2'}, \]

(2·7)

where \( \kappa \) denotes the quantum number \((\sigma, i)\) and \( |\kappa_1 \kappa_2 \rangle \) is an antisymmetrized state, and so the following relations hold,

\[ \langle \kappa_1 \kappa_2 | V | \kappa_1' \kappa_2' \rangle = -\langle \kappa_2 \kappa_1 | V | \kappa_1' \kappa_2' \rangle \]

\[ = -\langle \kappa_1 \kappa_2 | V | \kappa_2' \kappa_1' \rangle = \langle \kappa_2 \kappa_1 | V | \kappa_1' \kappa_2' \rangle. \]  

(2·8)

As is mentioned in § 1, we assume that the matrix elements of Hamiltonian is real, that is,

\[ \langle \kappa_1 \kappa_2 | V | \kappa_1' \kappa_2' \rangle = \langle \kappa_1' \kappa_2' | V | \kappa_1 \kappa_2 \rangle. \]  

(2·9)

Time reversal invariance of Hamiltonian says that there exist bases which satisfy this condition. But this condition somewhat restricts the choice of the bases. For wider applicability, we consider the above condition as an assumption. Then we can restrict the transformations to proper real orthogonal ones in the \((\alpha, \alpha_i, \beta)\) space. This is the reason why we restrict the Lie algebra (2·2) to the antihermicity operators with real coefficients.

Because of the indefiniteness of the particle number of the trial state (2·3), a chemical potential \( \lambda_i \) should be introduced. Now we should treat the Hamiltonian

\[ H = H - \sum_i \lambda_i a_i^* a_i = \sum_i \eta_i a_i^* a_i + V, \]  

(2·10)

with \( \eta_i = \epsilon_i - \lambda_i \), subject to the condition that the expectation values of number of the same kind of particles are given one.

Now the expectation value \( W \) of the Hamiltonian \( H \) is found out to be

\[ W = \langle H \rangle \]

\[ = \sum_{i, j} \eta_{\alpha_i \alpha_j} h_{ij}^\alpha \]
where
\[ v^{ijkl} = \langle \sigma i \sigma j k | V | \sigma^j \sigma^l \rangle r^{ijkl}, \]

To find the minimum of \( W \), we vary the coefficients \((\alpha, \alpha_{ij}, \beta)\) under the condition (2.5) and the antisymmetry of \( \alpha_{ij} \). Then we obtain the following variational equation for each \( \sigma \):
\[ H \alpha = X \alpha, \]

where \( \alpha \) is a column vector
\[ \alpha = (\alpha, \alpha_{12}, \alpha_{33}, \alpha_{44}, \alpha_{55}, \alpha_{66}, \beta), \]
(written in row form to save the space).

The matrix elements of \( H \) are given by
\[ H_{\alpha \alpha} = H_{\alpha \beta} = H_{\beta \alpha} = 0, \]
\[ H_{\beta \beta} = \sum_{i=1}^{4} (\eta_i + u_{ij}) + \sum_{i<j}^4 v_{ijkl}, \]
\[ H_{\alpha i,ij} = H_{\beta i,ij} = \mu_{ij}, \]
\[ H_{\beta i,ij} = H_{\gamma i,ij} = \frac{1}{2} \sum_{k,l}^4 \delta_{ik} \delta_{j} \mu_{kl}, \]
\[ H_{ij,kl} = (\eta_i + \eta_j) \delta_{ik} \delta_{j} + v_{ijkl} + \delta_{ik} u_{jl} - \delta_{jl} u_{ik} + \delta_{ik} u_{jl} - \delta_{jl} u_{ik}. \]

Equation (2.13) is a non-linear self-consistent one since \( u_{ij} \) and \( \mu_{ij} \) are also functions of \( \alpha e^\sigma (\sigma' = \sigma) \).

Now the linearized Hamiltonian for \( \sigma \) in the sense of Brémond and Valatin becomes
\[ \hat{H} = \text{const} + \sum_{\sigma} \hat{H}_\sigma, \]
with
\[ \hat{H}_\sigma = \sum_{i} \eta_i a_i^\sigma a_i + \sum_{i,j} u_{ij}^\sigma a_i^\sigma a_j + \frac{1}{2} \sum_{i,j} \mu_{ij}^\sigma (a_i^\sigma a_j + a_j a_i). \]
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\[ + \frac{1}{4} \sum_{i \neq j, k} \varepsilon_{i j k l} a_{i \sigma}^* a_{j \sigma}^* a_{k \sigma} a_{l \sigma}, \]  

(2.17)

where \( u_{ij} \) and \( \mu_{ij} \) are expectation values for the ground state. This is still non-linear with respect to the state \( i \).

Then the even particle excited states are determined by the equation

\[ \hat{H}_\sigma \Phi_\sigma^{\text{odd}} |0\rangle = X^\sigma \Phi_\sigma^{\text{odd}} |0\rangle, \]  

(2.18)

where the state with odd particles in \( \sigma \) states is written by

\[ \Phi_\sigma^{\text{odd}}(\sigma) = \Phi_\sigma^{\text{odd}} \prod_{\sigma \neq \sigma} |0\rangle, \]  

(2.19)

with

\[ \Phi_\sigma^{\text{odd}} = \sum \alpha_{i \sigma}^* a_{i \sigma} + \frac{1}{3!} \sum \beta_{i \sigma}^* \varepsilon_{ijkl} a_{i \sigma}^* a_{j \sigma}^* a_{k \sigma} a_{l \sigma}^*. \]  

(2.20)

Thus the low lying excitation energies are approximately given by the difference between the eigenvalues \( X \) of Eqs. (2.13) and (2.18) while the ground state is given by the solution belonging to the lowest eigenvalue \( X_0 \) of Eq. (2.13).

