ditions, is readily constructed from the known solution[5] of a rectangular plate, simply supported along the edges and bent by uniform moments along two opposite edges. In view of this known solution, the center static deflection for the present problem is

$$(v_{st})_{max} = M_T \frac{2a^2}{\pi^3 D} \sum_{m=1,3,5}^{\infty} (-1)^{\frac{m-1}{2}} \frac{1}{m^3} \left[\frac{\alpha_m \tanh \alpha_m}{\cosh \alpha_m} + \frac{b^2}{a^2} \frac{\beta_m \tanh \beta_m}{\cosh \beta_m}\right]$$

where

$$\alpha_m = \frac{m\pi b}{2a}, \qquad \beta_m = \frac{m\pi a}{2b}$$

The static deflection just given bears little resemblance to the static deflection presented in Equation [9] of the paper. As a matter of fact, for the case of a square plate, the foregoing deflection is about three fourths of the deflection found by the authors.

From the form of Equation [9] of the paper, it appears that the procedure of derivation used by the authors was to assume a double Fourier expansion of both the thermal moment and the deflection, i.e.

$$M_T = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi z}{b}$$

$$v_{st} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} d_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi z}{b}$$

The coefficients c_{nm} are readily determined, and direct substitution in the equilibrium Equation [7] immediately determines the coefficients d_{nm}. Evaluation of the resulting series for v_{st} at the center of the plate leads to Equation [9] of the paper. Observe, however, that this assumed form of the solution violates the stated boundary conditions of the problem in that the edges are now free from moments.

Since it appears that the static solution presented by the authors violates the stated boundary conditions one may presume that their dynamic solution also will suffer this defect. It seems that the ratio of these "solutions" can yield only a very doubtful appraisal of the effect of inertia under rapid heating.

It also should be pointed out that Equations [7], [7*a*], [8*a*], [8*b*] should be multiplied on the right-hand side by

$$(1-\nu), \quad \left(\frac{1+\nu}{2}\right), \quad (-1/D), \quad \text{and} \quad \left(\frac{1+\nu}{2}\right)$$

respectively, to obtain a formulation of the problem consistent with their reference (5).[6]

Authors' Closure

The authors wish to thank the discussers for bringing to their attention some errors in Equations [7] and [8] of the paper; in checking back to their original work, they fortunately found that the calculations were indeed performed on the basis of the correct equations, and that therefore these errors are merely typographical. Aside from this, the doubts raised by the second discusser are unfounded.

The static solution of the plate problem was obtained by Maulbetsch's method.[7] The deflection can be easily expressed either in terms of circular trigonometric functions along one Cartesian co-ordinate direction in the plane of the plate and hyperbolic in the other, or in terms of circular functions in both directions; the latter method was adopted in the paper. It should be noted that, contrary to the second discusser's suggestion, there is no contradiction between the use of a double sine series (whose second derivations ostensibly are zero at $x = 0$, a and $y = 0$, b) and a nonzero moment (which would require nonzero values for these derivatives). The sine series (for example, for the edge $x = a$) converges in fact to the correct value at $x = a - 0$, though it converges, at $x = a$ itself, to the average (i.e., zero) of the values at $x = a - 0$ and $x = a + 0$. The behavior of Fourier series in the neighborhood of such discontinuities is well known;[8] the present approach was adequate for the purposes of the computations, required for the paper, pertaining to the interior of the plate.

It was pointed out, at the time of the oral presentation of this paper, by the chairman of the session, Dr. R. Plunkett, that both the static and dynamic solutions may be constructed with the aid of isothermal solutions corresponding to prescribed edge moments. Such moments must be applied over all four edges and not, as the second discussion seems to imply, over two opposite edges only.

[5] "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Company, Inc., New York, N. Y., 1940, pp. 199–202.

[6] "Similarity Laws for Stressing Heated Wings," by H-S. Tsien, *Journal of the Aeronautical Sciences*, vol. 20, January, 1953, pp. 1–11.

[7] See "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, p. 176ff.

