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Reservoir Mechanism
Analysed by the Method of Galerkin and Orthogonal Eigenfunctions

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The physical problem dealt with in this article is the unsteady groundwater flow in an isotropic, nonhomogenous aquifer of limited horizontal extent and arbitrary boundary shape.
The unsteady ground-water level is described by a differential equation which is solved by the Galerkin finite element method, combined with orthogonal eigenfunctions. Unlike many numerical procedures the physical properties of the aquifer are not lost. Boundary conditions are provided by nature in the form of watertight rocks, rivers, lakes or any kind of constant water level in hydraulic contact with the aquifer, or given boundary inflow or outflow.

Introduction

In this article we are dealing with the unsteady groundwater flow in an isotropic, nonhomogenous aquifer of limited horizontal extent and arbitrary boundary shape. In nature we do not have such ideal aquifers and we usually have a very complicated system of aquifers, but we use the differential equation (2.1) as an average. The constants in the differential equation are then chosen to fit the model to field data. A suitable norm should therefore be introduced to measure the difference between the model and the data.

The differential equation is solved by using the Galerkin finite element method in the x and y coordinates but keeping the time continuous.

Because the time is kept continuous the physical properties of the aquifer are not lost, see Eliasson (1971) and Eliasson et al. (1973a,b), as they would be in case of using e.g. the Crank-Nicholson finite difference method for the time derivative, see Pinder and Frind (1972), or some three dimensional finite element methods, see Gray et al.

(1974) and Huang and Sonnenfeld (1974). We solve instead the eigenvalue problem for the matrices involved in the finite element method. The eigenvalues and the eigenvectors then give us the geometrical and physical properties of the aquifer and need only be solved once and for all for the same aquifer. If the input function changes we can use the already calculated properties to give us the groundwater level. If we had used the finite difference method for the time derivative we would have to solve the whole problem again if the input function changes.

The flexibility of the finite element method gives you the opportunity to use geological and hydrological insight when constructing the finite element mesh.

The accuracy of the eigenfunctions expansion depends upon how well they approximate the input function, the gradients and the boundary conditions. As the gradients in most problems are small, and the input function does not fluctuate very much over the area, one can be satisfied with a very limited number of eigenfunctions.

Mathematical Formulation

We assume that the groundwater flow is practically horizontal and that the aquifer is isotropic and nonhomogenous. This kind of flow is governed by the following equations. In the following the groundwater flow is supposed to take place in Ω, and Ω is bounded by $\partial\Omega_1$ and $\partial\Omega_2$.

Differential equation

$$\nabla(T\cdot\nabla h) = S\frac{\partial h}{\partial t} - R - Q_i\delta(x-\xi_i) \qquad x \in \Omega \qquad (2.1)$$

gradient of internal flow = storage changes - infiltration - sources and sinks.

Boundary conditions

$$h(x,t) = g(x,t) \qquad x \in \partial\Omega_1 \qquad (2.2)$$

water level = known function on the boundary $\partial\Omega_1$

$$T\frac{\partial h}{\partial n} = -q(x,t) \qquad x \in \partial\Omega_2 \qquad (2.3)$$

outflow = known function on the boundary $\partial\Omega_2$
(watertight boundary $\partial\Omega_2$ makes $q(x,t) = 0$)

Initial condition

$$h(x,t_0) = f(x) \qquad x \in \Omega \qquad (2.4)$$

known waterlevel at time = t_0

$h(x, t)$	:	groundwater level, L (length)
t	:	time, s
x, ξ_i	:	coordinate vectors, L
S, S_0	:	storage coefficients
T, T_0	:	transmissivity, L^2/S
$R(x, t)$	:	infiltration, $L^3/S/L^2 = L/S$
Q_i	:	source input L^3/S
$\delta(x-\xi_i)$	:	Dirac's delta function, $1/L^2$

T and S vary within Ω but T_0 are supposed to be constant reference values.