2.2 Discriminant and relation between \( U_{\text{B.V}} \) and \( U \)

Now we shall study the structure of the state vector (2.3) concerning four body correlation. Unless the coefficient \( \alpha \) vanishes, the trial ground state vector is expressed by

\[ \Phi = \alpha \exp \left[ \frac{1}{2} \sum_{i \neq j} \frac{\alpha_{ij} a_{i \sigma}^* a_{j \sigma}^*}{\alpha} \right] \times \exp \left[ \frac{\alpha \beta - (1/8) \sum \varepsilon_{ijkl} \alpha_{ij} \alpha_{kl}}{\alpha^2} a_{i \sigma}^* a_{j \sigma}^* a_{k \sigma} a_{l \sigma}^* \right] |0\rangle, \]  

(2.21)

where we have omitted the suffix \( \sigma \) again. Thus if the following relation holds for every \( \sigma \),

\[ \alpha \beta - \frac{1}{8} \sum \varepsilon_{ijkl} \alpha_{ij} \alpha_{kl} = 0, \]  

(2.22)

the wave function of any number of particles can be represented by the product of two particle wave function.

More generally the relation (2.22) holds for the ground state given by the G.B.T. such that

\[ \Phi_{\text{G.B.T.}} = U_{\sigma}^{\text{G.B.T.}} |0\rangle, \]  

(2.23)

where
To prove this, we notice that the operator $D$

$$D = \frac{1}{2} (a_i^* a_j^* a_k^* a_l^* + a_i a_j a_k a_l)$$

satisfies the following commutation relations,

$$[D, a_i^* a_j^* - \text{h.c.}] = [D, a_i^* a_j - \text{h.c.}] = 0.$$  \hspace{1cm} (2.26)

Therefore the operator $D$ is invariant under the transformation (2.24). The expectation value $D$ of the operator $D$ with respect to the ground state is just given by

$$D = \alpha \beta - \frac{1}{8} \sum \varepsilon_{ijkl} \alpha_{ij} \alpha_{kl},$$  \hspace{1cm} (2.27)

which corresponds to the quantity $\gamma$ of reference 11, and will be called “discriminant”.* Since $\langle 0 | D | 0 \rangle = 0$, the discriminant $D$ for the state (2.23) also vanishes. Thus we have proved the above statement.

On the contrary, the commutators

$$[D, a_i^* a_j^* - \text{h.c.}],$$

$$[D, a_i^* a_j - \text{h.c.}],$$

etc., do not vanish. That is, the operator $D$ is not invariant under all transformations (2.1). Therefore, the discriminant $D$ for the state $U|0\rangle$ is usually not zero.

Moreover, we can prove that the $D$ invariant transformation group is just the group of the transformation (2.24). (The proof will be given in the Appendix.) This fact gives us the criterion whether all solutions of the variational equation (2.13) can be obtained by the simpler transformation (2.24) from the eigenstates of the free Hamiltonian, but unfortunately the condition of $D$ invariance is represented by a set of a great number of equations (36 for each $\sigma$) for the solutions $\alpha^{(\sigma)}$ of the variational equation. One of them is that the discriminant $D$ vanishes, only to which we restrict our attention afterwards.

Further the trial ground state vector with $D=0$ has just the same structure as that of the G.B.T. In fact, we can construct the quasiparticle operator $\xi$ by $U_{0,0}$ for the ground state $\Phi_0$ with $D=0$ as follows: We write the quasiparticle annihilation operator as

$$\xi_i = \sum_j \{(\text{ch} \theta)_{ij} a_j - (\text{sh} \theta)_{ij} a_j^*\},$$  \hspace{1cm} (2.29)

and define the antisymmetric tensor $\theta = (\theta_{ij})$ by an equation

\* The word “discriminant” used here has no relation to that in mathematics.
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\[
\begin{align*}
(tanh \theta)_{ij} &= \alpha_{ij} / \alpha \quad \text{if } \alpha \neq 0, \\
(coth \theta)_{ij} &= -\frac{\sum \varepsilon_{jkl} \alpha_{kl}}{2\beta} \quad \text{if } \beta \neq 0,
\end{align*}
\]

while if \( \alpha = \beta = 0 \), we define

\[
\xi_{1,2} = b_{1,2}, \quad \xi_{3,4} = b_{3,4}^\dagger,
\]

with

\[
b_i = T_1(a) a_s T_1^{-1}(a),
\]

where the transformation \( T_1(a) \) brings the ground state \( \left( \frac{1}{2} \sum \alpha_{ij} a_i a_j^* \right) |0\rangle \) into the canonical form \( \left( \sum \varepsilon_{ijkl} \alpha_{kl} \right) |0\rangle \), and the condition \( D=0 \) into \( \alpha_{ij} \alpha_{kl}^* = 0 \), and we choose the transformation \( T_1(a) \) with \( \alpha_{ij} = 0 \) for Eq. (2·32).

Using the definition of \( \theta \), the condition \( D=0 \) and the identity

\[
\sum_{j,k} \alpha_{ij} \alpha_{kl} \varepsilon_{jklm} = \text{const} \cdot \delta_{lm},
\]

we obtain relations

\[
\alpha (sh \theta)_{ij} = \sum_k (ch \theta)_{ij} \alpha_{kl},
\]

\[
\sum_{k,l,j} (sh \theta)_{ij} \alpha_{kl} \varepsilon_{jklm} = -2\beta (ch \theta)_{im},
\]

which are just the conditions that \( \xi \) is an annihilation operator, i.e. \( \xi \theta_0 = 0 \).

Thus the condition \( D=0 \) means that the same state with \( D=0 \) is given both by \( U_{0,B} \) and by \( U \), but \( U \) is not necessarily equivalent to \( U_{0,B} \).