[8] See, for example, "Fourier Series and Boundary Value Problems," by R. V. Churchill, McGraw-Hill Book Co., Inc., New York, N. Y., 1941.

Stress-Strain Relations and Vibration of a Granular Medium[1]

C. W. Thurston.[2] It may be of interest to extend the derivation of the differential stress-strain relations given by the authors to the case in which the array of spheres is subjected to an arbitrary initial state of stress. This may be done by the following method suggested to the writer by the authors: The unit cube (authors' Fig. 1) is taken to be in a sequence of states of strain in each of which only one component, in turn, is different from zero. For each state, the relations between all the components (zero and nonzero) of strain and the corresponding components of relative displacement between centers of contiguous spheres are then written in accordance with the expressions immediately preceding the authors' Equations [16] and [17]. This gives, in each case, six equations in terms of the 18 (unknown) components of displacement. Three more equations are obtained from the conditions for the equilibrium of moments (authors' Equations [10]) in which the authors' Equations [12] are inserted. The final nine equations are the conditions of compatibility of the relative displacements (authors' Equations [11]). The solution of each set of 18 equations (12 of which are common to each set) yields the components of relative displacement in terms of the different nonzero strain components; hence the sum of the solutions of the 6 sets yields the relations between each component of displacement and all the components of strain.

The stress and strain may now be related by substituting the expressions found in the foregoing into the authors' Equations [12] and the result, in turn, into the authors' Equations [9]. The final expressions, referred to the principal axes of the unit cube, are of the form appropriate to a triclinic crystal

$$d\sigma_m = c_{mn} d\epsilon_n \qquad m, n = 1, 2, \ldots, 6$$

[1] By J. Duffy and R. D. Mindlin, published in the December, 1957, issue of the JOURNAL OF APPLIED MECHANICS, TRANS. ASME, vol. 79, pp. 585–593.

[2] Instructor in Civil Engineering, Columbia University, New York, N. Y.

where c_{mn} represents the symmetric array $(c_{mn} = c_{nm})$ of 21 independent coefficients whose elements are

$$4\sqrt{2}Rc_{11} = c_{zx} + c_{xy} + s_{zx} + s_{xy} - 2(\bar{s}_{zx}{}^2\Lambda_{zx} + \bar{s}_{xy}{}^2\Lambda_{xy} - 2\bar{s}_{zx}\bar{s}_{xy}\bar{\Lambda}_{yz})/\Delta$$

$$4\sqrt{2}Rc_{12} = c_{xy} - s_{xy} + 2(\bar{s}_{xy}{}^2\Lambda_{xy} + \bar{s}_{yz}\bar{s}_{zx}\bar{\Lambda}_{xy} - \bar{s}_{zx}\bar{s}_{xy}\bar{\Lambda}_{yz} - \bar{s}_{xy}\bar{s}_{yz}\bar{\Lambda}_{zx})/\Delta$$

$$4\sqrt{2}Rc_{14} = 2(\bar{s}_{yy}\bar{s}_{xy}\Lambda_{xy} - \bar{s}_{yy}\bar{s}_{zx}\bar{\Lambda}_{yz})/\Delta + 2(s_{yy} + s_{yz})(\bar{s}_{zx}\bar{\Lambda}_{xy} - \bar{s}_{xy}\bar{\Lambda}_{zx})/\Delta$$

$$4\sqrt{2}Rc_{15} = \bar{c}_{zx} - \bar{s}_{zx} + 2(\bar{s}_{zz}\bar{s}_{xy}\bar{\Lambda}_{zx} - \bar{s}_{zz}\bar{s}_{zx}\bar{\Lambda}_{xy})/\Delta + 2(s_{zz} + s_{zx})(\bar{s}_{zx}\Lambda_{zx} - \bar{s}_{xy}\bar{\Lambda}_{yz})/\Delta$$

$$4\sqrt{2}Rc_{16} = \bar{c}_{xy} + \bar{s}_{xy} - 2[\bar{s}_{xx}\bar{s}_{zx}\Lambda_{zx} + (s_{xx} + s_{xy})\bar{s}_{xy}\Lambda_{xy}]/\Delta + 2(s_{xx}\bar{s}_{zx} + \bar{s}_{zx}s_{xy} + \bar{s}_{xx}\bar{s}_{xy})\bar{\Lambda}_{yz}/\Delta$$