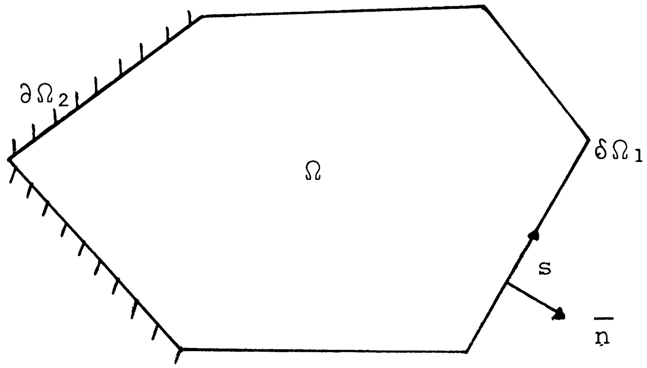


Fig. 1

Methods of Solution

We use the Galerkin finite element numerical solution. We multiply the differential equation by $\Phi=\Phi(x)$ and integrate over the area.
Φ is chosen to be zero on $\partial\Omega_1$ so we get:

$$S_0 \int_\Omega \frac{S}{S_0} \frac{\partial h}{\partial t} \Phi dx = \int_\Omega R\Phi dx - \int_{\partial\Omega_1} q\Phi ds - T_0 \int_\Omega \frac{T}{T_0} \nabla h \cdot \nabla\Phi dx + Q_i \cdot \Phi(\xi_i) \tag{2.5}$$

We now approximate the area and the boundary by a set of triangles (finite elements).

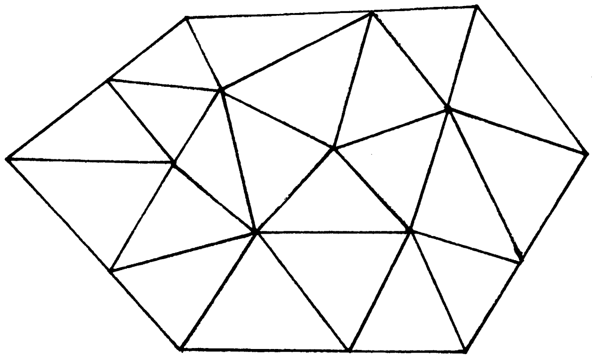


Fig. 2.

We approximate h as a linear function over each triangel and S and T as constant over each triangel.

This means that if we are able to calculate h in each of the intersection points then we can easily find h in every point within Ω by linear interpolation. This method can be made as accurate as we wish, simply by choosing a sufficient number of intersection points.

The necessary number of intersection points will be discussed later; until then we define:

N_1: Number of unknown points in Ω and on $\partial\,\Omega_2$ that is within the reservoir and on the part of the boundary where $h(t)$ is not given.

N_2: Number of known points on $\partial\,\Omega_1$ that is the remaining part of the boundary.

In the forthcoming we use the following vector and matrix symbols:

$\underline{h}$ means the vector h. For instance the set of numbers indicating the groundwater level at each intersection point at a given time.

$\underline{\underline{C}}$ means the matrix C.

$\underline{\underline{C}}^T$ means the transposed of the matrix C.

$\underline{\underline{C}}^{-1}$ means the inverse of the matrix C.

If Φ is now defined as a pyramidal-shaped function, (2.5) can be integrated within each triangle. The result is a system of N_1 ordinary differential equations where the N_1 unknowns are the groundwater levels at each intersection point. Instead of going into the rather ponderous mathematics we only give the resulting equation:

$$S_o \underline{\underline{C}}_1 \frac{\partial \underline{h}}{\partial t} = - T_o \underline{\underline{B}}_1 \underline{h} + \underline{\underline{D}}\,\underline{R} + \underline{Q} = (S_o \underline{\underline{C}}_2 \frac{\partial \underline{g}}{\partial t} + T_o \underline{\underline{B}}_2 \underline{g} + \underline{W}) \qquad (2.6)$$

Here, the following new vectors and matrices emerge:

$\underline{\underline{B}}_1$ (N_1, N_1) this matrix gives the internal flow between the triangular elements, caused by the variation with time of the groundwater level at each intersection point. It also includes the variation in the transmissivity (T) within the reservoir). Ω.

$\underline{\underline{B}}_2$ *(N_1, N_2)* this matrix gives the internal flow caused by the known variation of the groundwater level at the boundary. It contains the variation of T too.