When \( D=0 \) and \( U \neq U_{0,B} \), there are two interpretations. The first is that every state has four body correlation, and the second is that the ground state has not four body correlation, but some of the excited states have four quasiparticle correlation (where the quasiparticle is determined by \( U_{0,B} \)). This is only the question of terminology, but because of complexity of identification between \( U \) and \( U_{0,B} \), we shall take the second point of view.

On the contrary we cannot construct quasiparticles by the transformation \( U_{0,B} \) for the trial ground state with \( D=0 \) which usually has four body correlation. In this case this new method has essential differences from the G.B.T. Thus we have a special interest in whether the discriminant \( D \) vanishes or not.

Here we should remark that the G.B.V.T. is reduced into the G.B.T. when we deal with an infinite system,\(^*\) the last term of Eq. (2·17) can be neglected for infinite system, because \( v_{ijkl} / \gamma_i \sim v_{ijkl} / n_i v_{ijkl} / \mu_{ij} \sim \Omega (1 / \Omega) \) where \( \Omega \) is the volume of the system, and so the linearized Hamiltonian \( \hat{H} \) can be diagonalized by the G.B.T. (2·24).

Next we consider the relation between the transformations \( U_{0,V} \) and \( U \).

\(^*\) The author is indebted to Dr. S. Yamazaki for informing of this point.
As is mentioned in reference 11), the trial ground state vector (2·3) can be brought into the canonical form

\[ \langle \alpha + \alpha_1 b_1^* + \alpha_2 b_2^* + \alpha_3 b_3^* + \beta b_1^* b_2^* + b_3^* \rangle |0\rangle \] (2·35)

by means of a suitable transformation \( T_1 (a) \) like (2·32). But even in \( b \)-field representation, other excited states given by the transformation \( U \) have not the canonical form unless the linearized Hamiltonian (2·17) has special symmetry such that it is invariant under any transformation of the type \( T_1 (a) \), and so has a degeneracy in the form of the state vector.\(^{11}\)

Therefore even after we find a certain set of quasiparticle operators for the given ground state by the transformation \( U_{B.V} (b) \) as \( \xi = U_{B.V} (b) b U_{B.V}^{-1} (b) \), the linearized Hamiltonian is not yet diagonalized for excited states and has the complex form

\[ \hat{H}_e = X_0 + \sum_{ij} \xi_i^* \xi_j + \sum \xi_i^* H_{i} (\xi^*, \xi), \] (2·36)

where \( H_{i} (\xi^*, \xi) \) is hermitic and a polynomial of \( \xi^* \) and \( \xi \) without zero and odd products.

If the linearized Hamiltonian (2·36) can be diagonalized by the transformation \( T_1 (\xi) \), the relation between \( U \) and \( U_{B.V} \) is similar to the relation (1·5) between \( U_{G.B.} \) and \( U_{n} \), i.e.

\[ U = T_1 (\xi) U_{B.V} (b) T_1 (a), \] (2·37)

but otherwise, \( U \) and \( U_{B.V} \) have not so simple relation, and the G.B.V.T. turns out essential.

Finally we make some remarks on the diagonal form of the linearized Hamiltonian. When we find the transformation \( U \) and define the quasiparticle operators and their number operators as

\[ \zeta_i = U (a) a_i U^{-1} (a), \] (2·38)

we obtain the diagonal form

\[ \hat{H}_e = X_0 + \sum_i E_i \mathcal{N}_i + \tilde{H} (\mathcal{N}_i), \] (2·40)

where \( \tilde{H} (\mathcal{N}_i) \) is non-linear with respect to the operator \( \mathcal{N}_i \). Thus there are four kinds of single quasiparticle states of each \( \sigma \), and the energy of a state which corresponds to the excitation of two or more quasiparticles of the same \( \sigma \) is not the sum of energies of single quasiparticles.

\[ \mathsection 3. \textbf{System with charge independent pairing interaction} \]

\subsection*{3.1 System with charge independent pairing interaction}

In the application of the G.B.V.T. to the nuclear problem, there are some kinds of possible assignment of the quantum numbers \((\sigma, i)\). The natural choice may be
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\[ 1 \leftrightarrow p + \]
\[ 2 \leftrightarrow p - \]
\[ 3 \leftrightarrow n + \]
\[ 4 \leftrightarrow n - , \]  

(3.1)

where \( p \) and \( n \) denote proton and neutron respectively, and + and − denote the sign of the \( z \)-component of total spin \( j \) or conjugate states defined by S. T. Belyaev. Other choices are, of course, possible. Anyway a suitable choice depends on the symmetry of Hamiltonian.

Now let us study some consequences of \( n-p \) interaction to the usual like particle pairing scheme by means of the G.B.V.T. For this purpose we consider a simple model, i.e. a spherical nucleus not only with \( p-p \) and \( n-n \) pairing interactions but also with \( n-p \) pairing one in a charge independent manner.

The Hamiltonian of the system is given by

\[
H = H_s + \lambda_p N_p - \lambda_n N_n + V_{\text{pair}}
\]

\[
= \sum_{\sigma} \gamma_{\sigma p} (a_{\sigma p}^+ a_{\sigma p}^+ + a_{\sigma p}^- a_{\sigma p}^-) + \sum_{\sigma} \gamma_{\sigma n} (a_{\sigma n}^+ a_{\sigma n}^+ + a_{\sigma n}^- a_{\sigma n}^-) + \sum_{\sigma, \sigma'} G(\sigma, \sigma') (a_{\sigma p}^+ a_{\sigma' p}^+ a_{\sigma' p}^- a_{\sigma p}^- + a_{\sigma p}^+ a_{\sigma p}^- a_{\sigma' n}^+ a_{\sigma' n}^- + a_{\sigma' n}^+ a_{\sigma' n}^- a_{\sigma p}^+ a_{\sigma p}^-)
\]

\[
+ \frac{1}{\sqrt{2}} (a_{\sigma p}^+ a_{\sigma' n}^+ a_{\sigma' n}^- a_{\sigma p}^- + a_{\sigma p}^+ a_{\sigma p}^- a_{\sigma' n}^+ a_{\sigma' n}^-) \frac{1}{\sqrt{2}} (a_{\sigma' n}^+ a_{\sigma' p}^+ a_{\sigma' p}^- a_{\sigma' n}^- + a_{\sigma' n}^+ a_{\sigma' n}^- a_{\sigma' p}^+ a_{\sigma' p}^-),
\]  

