Minimum Diameter of Deregular Digraphs of Degree 2*

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In this paper we study some relationships between the number of vertices, degree and diameter of finite digraphs. The three problems considered are \( N(d, k) \), \( K(n, d) \) and \( D(n, k) \); that is, find the maximum possible number of vertices given degree \( d \) and diameter \( k \), find the minimum possible diameter given the number of vertices \( n \) and degree \( d \), and find the minimum possible degree given the number of vertices \( n \) and diameter \( k \) respectively. These three problems are related but as far as we know not equivalent. In this paper we restrict our attention to digraphs of degree 2. Using new techniques we improve upon the bounds for \( N(d, k) \) and \( K(n, d) \) in the case \( d = 2 \). The current state of knowledge of \( N(2, k) \) and \( K(n, 2) \) \((n \leq 100)\) is given in Tables 1 and 2. The paper concludes with eight open problems in the area.

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INTRODUCTION

The problem of designing a processor network which is in some sense optimal is an open problem for many of the possible criteria of optimisation. Assuming a network consists of \( n \) processors, each processor with \( d_i \) incoming and \( d_i' \) outgoing ports, we can represent such a network as a directed graph, or a digraph, with \( n \) vertices (i.e. of order \( n \)), with \( d_i \) arcs coming into vertex \( i \) and \( d_i' \) arcs going from \( i \). In this paper we consider networks such that \( d_i = d_i' \) \((= d)\) for every processor \( i \) in the network, that is, we restrict our attention to digraphs of degree \( n \) and degree \( d \). For the optimisation criterion we will require that the distance (i.e. the shortest path) between any two processors be minimal, that is, that the number of times a message has to be relayed to reach its destination be as small as possible. Calling the overall maximum distance in a digraph \( G \) the diameter of \( G \), we will denote a digraph of digraph of degree \( d \), order \( n \) and diameter \( k \) by \( G(n, d, k) \). In particular, the following three problems arise for digraphs \( G(n, d, k) \).

(1) The \( N(d, k) \) problem: given \( d \) and \( k \), find the maximum possible order \( N(d, k) \). This problem has also been called the \( (d, k) \) problem.

(2) The \( K(n, d) \) problem: given \( n \) and \( d \), find the minimum possible diameter \( K(n, d) \).

(3) The \( D(n, k) \) problem: given \( n \) and \( k \), find the minimum possible degree \( D(n, k) \).

These three problems are related, but as far as we know not equivalent. In this paper we will consider these problems in more detail mainly for the particular case when \( d = 2 \). However, we will mention some other related optimising problems in the Conclusion.

1. DISCUSSION OF THE THREE PROBLEMS

We allow loops and multiple arcs in a digraph \( G \). If \( A \) is the vertex-adjacency matrix of \( G = G(n, d, k) \) then

\[ I + A + A^2 + \ldots + A^k \geq E \]

where \( I \) is the \( n \times n \) identity matrix and \( E \) is the \( n \times n \) unit matrix (i.e. the \( n \times n \) matrix whose entries are all 1s). Thus

\[ 1 + d + d^2 + \ldots + d^k \geq n \]

must hold whenever \( G(n, d, k) \) exists and

\[ N(d, k) \leq 1 + d + d^2 + \ldots + d^k \]

Bridges and Toueg\(^*\) showed that for \( d \geq 1 \) and \( k > 1 \) a digraph of degree \( d \) cannot exist. Consequently, for \( d > 1 \), \( k > 1 \)

\[ N(d, k) \leq d + d^2 + \ldots + d^k \]

If \( d = 1 \) then \( G(n, 1, k) \) exists for \( n = k + 1 \); it is the elementary circuit of \( k + 1 \) vertices. Thus \( N(1, k) = k + 1 \). If \( k = 1 \) then \( G(n, d, 1) \) exists for \( n = d + 1 \); it is the complete digraph on \( d + 1 \) vertices. Thus \( N(d, 1) = d + 1 \). Now for \( d \geq 1 \) and \( k > 1 \) we can construct a digraph

\[ G = G(d^{k-1}(d+1), d, k)^{k-1} \]

and so for \( k = 2 \), \( N(d, 2) = d + d^2 \) and for \( d > 1 \), \( k > 2 \), \( N(d, k) \geq d^{k-1}(d+1) \).