$$4\sqrt{2}Rc_{44} = c_{yz} + s_{yz} + 2s_{yy} - 2[\bar{s}_{yy}{}^2\Lambda_{xy} + (s_{yy} + s_{yz})^2\Lambda_{yz} - 2(s_{yy} + s_{yz})\bar{s}_{yy}\bar{\Lambda}_{zx}]/\Delta$$

$$4\sqrt{2}Rc_{45} = 2[(s_{yy} + s_{yz})\bar{s}_{zz}\Lambda_{yz} + (s_{zz} + s_{zx})\bar{s}_{yy}\bar{\Lambda}_{yz}]/\Delta - 2[\bar{s}_{yy}\bar{s}_{zz}\bar{\Lambda}_{zx} + (s_{yy} + s_{yz})(s_{zz} + s_{zx})\bar{\Lambda}_{xy}]/\Delta$$

where

$$\Lambda_{ij} = s_{ii}s_{jj} + s_{jj}s_{kk} + s_{kk}s_{ii} + s_{kk}{}^2 - \bar{s}_{kk}{}^2 + 2(s_{ii} + s_{jj})s_{jk} + 2(s_{jj} + s_{kk})s_{ij} + 4s_{jk}s_{ki}$$

$$\bar{\Lambda}_{ij} = \bar{s}_{ii}\bar{s}_{jj} + \bar{s}_{kk}(s_{ii} + s_{jj} + 2s_{ij})$$

$$\begin{aligned}\Delta = {} & 8s_{jk}s_{ki}s_{ij} + 2(s_{ii}s_{jj}s_{kk} - \bar{s}_{ii}\bar{s}_{jj}\bar{s}_{kk}) \\ & + (s_{ii}{}^2 - s_{jj}{}^2)(s_{jj} + s_{kk} + 2s_{jk}) + 4s_{ii}(s_{ij}s_{jk} + s_{jk}s_{ki}) \\ & + (s_{jj}{}^2 - s_{kk}{}^2)(s_{kk} + s_{ii} + 2s_{ki}) + 4s_{jj}(s_{jk}s_{ki} + s_{ki}s_{ij}) \\ & + (s_{kk}{}^2 - s_{ii}{}^2)(s_{ii} + s_{jj} + 2s_{ij}) + 4s_{kk}(s_{ki}s_{ij} + s_{ij}s_{jk}) \\ & + 2(s_{ii}s_{jj} + s_{jj}s_{kk} + s_{kk}s_{ii})(s_{jk} + s_{ki} + s_{ij})\end{aligned}$$

and

$$c_{ij} = 1/C_{ij} + 1/C_{ij}' \qquad \bar{c}_{ij} = 1/C_{ij} - 1/C_{ij}'$$

$$s_{ij} = 1/S_{ij} + 1/S_{ij}' \qquad \bar{s}_{ij} = 1/S_{ij} - 1/S_{ij}'$$

$$s_{ii} = 1/S_{ii} + 1/S_{ii}' \qquad \bar{s}_{ii} = 1/S_{ii} - 1/S_{ii}'$$

The remaining coefficients may be written by cyclical interchange of subscripts, as follows

$$c_{11} \to c_{22} \to c_{33} \qquad c_{12} \to c_{23} \to c_{31}$$

$$c_{14} \to c_{25} \to c_{36} \qquad c_{15} \to c_{26} \to c_{34}$$

$$c_{16} \to c_{24} \to c_{35} \qquad c_{44} \to c_{55} \to c_{66}$$

$$c_{45} \to c_{56} \to c_{64}$$

Influence of Width on Velocities of Long Waves in Plates[1]

E. Volterra.[2] In a recent report (1),[3] E. C. Zachmanoglou and the writer developed a second approximation of the method of internal constraints by taking into account second-order terms in the equation of constraints and satisfying the requirement that normal stresses vanish on the surface of the bar or of the plate under consideration. This approach has the main advantage that,