(3.2)

where \( N_p \) and \( N_n \) are proton and neutron number operators, and + and − denote the upward and downward directions of \( j_z \) and \( \sigma \) denotes the quantum numbers \( (n, j, l, |m|) \) of the single particle states. The pairing of the type (3.2) means the \( j-j \) coupling pair like Kisslinger and Sorensen's, and the coupling strength is given by

\[
G(\sigma, \sigma') = G(\sigma', \sigma) = \frac{(-)^l_{J+J'} l_{|m|} |m'|}{\sqrt{(2J+1)(2J'+1)}} \langle j, J, J' = 0 | T = 1 | V | j', J, J' = 0 | T = 1 \rangle.
\]  

(3.3)

Their force is \( G(\sigma, \sigma') = (-)^l_{J+J'} l_{|m|} |m'| g \) where \( g \) is a negative coupling constant (which they call \( -G \)).

From the symmetries of the Hamiltonian it is convenient to write the trial state vector of even particles as follows:

\[
\Theta_s = \alpha + \alpha_p a_{\sigma p}^+ a_{\sigma p}^+ + \alpha_n a_{\sigma n}^+ a_{\sigma n}^+ + \alpha_n a_{\sigma n}^+ a_{\sigma n}^+ a_{\sigma n}^+ a_{\sigma n}^-
\]

\[
+ \alpha_0 (a_{\sigma p}^+ a_{\sigma n}^- + a_{\sigma n}^+ a_{\sigma p}^-) / \sqrt{2} + \alpha_0 (a_{\sigma p}^+ a_{\sigma n}^- a_{\sigma n}^- a_{\sigma p}^-) / \sqrt{2}
\]

\[
+ \alpha_0 a_{\sigma p}^+ a_{\sigma n}^+ a_{\sigma n}^- a_{\sigma n}^- + \alpha_n a_{\sigma n}^+ a_{\sigma n}^- a_{\sigma n}^+ a_{\sigma n}^-
\]

\[
+ \beta a_{\sigma p}^+ a_{\sigma n}^+ a_{\sigma n}^- a_{\sigma n}^-,
\]  

(3.4)
where we omitted the suffix $\sigma$ in the right-hand side, and $\alpha_i$ and $\alpha_0$ terms denote the $p$-$n$ states with isotopic spin one and zero respectively. The discriminant $D$ becomes

$$ D = \alpha_1 \beta - \alpha_p \alpha_n + 1/2 (\alpha_1^2 - \alpha_0^2) + \alpha_i \alpha_n. $$

(3.5)

It is easily seen that the components $\alpha_i$, $\alpha_-$, and $\alpha_0$ are decoupled from others in the variational equations for the normalized column vector

$$ \mathbf{a} = (\alpha, \alpha_p, \alpha_1, \alpha_n, \alpha_i, \alpha_-, \alpha_0, \beta). $$

(3.6)

So we obtain the three degenerate eigenstates with the eigenvalue $X = \eta_p + \eta_n$, and a selection of state vector belonging to this eigenvalue is

i) $\alpha_1 = 1$ others $= 0$ with $D = 0$

ii) $\alpha_- = 1$ others $= 0$ with $D = 0$

iii) $\alpha_0 = 1$ others $= 0$ with $D = -1/2$.

In such a degenerate case, the discriminant $D$ has not definite value and depends on the choice of the state vector.

Now the variational equation is reduced to

$$ \mathcal{H}' \mathbf{a} = Y \mathbf{a}, $$

with

$$ \mathbf{a} = (\alpha, \alpha_p, \alpha_1, \alpha_n, \beta), $$

$$ Y = X - (\eta_p + \eta_n + G), $$

$$ G = G(\sigma, \sigma'). $$

(3.7)

(3.8)

(3.9)

And the matrix $\mathcal{H}'$ is given by

$$ \mathcal{H}' = \begin{bmatrix} - (\eta_p + \eta_n + G) & \mu_p & \mu_1 & \mu_n & 0 \\ \mu_p & \eta_n - \eta_p & 0 & 0 & \mu_n \\ \mu_1 & 0 & 0 & 0 & - \mu_1 \\ \mu_n & 0 & 0 & \eta_n - \eta_p & \mu_p \\ 0 & \mu_n & - \mu_1 & \mu_p & \eta_p + \eta_n + 2G \end{bmatrix}, $$

(3.10)

with

$$ \mu_p = \sum_{\sigma', \sigma''} G(\sigma, \sigma') (\alpha^{\sigma'} \alpha^{\sigma''}_p + \alpha^{\sigma''}_n \beta^{\sigma''}), $$

$$ \mu_n = \sum_{\sigma', \sigma''} G(\sigma, \sigma') (\alpha^{\sigma'} \alpha^{\sigma''}_n + \alpha^{\sigma''}_p \beta^{\sigma''}), $$

$$ \mu_1 = \sum_{\sigma', \sigma''} G(\sigma, \sigma') \alpha^{\sigma''}_1 (\alpha^{\sigma'} - \beta^{\sigma'}). $$

(3.11)

For simplicity, we shall restrict ourselves to a more symmetric case where $\eta_p = \eta_n = \eta$ and $\mu_p = \mu_n = 2^{-1/2} \mu$, of which restriction satisfies the self-consistency.\(^*\)