Next, it is easy to see that the following implications hold.

\[ \begin{align*}
(a) \quad d_1 < d_2 & \Rightarrow N(d_1, k) < N(d_2, k) \\
(b) \quad k_1 < k_2 & \Rightarrow N(d, k_1) < N(d, k_2) \\
(c) \quad d_1 < d_2 & \Rightarrow K(n, d_1) \geq K(n, d_2)
\end{align*} \]

On the other hand, we do not know whether or not any of the implications (i)-(iii) hold.

\[ \begin{align*}
(i) \quad k_1 < k_2 & \Rightarrow D(n, k_1) \geq D(n, k_2) \quad (k_1, k_2 \leq n-1) \\
(ii) \quad n_1 < n_2 & \Rightarrow K(n_1, d) \leq K(n_2, d) \\
(iii) \quad n_1 < n_2 & \Rightarrow D(n, k_1) \leq D(n, k_2) \quad (n_1, n_2 \geq k + 1)
\end{align*} \]

If (i) were true then \( D(n, k) = \min \{d : K(n, d) \geq k\} \), that is, the solution of the \( K(n, d) \) problem, would give the solution of the \( D(n, k) \) problem. Similarly, if (ii) were true the solution of the \( N(d, k) \) problem would give the solution of the \( K(n, d) \) problem. Finally, if (iii) were true the solution of the \( N(d, k) \) problem would give the solution of the \( D(n, k) \) problem. However, as the situation is at present, we can deduce from (a), (b) and (c):

(A) the solution of the \( K(n, d) \) problem will give the solution of the \( N(d, k) \) problem;

(B) the solution of the \( D(n, k) \) problem will give the solutions of both the \( N(d, k) \) and the \( K(n, d) \) problems.

Thus it would be best to try to solve the \( D(n, k) \) problem. Unfortunately, of the three problems, the

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The \( D(n, k) \) problem is the most difficult one to handle. Consequently, we will only deal with \( N(d, k) \) and \( K(n, d) \) problems.

The following theorem, due to Fiol, Alegre and Yebra\(^a\), will be useful in further discussion.

**Theorem 1**

If \( d > 1 \) and \( G = G(a, d, k) \) exists then \( L(G) = G(dn, d, k+1) \) also exists.

This theorem is proved using the technique of ‘line digraph’ iterations. Starting from a digraph \( G \), we form the line digraph of \( G \), \( L(G) \) by turning arcs of \( G \) into vertices of \( L(G) \); and paths of length 2 of \( G \) into arcs of \( L(G) \). In this way we can obtain an infinite sequence of line digraph iterations

\[
G, L(G), L^2(G) = L(L(G)), \ldots, L^k(G) = L(L^{k-1}(G)), \ldots
\]

In the remainder of this paper we will consider the \( N(d, k) \) and \( K(n, d) \) problems for \( d = 2 \).

2. THE \( N(2, k) \) PROBLEM

If \( d = 2 \) we can prove that for \( k > 1 \), \( N(2, k) \leq 2 + 2^2 + \ldots + 2^k \) (the result of Bridges and Toueg\(^a\) for \( d = 2 \)) in a more direct way. This proof illustrates methods used to prove Theorems 3 and 4.

**Theorem 2**

If \( k > 1 \) then \( N(2, k) \leq 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 2 \).

**Proof**

Suppose \( k > 1 \) and \( G(2^{k+1} - 1, 2, k) \) exists. Denoting the vertices of \( G \) by \( 1, 2, \ldots, 2^{k+1} - 1 \) we can partly draw \( G \) (Fig. 1). Now to reach every other vertex from vertex 2 in at most \( k \) steps there must be arcs from the vertices \( 2^k, 2^k + 1, \ldots, 3 
\times 2^k - 1 \) to the vertices \( 1, 3, 6, 7, 12, 13, 14, 15, \ldots, 3 \times 2^k - 1, 2^{k+1} - 1 \). Similarly, to reach every other vertex from vertex 3 in at most \( k \) steps there must be arcs from the vertices \( 3 \times 2^k - 1, 3 \times 2^k - 1 + 1, \ldots, 2^{k+1} - 1 \) to the vertices \( 1, 2, 4, 5, 8, 9, 10, 11, \ldots, 2^k, 2^k + 1, \ldots, 3 \times 2^k - 1 \). Two arcs go to every vertex and two arcs come from every vertex.

Let \( x \rightarrow 2(x \neq 1) \) and let \( y \) be the end point of the second arc from \( x \). Then \( y \in \{1, 2, 4, 5, 8, 9, 10, 11, \ldots, 2^k, 2^k + 1, \ldots, 3 \times 2^k - 1 \} \). But then we reach the vertex \( y \) from \( x \) in two different ways in \( k \) steps or less. Then there must be a vertex \( z \in \{1, 3, 6, 7, 12, 13, 14, 15, \ldots, 3 \times 2^k - 1, \ldots, 2^{k+1} - 1 \} \) which cannot be reached from \( x \) in \( k \) steps or less and so the diameter of \( G \) is not \( k \).