$\underline{\underline{C}}_1$ *(N_1, N_1)* this matrix gives the variation of water storage caused by the variation of the groundwater level. It contains the variation of the storage coefficient S.

$\underline{\underline{C}}_2$ *(N_1, N_2)* this matrix gives the variation of water storage caused by g. It contains the variation in S too.

$\underline{\underline{D}}$ *(N_1, N_1+N_2)* this matrix gives the effect of the infiltration at each point on the groundwater level in every point.

$\underline{Q}$ *(N_1)* this vector is the source input at each point. E.g. if there is a well it is wise to put an intersection point there and then the $\underline{Q}$-value at that point is the pumping rate.

$\underline{W}$ *(N_1)* this vector gives the effect of the outflow q on the groundwater level at each point.

$\underline{g}$ *(N_2)* this vector is the known groundwater level at the boundary.

$\underline{R}$ *($N_1 + N_2$)* this vector is the infiltration rate.

$\underline{h}$ *(N_1)* this vector is the groundwater level at each point, the unknowns of our equations.

Our vectors and matrices can be classified as follows:

1. Boundary values the vectors $\underline{Q}$, $\underline{W}$ and g.

2. Space matrices the matrices $\underline{\underline{B}}_1$, $\underline{\underline{B}}_2$, $\underline{\underline{C}}_1$, $\underline{\underline{C}}_2$, and $\underline{\underline{D}}$.

These matrices include all the necessary information about the shape of the groundwater reservoir, the layout of the triangular mesh and the variation in the reservoir properties T and S. They are all independent of time, which is a great advantage.

Calculation of these vectors and matrices is easily performed by any mathematician familiar with the Galerkin method. As the $\underline{\underline{C}}$ and $\underline{\underline{D}}$ matrices have the dimension L^2 it is convenient to normalize them with respect to the overall area of the reservoir A (divide all their values with A); we also multiply through with $(S_0\underline{\underline{C}}_1)^{-1}$ and get

$$\frac{\partial \underline{h}}{\partial t} = -\frac{T_0}{A\,S_0}\underline{\underline{C}}_1^{-1}\underline{\underline{B}}_1\underline{h} + \frac{1}{S_0}\underline{\underline{C}}_1^{-1}\underline{\underline{D}}\underline{R} + \frac{1}{A\,S_0}\underline{\underline{C}}_1^{-1}\underline{Q} - \underline{\underline{C}}_1^{-1}\left(\underline{\underline{C}}_2\frac{\partial \underline{g}}{\partial t} + \frac{T_0}{A\,S_0}\underline{\underline{B}}_2\underline{g} + \frac{1}{A_0 S_0}\underline{W}\right) \quad (2.7)$$

This equation is a set of N_1 ordinary differential equations with the initial condition:

$$\underline{h}(t_0) = \underline{f} \quad (2.8)$$

For our purpose it seems practical to split (2.7) into two problems, a stationary part and a transient part. We have for the stationary part

$$\underline{\underline{B}}_1\underline{h}_0 = \frac{A}{T_0}\underline{\underline{D}}\overline{\underline{R}} - \underline{\underline{B}}_2\overline{\underline{g}} - \frac{1}{T_0}\overline{\underline{W}} \quad (2.9)$$

where $\overline{\underline{R}}$, $\overline{\underline{g}}$, $\overline{\underline{W}}$ mean long time average values. Long time average values of $\underline{Q}$, pumping or recharge, are in most cases equal to zero.

For the transient part we get:

$$\underline{\underline{C}}_1 \frac{d\underline{h}}{dt} = -\frac{T_o}{A\,S_o} \underline{\underline{B}}_1 \underline{h}_1 + \frac{1}{S_o} \underline{\underline{D}}(\underline{R}-\overline{\underline{R}}) + \frac{1}{A\,S_o} \underline{Q}$$

$$-\{\underline{\underline{C}}_2 \frac{d(\underline{g}-\overline{\underline{g}})}{dt} + \frac{T_o}{A\,S_o} \underline{\underline{B}}_2 (\underline{g}-\overline{\underline{g}}) + \frac{1}{A\,S_o} (\underline{W}-\overline{\underline{W}})\} \quad (2.10)$$