\(^*\) Another plausible symmetry is $\mu_p = - \mu_n$, for which a similar discussion can be developed, but the results have a different feature.
Then we can easily get an eigenstate

\[ \alpha_{\nu} = -\alpha_{\alpha} = 1/\sqrt{2}, \quad Y = 0 \text{ with } D = 1/2. \quad (3.12) \]

And the variational equation is reduced further to

\[ \mathcal{H}' \alpha' = Y \alpha', \quad (3.13) \]

where

\[ \alpha' = (\alpha, \alpha, \alpha', \beta), \]

with

\[ \alpha' = (\alpha_{\nu} + \alpha_{\alpha})/2^{1/2} \]

and

\[ \mathcal{H}' = \begin{pmatrix} - (2\eta + G) & \mu_1 & \mu & 0 \\ \mu_1 & 0 & 0 & -\mu_1 \\ \mu & 0 & 0 & \mu \\ 0 & -\mu_1 & \mu & 2\eta + 2G \end{pmatrix}. \quad (3.14) \]

And the discriminant for this subspace becomes

\[ D = \alpha \beta - 1/2 (\alpha'' - \alpha'^{2}). \quad (3.15) \]

The secular equation for Eq. (3.13) is given by

\[
\text{det} (\mathcal{H}' - Y I) = Y^4 - GY^3 - 2 \left( (\eta + G) (2\eta + G) + (\mu^2 + \mu_1^2) \right) Y^2 \\
+ G (\mu^2 + \mu_1^2) Y + 4 \mu^2 \mu_1^2 = 0. \quad (3.16)
\]

We assume the case where \( \mu \mu_1 \neq 0 \), because we are not concerned with the case where \( \mu \mu_1 = 0 \), and in this case the solution can be easily obtained. If \( \mu \mu_1 \neq 0 \), \( \alpha \) and \( \beta \) do not vanish simultaneously.

Unless \( \alpha = \beta = 0 \), the eigenvector belonging to the eigenvalue \( Y \) is formally given as follows:

(i) in the case \( \alpha \neq 0 \)

\[ \alpha_1 = \frac{\mu_1 [Y^2 - 2(\eta + G) Y - 2\mu^2]}{Y[Y^2 - 2(\eta + G) Y - (\mu^2 + \mu_1^2)]} \alpha, \]

\[ \alpha' = \frac{\mu [Y^2 - 2(\eta + G) Y - 2\mu^2]}{Y[Y^2 - 2(\eta + G) Y - (\mu^2 + \mu_1^2)]} \alpha, \]

\[ \beta = \frac{\mu^2 - \mu_1^2}{Y^2 - 2(\eta + G) Y - (\mu^2 + \mu_1^2)} \alpha, \quad (3.17) \]

(ii) in the case \( \beta \neq 0 \)

\[ \alpha = \frac{\mu^2 - \mu_1^2}{Y^2 + (2\eta + G) Y - (\mu^2 + \mu_1^2)} \beta, \]

\[ \alpha_1 = \frac{\mu_1 [Y^2 + (2\eta + G) Y - 2\mu^2]}{Y[Y^2 + (2\eta + G) Y - (\mu^2 + \mu_1^2)]} \beta, \quad (3.18) \]
\[ \alpha' = \frac{\mu [Y^2 + (2\eta + G) Y - 2\mu_i^3]}{Y [Y^2 + (2\eta + G) Y - (\mu^2 + \mu_i^3)]} - \beta. \]

From Eqs. (3·15), (3·16), (3·17) and (3·18), the discriminant becomes

\[ D = \frac{G (\mu^2 - \mu_i^3)}{2Y (Y^2 - 2(\eta + G) Y - (\mu^2 + \mu_i^3))} \alpha^2 \quad \text{for} \quad \alpha \neq 0, \]

\[ = \frac{G (\mu^2 - \mu_i^3)}{2Y (Y^2 + (2\eta + G) Y - (\mu^2 + \mu_i^3))} \beta^2 \quad \text{for} \quad \beta \neq 0. \quad (3·19) \]

Therefore, the discriminant \( D \) vanishes for any eigenvalue when and only when the following relation holds,

\[ \mu^2 = \mu_i^3, \]

i.e.

\[ 2\mu_p^2 = 2\mu_n^3 = \mu_i^3 \quad \text{from assumed symmetry.} \quad (3·20) \]

Now we shall study the case of \( \mu^2 = \mu_i^3 \) in more detail, because the solution of this case is expected to resemble closely to those of the G.B.T.

In this case, the eigenvalues \( Y \) are given by

\[ Y_{1,2} = (\eta' + G/4) \pm \sqrt{(\eta' + G/4)^2 + 2\mu^2}, \]

\[ Y_{3,4} = (-\eta' + G/4) \pm \sqrt{(-\eta' + G/4)^2 + 2\mu^2}, \quad (3·21) \]

with \( \eta' = \eta + 3/4 \, G \).