**Corollary**

\( N(2, 2) = 6 \).

For \( k > 2 \) we have the following.

**Theorem 3 (Miller\(^a\))**

If \( k > 2 \) then \( N(2, k) \leq 2^{k+1} - 3 \).

The proof of this theorem is done essentially by brute force and is too long to be included here.

**Theorem 4 (Miller\(^a\))**

\( N(2, 3) \neq 13 \).

**Outline of the proof**

Suppose \( G \in G(13, 2, 3) \) exists.

Then every vertex of \( G \) reaches itself in 3 or 4 steps in exactly two ways. That is, any vertex of \( G \) lies on

(a) two 4-circuits or

\[
\begin{align*}
2^k & \quad 2^{k+1} \ldots 3 \times 2^k - 1 \\
\cdots & \quad \cdots \\
3 \times 2^k - 2 & \quad \cdots \\
2^k - 1 & \quad 2^{k+1} - 1
\end{align*}
\]

**Figure 1**

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3. THE K(n, 2) PROBLEM

To determine the value of $K(n, 2)$ we can use all the results of Section 2. Indeed, if $N(2, k) = n$ then $K(n, 2) = k$. Thus since $N(2, 0) = 1$, $N(2, 1) = 3$, $N(2, 2) = 6$, $N(2, 3) = 12$ we have $K(1, 2) = 0$, $K(3, 2) = 1$, $K(6, 2) = 2$, $K(12, 2) = 3$. However, we still need to consider all $n$'s which are not in the range of $N(2, k)$.

**Theorem 5**

If $n \leq 2^s$ then $K(n, 2) \leq s$.

**Proof**

We construct a digraph $G(n, 2, k)$ with $n \leq 2^s$ and $k \leq s$ as follows. Labelling the vertices of $G$ as $0, 1, 2, \ldots, n - 1$, let $G$ contain arcs from vertex $i$ to vertices $-2i + 1$ and $-2i + 2 \pmod{n}$. Then $G$ is a digraph with degree 2. In step $s$ we reach from vertex $i$ the following vertices $\pmod{n}$:

$$-2^s i + 1, -2^s i + 2, \ldots, -2^s i + 2^s \quad \text{if } s \text{ odd}$$

$$2^s i - 0, 2^s i - 1, \ldots, 2^s i - (2^s - 1) \quad \text{if } s \text{ even}$$

In either case, if $n \leq 2^s$ then in the $s$th step we reach from $i$ exactly all the $n$ vertices of $G$. Hence $G$ has diameter at most $s$ and so $K(n, 2) \leq s$.

**Corollary**

If $n = 2^s + 1$ and $s$ odd then $K(n, 2) \leq s$.

**Proof**

Using the same construction as in the proof of Theorem 5, if $s$ is odd then $-2^s i + 1, -2^s i + 2, \ldots, -2^s i + 2^s \pmod{n}$ are all distinct vertices, $2^s + 1$ of them.

Note that Theorem 5 holds for any $d > 1$ with essentially the same proof. Now we have $K(3 \times 2^{k-1}, 2) = k$ and (i) for $k$ even

$$K(n, 2) = \begin{cases} k, & \text{if } N_k + 1 \leq n \leq 2k \\ k + 1, & \text{if } 2k + 1 \leq n \leq N_{k+1} \end{cases}$$

and (ii) for $k$ odd

$$K(n, 2) = \begin{cases} k, & \text{if } N_k + 1 \leq n \leq 2k + 1 \\ k + 1, & \text{if } 2k + 2 \leq n \leq N_{k+1} \end{cases}$$

where

$$N_k = \begin{cases} 1, & \text{if } k = 1 \\ 3, & \text{if } k = 2 \\ 6, & \text{if } k = 3 \\ 2^k - 3, & \text{if } k > 3 \end{cases}$$

Thus, for most practical purposes the $K(n, 2)$ problem is solved; for any $n$ we can construct a digraph with diameter at most 1 more than the possible minimum. (In fact, this is true for any $d \geq 1$.)

If a digraph $G = G(n, 2, k)$ contains $p$ digons (a digon is a pair of arcs $(u, v)$ and $(v, u)$, $u \neq v$) then we will also denote $G$ by $(n, 2, k) \langle p \rangle$.

**Theorem 6 (Culik)**

If $n > 2$ and $G = G(n, 2, k) \langle p \rangle$ exists then $G' = G(n - 1, 2, k') \langle p' \rangle$ exists, for some $k' \leq k$ and $p' \geq p - 1$.
Proof

Suppose \( G = G(n, 2, k) \langle p \rangle, \ p \geq 1 \). Then for some vertices \( u, v \neq u \) there exist arcs \( (u, v) \) and \( (v, u) \) in \( G \).