$$\underline{h}_1(t_o) = \underline{f} - \underline{h}_o \quad (2.11)$$

$$\underline{h} = \underline{h}_o + \underline{h}_1 \quad (2.12)$$

Eq. (2.9) can be solved simply by Cholesky's factorization. Solution to (2.10) is given by

$$\underline{h}_1(t) = e^{-(t-t_o)\underline{\underline{C}}_1^{-1}\underline{\underline{B}}_1\frac{T_o}{A\,S_o}} (\underline{f}-\underline{h}_o) + \frac{1}{S_o}\int_{t_o}^{t} e^{-(t-\tau)\underline{\underline{C}}_1^{-1}\underline{\underline{B}}_1\frac{T_o}{A\,S_o}}$$

$$\cdot \underline{\underline{C}}_1^{-1} \underline{\underline{D}}(\underline{R}-\overline{\underline{R}})\,d\tau + \frac{1}{A\,S_o}\int_{t_o}^{t} e^{-(t-\tau)\underline{\underline{C}}_1^{-1}\underline{\underline{B}}_1\frac{T_o}{A\,S}} \underline{\underline{C}}_1^{-1}\underline{Q}\,d\tau$$

$$-\int_{t_o}^{t} e^{-(t-\tau)\underline{\underline{C}}_1^{-1}\underline{\underline{B}}_1\frac{T_o}{A\,S_o}} \underline{\underline{C}}_1^{-1}$$

$$\left(\underline{\underline{C}}_2 \frac{d(\underline{g}-\overline{\underline{g}})}{dt} + \frac{T_o}{A\,S_o}\underline{\underline{B}}_2(\underline{g}-\overline{\underline{g}}) + \frac{1}{A\,S_o}(\underline{W}-\overline{\underline{W}})\right) d\tau \quad (2.13)$$

Eq. (2.13) is rather impractical due to the matrix nature of the exponential functions in the integrals. This difficulty is overcome by the use of orthogonal eigenfunctions, whereby the solution is transformed to a series of ordinary convolution integrals. To do this we first solve the following matrix eigenvalue problem.

$$\underline{\underline{B}}_1 \underline{\Phi} = \lambda\, \underline{\underline{C}}_1 \underline{\Phi} \quad (2.14)$$

As the structure of the matrixes $\underline{\underline{C}}_1$ and $\underline{\underline{B}}_1$ is (N_1, N_1) the solution to (2.14) is a set of N_1 orthogonal eigenvectors Φn, where n is an integer between 1 and N_1. To each set there is associated an eigenvalue λn. The transient part of the groundwater level, $h_1(t)$, can now be expanded in a series of these eigenvectors, just as any ordinary function of time can be expanded in a Fourier series.

To do so we first define the following N_1 time constants:

$$K_n = \frac{A\,S_o}{\lambda_n T_o}$$

Now we introduce these time constants and the eigenvectors Φn into (2.13). Furthermore, we assume that the infiltration $R(t)$ is known so far back in time that the influence of the initial waterlevels can be neglected. That is permissible if the term e^{-t/K_n} is sufficiently small for all values of n. We also assume the boundary values to be constant in time, this may be achieved by choosing the boundary to run along watertight formations or where constant groundwater levels or groundwater inflows are known to exist.

The calculations are somewhat lengthy, and are omitted here, the result is

$$\underline{h}(t) = \underline{h}_o + \sum_{1}^{N_1} \Phi n \frac{1}{S_o} \int_{o}^{\infty} \underline{\Phi n}^T \left(\underline{\underline{D}}(\underline{R}(t-\tau) - \overline{\underline{R}}) + \frac{1}{A}\underline{Q}(t-\tau) \right) e^{-\tau/K_n}\, d\tau \tag{2.15}$$

h_0 ia given by Eq. (2.9). $\Phi\, n^T$ means Φ_n transposed.
In practice the infiltration and the pumping or recharge are given as discrete series. In that case Eq. (2.15) can be simplified. We get:

$$\underline{h}_{1,n}^{(k)} = \frac{K_n}{S_o} \underline{\Phi n}(1-e^{-\Delta t/K_n}) \sum_{i=o}^{k-1} \underline{\Phi n}^T \left(\underline{\underline{D}}(\underline{R}_{k-i} - \overline{\underline{R}}) + \frac{1}{A}\underline{Q}_{k-i} \right) e^{-i\Delta t/K_n} \tag{2.16}$$