Eigenvectors are given by

(i) for \( Y_1 \) and \( Y_2 \)

\[ \alpha = 0, \]

\[ \alpha_1 = -\frac{\mu_i^3}{\sqrt{Y_i^2 + 2\mu^2}}, \]

\[ \alpha' = \frac{\mu}{\sqrt{Y_i^2 + 2\mu^2}}, \]

\[ \beta = \frac{Y_i}{\sqrt{Y_i^2 + 2\mu^2}}. \quad (3·22) \]

(ii) for \( Y_3 \) and \( Y_4 \)

\[ \alpha = \frac{Y_i}{\sqrt{Y_i^2 + 2\mu^2}}, \]

\[ \alpha_1 = \frac{-\mu_i^3}{\sqrt{Y_i^2 + 2\mu^2}}, \]

\[ \alpha' = \frac{\mu}{\sqrt{Y_i^2 + 2\mu^2}}, \]

\[ \beta = \frac{Y_i}{\sqrt{Y_i^2 + 2\mu^2}}. \]

*This relation becomes \( 2\mu_p^2 \mu_n^2 = \mu_i^3 \) when the restriction \( \mu_p = \mu_n \) is taken off. Equation (3·8) can be solved analytically also when only the restriction \( \eta_p = \eta_n = \eta \) is imposed but not \( \mu_p = \mu_n \), however the solutions are obtained from the case with the further restriction \( \mu_p = \mu_n \) only through the replacements \( \mu \rightarrow \mu = (\mu_p + \mu_n)/2^{1/2}, \) and \( \mu_1 \rightarrow \mu'_1 = (2\mu_l^3 + (\mu_p - \mu_n)^3)/2^{1/2}. \)
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\[ \alpha = \frac{\mu}{\sqrt{Y^2 + 2\mu^2}} , \quad (3.23) \]

\[ \alpha' = \frac{\mu}{\sqrt{Y^2 + 2\mu^2}} , \quad (3.23) \]

\[ \beta = 0. \]

The ground state comes out to be

\[ Y = Y_1 = (\eta' + G/4) - \sqrt{(\eta' + G/4)^2 + 2\mu^2} \quad \text{for} \quad \eta' \leq 0 , \quad (3.24) \]

\[ Y = Y_2 = (-\eta' + G/4) - \sqrt{(-\eta' + G/4)^2 + 2\mu^2} \quad \text{for} \quad \eta' \geq 0 . \]

For \( \eta' = 0 \), the two solutions are degenerate. The qualitative behaviour of the ground state wave function is shown in fig. 1.

![Fig. 1. Qualitative behaviour of the ground state wave function.](image)

Then, from Eqs. (3.11), (3.22), (3.23) and (3.24), the equations for the pairing potentials \( \mu_1 \) and \( \mu_1'(\eta') \) are obtained as

\[ \left( \begin{array}{c} \mu_1' \\ \mu_1'' \end{array} \right) = \sum_{\sigma=\sigma'} G(\sigma, \sigma') \left\{ \begin{array}{c} \alpha_1''(\sigma'' - \beta''') \\ \alpha_1''(\sigma'' - \beta'') \end{array} \right\} \\
= -\sum_{\sigma=\sigma'} G(\sigma, \sigma')(\mu_1') \frac{1}{2\sqrt{(\eta_1'' + G/4)^2 + 2\mu^2}} \\
-\sum_{\sigma=\sigma'} G(\sigma, \sigma')(\mu_1'') \frac{1}{2\sqrt{(-\eta_1'' + G/4)^2 + 2\mu^2}} . \quad (3.25) \]

These equations are not inconsistent with the condition \( \mu_1^2 = \mu_1'^2 \).

Next we consider the odd particle states. From Eq. (2.17), the linearized Hamiltonian of this model is given by

\[ \hat{H} = \eta (a_{p+}^* a_{p+} + a_{p+}^* a_{p-} + a_{n+}^* a_{n+} + a_{n-}^* a_{n-}) \\
+ (\mu/\sqrt{2}) (a_{p+}^* a_{p-} + a_{p+}^* a_{p-} + a_{p+}^* a_{p-} + a_{p-}^* a_{p-}) \\
+ (\mu_1/\sqrt{2}) (a_{n+}^* a_{n+} + a_{n-}^* a_{n-} + a_{n+}^* a_{n-} + a_{n-}^* a_{n-}) \\
+ G (a_{p+}^* a_{p-} a_{p+} + a_{n+}^* a_{n-} a_{n+} + a_{n+}^* a_{n-} a_{n-} \\
+ (1/\sqrt{2}) (a_{p+}^* a_{p+} + a_{p-}^* a_{p-}) (1/\sqrt{2}) (a_{n+} a_{n+} + a_{n-} a_{n-})). \quad (3.26) \]
From the symmetry of the Hamiltonian, odd particle states are separated into two subspace and they are written as

\[ \Phi_{\pm} = \alpha_{p\pm} a_{p\pm}^{\dagger} + \alpha_{n\pm} a_{n\pm}^{\dagger} + \beta_{p\pm} a_{p\pm} + \beta_{n\pm} a_{n\pm}^{\dagger} a_{n\pm} a_{p\pm}^{\dagger} a_{p\pm}. \] (3·27)

Now Eq. (2·18) becomes

\[
\begin{bmatrix}
\eta & 0 & \mu/\sqrt{2} & -\mu_i/\sqrt{2} \\
0 & \eta & -\mu_i/\sqrt{2} & \mu/\sqrt{2} \\
\mu/\sqrt{2} - \mu_i/\sqrt{2} & \eta + 2\eta' & 0 & \eta + 2\eta' \\
-\mu_i/\sqrt{2} & \mu/\sqrt{2} & 0 & \eta + 2\eta'
\end{bmatrix} \begin{pmatrix}
\alpha_{p\pm} \\
\alpha_{n\pm} \\
\beta_{p\pm} \\
\beta_{n\pm}
\end{pmatrix} = \begin{pmatrix}
X \\
0 \\
\beta_{p\pm} \\
\beta_{n\pm}
\end{pmatrix}, \quad (3·28)
\]

the eigenvalue of which are given by

\[ X = \eta + \eta' \pm \sqrt{\eta'^2 + (\mu \pm \mu_i)^2/2}. \] (3·29)

These four solutions become, taking account of \( \mu^2 = \mu_i^2 \),

\[ X = \eta + \eta' - \sqrt{\eta'^2 + 2\mu^2}, \]
\[ X = \eta, \]
\[ X = \eta + 2\eta', \]
\[ X = \eta + \eta' + \sqrt{\eta'^2 + 2\mu^2}. \] (3·30)