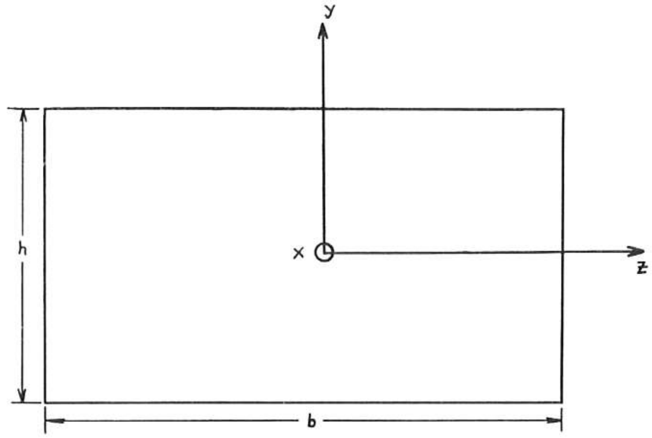

Fig. 1

while the degree of accuracy of the general method is increased, the equations of motion and corresponding boundary conditions, which univocally define the unknown functions of the problem, maintain the same simple mathematical form as in the cases in which only linear terms of the equations of constraint are taken into consideration (which cases were discussed in previously published papers (2–8)).

This method gives, in the case of a bar of rectangular cross section, Fig. 1 of this discussion, the following dispersion equation for longitudinal waves

$$\left.\begin{aligned}&\left(\frac{h}{L}\right)^4 \pi^4\beta^2 \left[\left(\frac{c}{c_0}\right)^2 - \frac{(1-\sigma)}{(1+\sigma)(1-2\sigma)}\left(1 - \frac{4}{3}\frac{\sigma^2}{(1-\sigma)}\right)\right]\left[\left(\frac{c}{c_0}\right)^2 - \frac{k}{2(1+\sigma)}\right]^2 \\ &- \left(\frac{h}{L}\right)^2 \pi^2(1+\beta^2)\frac{(1-\sigma)}{(1+\sigma)(1-2\sigma)} \left[\left(\frac{c}{c_0}\right)^2 - \frac{k}{2(1+\sigma)}\right]\left[\left(\frac{c}{c_0}\right)^2 - \frac{1}{(1-\sigma^2)}\left(1 - \frac{2}{3}\sigma^2\right)\right] + \frac{\left(\frac{c}{c_0}\right)^2 - 1}{(1+\sigma)^2(1-2\sigma)} = 0\end{aligned}\right\} \ldots [1]$$

In Equation [1] σ is Poisson's ratio, $\beta = b/h$, L the wave length, h and b the dimensions of the rectangular cross section of the bar, k has the same significance as Timoshenko's shear coefficient, c is the phase velocity, and $c_0 = (E/\rho)^{1/2}$ the bar velocity. The dispersion equation for longitudinal waves in the case of plane stress is obtained from Equation [1] by letting $\beta \to 0$ and is given by

$$\left(\frac{h}{L}\right)^2 = \frac{1}{\pi^2}\frac{1}{(1-\sigma)^2} \frac{\left(\frac{c}{c_0}\right)^2 - 1}{\left[\left(\frac{c}{c_0}\right)^2 - \frac{k}{2(1+\sigma)}\right]\left[\left(\frac{c}{c_0}\right)^2 - \frac{1}{(1-\sigma^2)}\left(1 - \frac{2}{3}\sigma^2\right)\right]} \quad \ldots [2]$$

In the plane-strain case the theory based on the method of internal constraints gives the following dispersion equation for longitudinal waves

[1] By D. C. Gazis and R. D. Mindlin, published in the December, 1957, issue of the JOURNAL OF APPLIED MECHANICS, TRANS. ASME, vol. 79, pp. 541–546.

[2] Professor of Engineering Mechanics, University of Texas, Austin, Texas. Mem. ASME.

[3] Numbers in parentheses refer to the Bibliography at the end of this discussion.