Suppose also that \( p \rightarrow u, q \rightarrow v \) and \( u \rightarrow r, v \rightarrow t \) (\( u, v, p, q, r, t \) not necessarily all distinct).

Construct \( G' \) from \( G \) by ‘gluing’ vertices \( u \) and \( v \) together, so that instead of vertices \( u \) and \( v \) we have a vertex \( uv \) in \( G' \) (Fig. 3). Then \( G' \) obviously has degree \( 2, n-1 \) vertices, diameter either \( k \) or \( k-1 \) and if \( n > 2 \), \( G' \) has at least \( p-1 \) digons.

Moreover, it can be shown that if \( n \geq 2^k + 2 \) then \( p' = p - 1 \).

Theorem 6 gives a ‘digon reduction’ scheme. Using this scheme together with the line digraph iteration scheme (and other results of Section 2) we can deduce \( K(n, 2) \) for many values of \( n \). For example,

\[
G(12, 2, 3) \langle 3 \rangle \Rightarrow G(11, 2, 3) \langle 2 \rangle \quad \text{by Theorem 6}
\]

\[
\Rightarrow G(22, 2, 4) \langle 2 \rangle \quad \text{by Theorem 1}
\]

and since the line digraph iteration scheme preserves the number of digons.

Digraphs \( G(1, 2, 0), G(2, 2, 1) \) and \( G(5, 2, 2) \langle 5 \rangle \) obviously exist. Using these together with Theorem 1 and Theorem 6 we can summarize our current state of knowledge of the \( K(n, 2) \) problem in Table 2 (for \( n \leq 100 \)).

CONCLUSION

In this paper we have proved some new results about the relationship between the degree, diameter and order of digraphs. In particular, for degree 2 we have improved upon the bounds for the order (given diameter); and for certain values of order we have improved upon the bounds for diameter (given order).

However, the \( N(2, k) \) and the \( K(n, 2) \) problems still remain open, as do the \( N(d, k), K(n, d) \) and \( D(n, k) \) problems in general. To get nearer to the solutions we may try to answer some simpler questions, such as:

1. Is \( K(n, d) \) monotonic in \( n \)?
2. Is \( D(n, k) \) monotonic in \( n \)?
3. Is \( D(n, k) \) monotonic in \( k \)?
4. Does \( K(n, d) = k \) imply \( K(nd, d) = k+1 \)?

Certainly, \( K(n, d) = k \) implies \( K(nd, d) \leq k+1 \) using the line digraph iteration scheme of Theorem 1. On the other hand, a similar implication for \( N(d, k) \) does not hold, as for example \( N(2, 3) = 12 \) and \( N(2, 4) \geq 25 \).

5. Is \( K(n, d) \) monotonic for some intervals of \( n \)? In particular,

(a) Is \( K(n, d) \) monotonic for all \( n \) such that \( (d^k - 1)/(d - 1) < n \leq N(d, k) \)?

(b) Is \( K(n, d) \) monotonic for all \( n \) such that \( (d^k - 1)/(d - 1) < n \leq (d+1)^k - 1 \)?

An affirmative answer to any of these questions would much advance the solution of the \( N(d, k) \), \( K(n, d) \) and \( D(n, k) \) problems.

Apart from the three problems treated in this paper, there are many other related open problems. We will mention just a few of these.

6. The problem of finding the minimum average diameter \( \bar{K}(n, d) = \min_{G(n, d, k)} \bar{K} \) given \( n, d, k \), where \( \bar{K} \) is the average diameter of \( G \),

\[
\bar{K} = \frac{\sum_{i,j \in E} d_{ij}}{n(n-1)}
\]

where \( d_{ij} \) is the length of the shortest path from vertex \( i \) to vertex \( j \) in the digraph \( G \).

Table 3

<table>
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<th>n</th>
<th>k</th>
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(7) The problem of finding all the optimal non-isomorphic digraphs of a given order, diameter and degree.

Bowen* found (with the use of a computer) all the optimal non-isomorphic diregular digraphs for degree 2, and order up to 12. Table 3 gives the number of nonisomorphic diregular digraphs of degree 2, \( I(n, k) \), for given order \( n \) and diameter \( k \).

(8) The problem of finding digraphs which are reducible, that is given a digraph \( G \), we wish to find subdigraphs \( G_1, G_2, \ldots, G_t = G \) such that \( G_1 \subseteq G_2 \subseteq \cdots \subseteq G_t \) and each \( G_i \) keeps certain given desired property (e.g. minimum distance or minimum average distance). Conversely, we could require a digraph to be extensible.

REFERENCES