Now we have obtained the whole eigenvalues of the linearized Hamiltonian (3·26), which are listed in Table I.

| Table I. Eigenvalues of the linearized Hamiltonian. |
|----------------|----------------|----------------|
| i | Eigenvaules \( X_i \) | Degeneracy | Number of quasiparticle \( \eta' > 0 \) | Number of quasiparticle \( \eta' < 0 \) |
| 0 | \( (\eta' - G/4) - \sqrt{(\eta' - G/4)^2 + 2\mu^2} \) | 1 | 0 | 2 |
| 1 | \( (2\eta' - 3/4G) - \sqrt{\eta'^2 + 2\mu^2} \) | 2 | 1 | 1 |
| 2 | \( \eta' - 3/4G \) | 2 | 1 | 3 |
| 3 | \( 4\eta' - (\eta' + G/4) - \sqrt{(\eta' + G/4)^2 + 2\mu^2} \) | 1 | 2 | 0 |
| 4 | \( 2(\eta' - G/4) \) | 1 | 2 | 2 |
| 5 | \( (\eta' - G/4) + \sqrt{(\eta' - G/4)^2 + 2\mu^2} \) | 1 | 2 | 4 |
| 6 | \( 2(\eta' - (3/4)G) \) | 3 | 2 | 2 |
| 7 | \( 3\eta' - (3/4)G \) | 2 | 3 | 1 |
| 8 | \( 2\eta' - (3/4)G + \sqrt{\eta'^2 + 2\mu^2} \) | 2 | 3 | 3 |
| 9 | \( 4\eta' - (\eta' + G/4) + \sqrt{(\eta' + G/4)^2 + 2\mu^2} \) | 1 | 4 | 2 |

3·2 Discussions

Our model Hamiltonian (3·2) with \( \eta_p = \eta_n \) is very simple and special, but it may give us some insight into the consequences of \( n-p \) interaction. Moreover
it may represent some important aspects of residual interaction in analogy with Kisslinger and Sorensen's work.\(^1\)

We have shown that if the linearized Hamiltonian has symmetries \(\eta_p = \eta_n\) and \(\mu_1 = 2\mu_p, \mu_n\), the ground state vector has the same structure as that of the G.B.T., i.e. it has no four body correlation. This condition \(\mu_1 = 2\mu_p, \mu_n\) is not inconsistent with the pairing potential equation, although other solutions besides the normal solution \((\mu = \mu_n = 0)\) may be possible (the general equation is a little more complex than Eq. (3.25)).

Unfortunately we cannot easily decide which solution of the pairing potential equations just corresponds to the minimum energy (physical solution). Such being the case, we have limited our discussions to the simple but not trivial case \(\mu_1 = 2, \mu_p, \mu_n \neq 0\). Another argument for choosing this condition has already been given in §3.1.

Equation (3.25) for the pairing potential resembles that of Kisslinger and Sorensen's\(^1\),\(^2\) (the gap equation), which is given by

\[
\mu_{p,n} = -\sum_{\sigma'} G(\sigma, \sigma') \frac{\mu_{p,n}}{2V \eta_{\sigma'}^2 + \mu_{p,n}^2}.
\]

Equation (3.31) appears when we deal with the \(n-p\) interaction by the G.B.T.\(^9\),\(^15\) But this fact does not necessarily mean that the gap with \(n-p\) interaction is twice as large as the gap without \(n-p\) interaction, since the value of \(\mu_n\) in the G.B.V.T. is not the same as that in the B.T.\(^1\)

Next we investigate the energy spectrum given in Table I. For simplicity, we consider the case of \(\eta' > 0\). (The same discussion is available for \(\eta' < 0\).)

There are two kinds of single quasiparticle excitation energy such as

\[
E_1 = X_1 - X_0 = \left(\eta' - G \right) + \sqrt{\left(\eta' - G/4\right)^2 + 2\mu_p^2} - \sqrt{\eta' + 2\mu_p^2},
\]

\[
E_2 = X_2 - X_0 = \sqrt{\left(\eta' - G/4\right)^2 + 2\mu_p^2} \left(1 - \frac{G}{2\sqrt{\left(\eta' - G/4\right)^2 + 2\mu_p^2}}\right).
\]

The gaps are given by putting \(\eta' = 0\) in Eq. (3.32). The first excitation energy \(E_1\) has only a small gap since the vacuum polarization effects are nearly the same for both ground and excited states, and cancel each other. The second excitation energy \(E_2\) has a large gap and corresponds to the usual one given by the Bogoliubov transformation. The factor \(1 - \left\{G/2\sqrt{\left(\eta' - G/4\right)^2 + 2\mu_p^2}\right\} \right)\) comes from the difference of the definition, i.e. the pairing potential of the G.B.T.
\( \mu_{a,b} \) so corresponds to our pairing potential as
\[
\mu_{a,b} \rightarrow \mu' + G \alpha_\alpha' (\alpha^* + \beta^*)
\]
\[
= \mu' \left( 1 - \frac{G}{2\sqrt{(\eta' - G/4)^2 + 2\mu'^2}} \right).
\]

The existence of the single quasiparticle excitation energy \( E_1 \) is one of the remarkable consequences of the n-p interaction, if this special symmetric solution is physical.

In spite of \( D=0 \), the energy spectra given in Table I are not represented by the addition of single quasiparticle energies \( E_1 \) and \( E_2 \). As is mentioned in § 2.2, this fact is due to the \( G \)-term in the linearized Hamiltonian (3·26). According to K.S., the order of magnitude of the relevant quantities for Ni isotopes has been estimated as
\[
G \approx 0.33 \text{ Mev}, \quad |\mu_n| \approx 0.8 \sim 1.1 \text{ Mev}.
\]

For lighter nuclei, \( G \) and \( \mu \) may have the same order of magnitude. In this case our treatment should become more suitable than the G.B.T.