$$\underline{h}_{1,n}^{(k+1)} = e^{-\Delta t/K_n}\, \underline{h}_{1,n}^{(k)} + \frac{K_n}{S_o} \underline{\Phi n}(1-e^{-\Delta t/K_n})\, \underline{\Phi n}^T \left(\underline{\underline{D}}(\underline{R}_{k+1} - \overline{\underline{R}}) + \frac{1}{A}\underline{Q}_{k+1} \right) \tag{2.17}$$

$$h^{(k)} = h_o + \sum_{n=1}^{N_1} h_{1,n}^{(k)} \tag{2.18}$$

where $\underline{h}^{(k)}$ means the groundwater level in time period k.
From Eqs. (2.16) - (2.18) we see that in practical calculations it could be enough to use just the few first eigenfunctions in the expansion. We, thus, introduce some truncation error.

We now see the reason why we split Eq. (2.7) into two problems. Eq. (2.9) for the stationary problem can be solved without any truncation error. Acceptable truncation error in the transient term could give too large errors in the stationary part, if it had been solved by the eigenfunction expansion too.

If the input function is not very rich in the higher harmonics, that is to say it does not fluctuate very much over the area, one can be satisfied with a very limited number of eigenfunctions.

Let us now take a simple example to demonstrate the principles. Let our area, Ω, be given by a rectangle. See Fig. 3. Our source function is constant pumping in point $(\xi, \eta) = (½, ½)$ and we have an observation point in $(x, y) = (^1/_4, ½)$

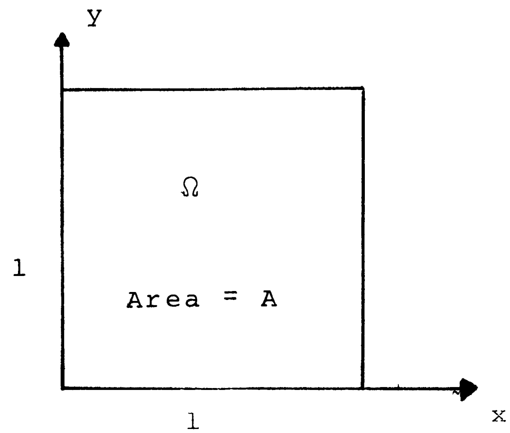


Fig. 3.

Let us use the following numerical values:

$A = 10^8 m^2$, $1 = 10^4 m$, $T = 0{,}2\ m^2/s$, $S = 0.06$, $Q = 1{,}0\ m^3/s$

$g = h_0$ constant on the boundary curve.

The theoretical solution is given by:

See Eliasson et al. (1973a,b)

$$h(x,y,t) = h_o - \frac{4Q}{\Pi^2 T} \sum_{n,m} \frac{\sin\frac{m\Pi x}{l}\ \sin\frac{n\Pi y}{l}\ \sin\frac{m\Pi \xi}{l}\ \sin\frac{n\Pi \eta}{l}}{(m^2+n^2)} \left(1 - e^{-\frac{\Pi^2 (n^2+m^2) T}{AS} t}\right) \qquad (2.19)$$