At last we should refer to another symmetric case \( \mu_p = -\mu_n \), for which \( D \) never vanishes unless \( G=0 \), but in the limit of \( G \rightarrow 0 \) (i.e. \( Q \rightarrow \infty \)) the solution has only one type of single quasiparticle excitation which corresponds to the solution given in reference 15).

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Appendix

Let us briefly study the algebraic structure of transformation (2·1). It conserves particle number parity, i.e. even or odd character of particle number, and its Lie algebra consists only of antihermitic operators with real coefficients.

Therefore, we should only seek the complete set of antihermitic operators constructed by the product of even number of operators \( \{a_i^*, a_i\} \) \((i=1, 2, 3, 4)\).

For this purpose, we define the following operators,
\[
N_{ij} = a_i^* a_j - a_j^* a_i, \quad \bar{N}_{ij} = a_i^* a_j + a_j^* a_i,
\]
\[
B_{ij} = a_i^* a_j^* - a_j a_i, \quad \bar{B}_{ij} = a_i^* a_j^* + a_j a_i, \quad (A·1)
\]

and get the following commutation relations,
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\[ [N_{ij}, N_{kl}] = [B_{ij}, B_{kl}] = -[\overline{B}_{ij}, \overline{B}_{kl}] \]

\[ = \delta_{jk} N_{il} + \delta_{il} N_{jk} + \delta_{ik} N_{lj} - \delta_{jl} N_{ki}, \]

\[ [\overline{N}_{ij}, \overline{N}_{kl}] = \delta_{jk} N_{il} + \delta_{il} N_{jk} - \delta_{ik} N_{lj} - \delta_{jl} \overline{N}_{ki}, \]

\[ [\overline{N}_{ij}, N_{kl}] = \delta_{jk} \overline{N}_{il} - \delta_{il} \overline{N}_{jk} + \delta_{ik} \overline{N}_{lj} + \delta_{jl} \overline{N}_{ki}, \]

\[ [B_{ij}, N_{kl}] = \delta_{jk} B_{il} + \delta_{il} B_{jk} + \delta_{ik} B_{lj} + \delta_{jl} \overline{B}_{ki}, \]

\[ [B_{ij}, \overline{N}_{kl}] = -\delta_{jk} \overline{B}_{il} + \delta_{il} \overline{B}_{jk} - \delta_{ik} \overline{B}_{lj} + \delta_{jl} B_{ki}, \]

\[ [\overline{B}_{ij}, \overline{N}_{kl}] = -\delta_{jk} \overline{B}_{il} + \delta_{il} \overline{B}_{jk} - \delta_{ik} \overline{B}_{lj} + \delta_{jl} B_{ki}, \]

\[ [\overline{B}_{ij}, B_{kl}] = \delta_{jk} \overline{N}_{il} + \delta_{il} \overline{N}_{jk} - \delta_{ik} \overline{N}_{lj} - \delta_{jl} \overline{N}_{ki}. \]

From these relations, we can see that the G.B.T. forms a group.

We can easily prove that all antihermitic operators with even number of operators \( \{ a_i^*, a_i \} \) can be given by a linear combination of the following set of operators,

\[ N_{ij}, B_{ij}, \]

\[ \overline{N}_{ij}, N_{kl}, \overline{N}_{ij}, \overline{N}_{kl}, \overline{B}_{ij}, B_{kl}, \]

\[ \overline{B}_{ij}, \overline{N}_{kl}, \overline{N}_{ij}, \overline{N}_{kl}, \overline{B}_{ij}, B_{kl}, \]

with \( i \neq k, l \) and \( j \neq k, l \).

For example, the operators in Eq. (2·2) are expressed by

\[ a_i^* a_j^* a_k^* a_l^* - \text{h.c.} = 1/2 (\overline{B}_{ij} B_{kl} + \overline{B}_{kl} B_{ij}) , \]

\[ a_i^* a_j^* a_k^* a_l^* - \text{h.c.} = 1/2 (B_{ij} \overline{N}_{kl} + \overline{N}_{kl} B_{ij}) . \]

Even if some of \((i, j)\) are identical with some of \((k, l)\), such antihermitean operator can be written by the combination of the above operators, as to example

\[ \overline{B}_{ij} B_{kl} - \text{h.c.} = \overline{N}_{ij} N_{kl} - N_{kl} . \]

Using the relations (A·4) and (A·2), we can obtain all bases of algebra by a sequence of commutations starting from the operators explicitly written in Eq. (2·2).

Finally we prove that the \( D \)-invariant transformation group is just that of the transformation (2·24). More precisely the former is isomorphic with the proper real orthogonal transformations in the 8-dimensional \( \alpha = (\alpha, \alpha_{12}, \alpha_{13}, \alpha_{14}, \beta, \alpha_{23}, \alpha_{34}, \alpha_{43}) \) space. (For the regular form of the following \( J \) matrix, the phase and arrangement of the components of \( \alpha \) are changed from Eq. (2·14).), which do not vary the bilinear form

\[ \langle \theta' D \theta \rangle = \langle \alpha' J \alpha \rangle, \]

(A·6)
The elements of the Lie algebra of such transformation group should be represented by

\[ A = \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}, \quad (A \cdot 8) \]

where \( X \) and \( Y \) are 4-dimensional real antisymmetric matrices, since \( 'A + A = 0 \) and \( 'AJ + JA = 0 \). The vector space constructed by the set of matrices \( A \) is 12-dimensional, and it certainly contains the 12 linear independent matrices \( N_{ij} \) and \( B_{ij} \). Therefore, the \( D \) invariant transformation group is just the group of the transformation \( U_{0,0} \), whose bases of the algebra are \( N_{ij} \) and \( B_{ij} \).

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