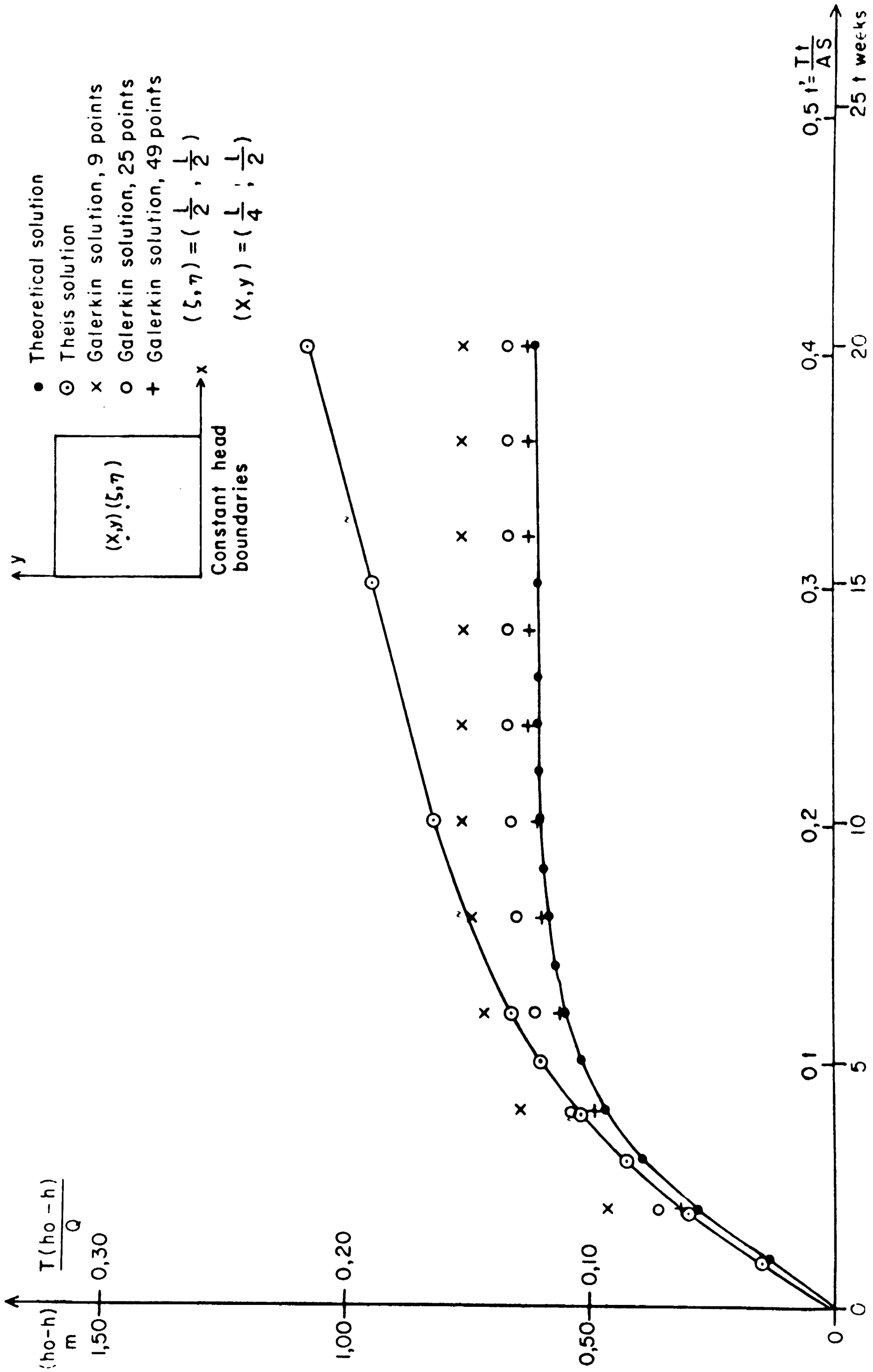
This equation is plotted in Fig. 4 together with Theis' solution for an infinite aquifer.

From (2.16) we get the approximate solution

$$\underline{h}^{(k)} = \underline{h}_o - \frac{Q}{T} \sum_{n=1}^{N_1} \frac{\underline{Q}_n}{\lambda_n} \underline{\Phi n}^T \underline{I} \left(1 - (e^{-\Delta t/K_n})^k\right) \qquad (2.20)$$

where we define

$$\underline{I} = \begin{Bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{Bmatrix} \qquad (2.21)$$



Theoretical solution
Theis solution
Galerkin solution, 9 points
Galerkin solution, 25 points
Galerkin solution, 49 points
(ξ,η) = (L/2 , L/2)
(X,y) = (L/4 ; L/2)
(X,y) (ξ,η)
Constant head boundaries
x
y
T(ho-h)/Q
(ho-h) m
0,30
0,20
0,10
1,50
1,00
0,50
0
0,1
0,2
0,3
0,4
0,5 t' = Tt/AS
5
10
15
20
25 t weeks

Fig. 4

with zero everywhere, except in the point where the pumping takes place. Eq. (2.20) is calculated, using 9, 25, and 49 internal, unknown points. The result is plotted in Fig. 4, with two scales on the axis. On the vertical axis we have the actual drawdown $S = (h - h_0)$ and the dimensionless drawdown $S^1 = TS/Q$. On the horizontal axis we have the actual time in weeks and the dimensionless time $t^1 = T{\cdot}t/A{\cdot}S$.

Then the diagram can be used for different values of Q, T, S, and A, by using the dimensionless scale on the axis.

It is readily seen that Eq. (2.15) represents the same model of parallel linear reservoirs as is set forth by Eliasson (1971) and Eliasson et al, (1973a,b). The present model is more elaborate as it takes into account the areal variations in infiltration and hydraulic properties of the aquifer. The basic concept of linear reservoirs with a definite time constant (the K_n's) attached to each reservoir is nevertheless preserved.

Model Construction and Calibration

In calibrating the model, the first thing is to construct the triangular network. For this there are two basic principles. First the network must be capable of representing all available information on the reservoir. Second, it must be fine enough to provide accurate results.

To cope with the first thing, it is undoubtedly wise to begin with collecting all available hydrological and geological information on the reservoir. Next, areas which are likely to have the same S and T should be drawn on a map. Furthermore, each well and observation drillhole should be intersection points, also springs and spots of concentrated recharge (sinks). A basic network can now be constructed by establishing a suitable number of intersection points along the borderlines of equal S and T so the triangle sides do not intersect these borderlines.

In this way we can obtain a kind of »basic triangular network«. To this network we must now add the necessary number of intersection points to meet the accuracy requirements. The leading principle in this work is that the network must be finer in places with sudden change of slope in the groundwater level than in places where the groundwater level is reasonably plane. This follows directly from the basic assumption that the groundwater level is supposed to be plane inside each triangle.

When the triangular network is established there remains the difficulty of finding the S and T values. Usually, we have considerable information on parameters S and T from pumping test. But S and T values from pumping tests are usually only local values, valid in the immediate neighbourhood of the tested well. In analysing the whole reservoir we need more global values. The only way to obtain these, is to compare observed $\underline{h}(t)$ values with measured values. In principle, it is possible to construct mathematical norms wherefrom the S and T values inside each triangle can be calculated. Such procedures must be used with the utmost care though, as such calculation procedures may possess considerable instability which means that the small deflections (errors) in the data (measured $\underline{h}(t)$ and $\underline{R}(t)$ values) cause great deflections in the computed S and T values. In any case, a method of trial and error must be recommended as the first step in the calibration process.

Epilogue

This method of analysing groundwater reservoirs is a further development of the theories set forth by Eliasson (1971) and Eliasson et al. (1973a,b). The difference is that we use the method of Galerkin to transform the partial differential equations into a finite set of ordinary differential equations. Then we use the eigenvalues and the eigenvectors of the matrices emerging from the Galerkin treatment to expand our problem into finite series of specially constructed convolution integrals.

The reason for doing this is twofold. First we must anticipate that the only way to handle all the available geological, meteorological and hydrological data available is in a digital computer. Second, discrete functions in finite spaces are much more easily handled in a computer than continous functions in infinite spaces as the treatment by Eliasson (1971) and Eliasson et al. (1973a,b) is based upon.

This method enables us to treat groundwater reservoir problems in a basic network of triangles that contains all the available field information (location of wells, springs, observation stations and hydrologically homogenous areas with equal S and T values) and transform this information into a set of constant matrices that do not change with time even though the infiltration, the pumping or other time dependent variables change. This makes it possible to test any specific utilization program before it is put in action, and thus reveals the properties of our reservoir and the consequences of the planned utilization before it is possibly too late.

In such an investigation the $\underline{R}(t)$ series in the formulas (2.9) and (2.15) and the input series $\underline{Q}(t)$ as well, do not have to be measured quantities. They may be planned or estimated values (artificial recharge or planned pumping), or stochastic infiltration series obtained from stochastic models of hydrological data. The method set forth here, is in short, a method to calculate the response of a given groundwater reservoir against any kind of hydrological action forced upon it from the environment.

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