Carter separable electromagnetic fields

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ABSTRACT

The purely electromagnetic analogue in flat space of Kerr’s metric in general relativity is only rarely considered. Here we carry out in flat space a programme similar to Carter’s investigation of metrics in general relativity in which the motion of a charged particle is separable. We concentrate on the separability of the motion (be it classical, relativistic or quantum) of a charged particle in electromagnetic fields that lie in planes through an axis of symmetry. In cylindrical polar coordinates \((t, R, \phi, z)\) the four-vector potential takes the form \((\Phi, A_\phi)\), where \(\phi\) is the unit toroidal vector. The forms of the functions \(\Phi(R, z)\) and \(A(R, z)\) are sought that allow separable motion. This occurs for relativistic motion only when \(A/R, \Phi\) and \(A^2 - \Phi^2\) are all of the separable form \(\zeta(\lambda) - \eta(\mu)/(\lambda - \mu)\), where \(\zeta\) and \(\eta\) are arbitrary functions, and \(\lambda\) and \(\mu\) are spheroidal coordinates or degenerations thereof. The special forms of \(A\) and \(\Phi\) that allow this are deduced. They include the Kerr metric analogue, with \(E + iB = -\nabla[ql(r - ia) \cdot (r - ia)]^{-1/2}\).

Rather more general electromagnetic fields allow separation when the motion is non-relativistic.

The investigation is extended to fields that lie in parallel planes. Connections to Larmor’s theorem are remarked upon.

Key words: black hole physics – magnetic fields – relativity – celestial mechanics, stellar dynamics.

1 INTRODUCTION

The problem of finding constants of the motion in classical mechanics, quantum mechanics and relativity has remained a fertile one since Newton (1687) initiated it. The method of separation of variables has been given remarkable new life through Sklyanin’s work (1995). He considers separation in the canonically invariant context in which the separation ‘coordinates’ are connected to the space coordinates by a general canonical transformation dependent on the momenta. This has allowed him and others to see most of the modern exact results on the solution of e.g. the Sine–Gordon and non-linear Schrödinger equations as consequences of such variable separations. Our work was stimulated by the discoveries of super-integrable cases in which all orbits exactly close (Gibbons & Manton 1986; Evans 1990, 1991; Lynden-Bell & Nouri-Zonoz 1998). We then found a simple theorem that is clearly due to Carter (1968), as it formed the basis of his studies of charged particle motions in general relativity. He found it by asking what properties of the Kerr metric had allowed him to separate the motions there.

Following Carter’s work, Teukolsky (1972, 1973) found how to separate the zero-mass wave equations in the Kerr metric, and Chandrasekhar (1972) generalized this to the Dirac equation with mass. Cook (1982) found separations of the Dirac equation in electromagnetic fields in flat space, and his tabulation is the fullest to date. It does not include the flat-space analogue of the Kerr–Newman metric, although Dirac’s equation does separate for it: see Carter & McLenaghan (1979), Page (1976) and Toop (private communication). Older tabulations may be found in Hautot (1973) and Lam (1970).

We shall not discuss the Dirac equation further.

Carter’s separation is done by coordinate transformation only. When the magnetic fields lie in planes through the axis of symmetry, this is all we need, but, when there are more complicated magnetic fields that cut through meridional planes, separation will require the more complicated momentum-dependent canonical transformations. Two known examples of separation of non-axially symmetric systems in rotating axes illustrate this, via Larmor’s analogy (Freeman 1966; Vandervoort 1979; Contopoulos & Vandervoort 1992). To make this paper as readable as possible, we start with the simplest case of classical non-relativistic mechanics and generalize later. Since our fields lie

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in meridional planes and $\text{div } \mathbf{B} = 0$, we may write $\mathbf{B} = \text{curl}(\mathbf{A} \hat{\phi}) = \nabla(AR) \times \nabla \phi$, where $A(R, z)\hat{\phi}$ is the vector potential. Notice that lines of constant $AR$ are lines of force in each plane $\phi = \text{constant}$ and $\nabla \phi = R^{-1} \hat{\phi}$.

The Lagrangian for a particle of mass $m$ and charge $q$ moving non-relativistically in an electromagnetic field described by a vector potential $\mathbf{A}$ and an electric potential $\Phi$ is

$$L = \frac{1}{2}m \mathbf{r}^2 + \frac{q}{c} \mathbf{r} \cdot \mathbf{A} - q\Phi.$$  \hspace{1cm} (1.1)

In axial symmetry $\Phi$ and $A_\phi = A$ are independent of $\phi$, so $\Phi$ is ignorable and $p_\phi = \partial L/\partial \dot{\phi}$ is constant:

$$\frac{\partial L}{\partial \phi} = mR^2 \dot{\phi} + \frac{q}{c} RA = p_\phi.$$  \hspace{1cm} (1.2)

We convert to the Hamiltonian

$$H = p \cdot \dot{\mathbf{r}} - L = \frac{1}{2}m^{-1}[p^2 + p_\phi^2 + (p_\phi R^{-1} - qAc^{-1})^2] + q\Phi.$$  \hspace{1cm} (1.3)

Evidently the Hamiltonian is of the form

$$H = \frac{1}{2}m^{-1}p^2 + [-cq'p_\phi AR^{-1} + q(\Phi + \frac{1}{2}q'A^2)].$$

where $q' = q/mc^2$ and $p^2 = p_R^2 + p_\phi^2 + p_z^2 R^{-2}$. Now $p_\phi$ is constant so the whole term in square brackets may be regarded as a new potential $q\Phi^*$. However, the separable potentials for the motion of a particle are already known to be (in axial symmetry)

$$\Phi^* = [\xi^*(\lambda) - \eta^*(\mu)]/\lambda - \mu$$  \hspace{1cm} (1.4)

(see e.g. Morse & Feshbach 1953), and the integrals (constants) of the motion in this potential are $H$, $p_\phi$, and $I$ where

$$I = \frac{1}{2m}[\frac{\mu p^2}{\mu^2} + \frac{\lambda p_\phi^2}{\lambda^2} + \frac{p_\phi^2}{R^2}(\lambda + \mu)] + \frac{q \mu \xi^*(\lambda) + \lambda \eta^*(\mu)}{\lambda - \mu}.$$  \hspace{1cm} (1.5)

[see e.g. Lynden-Bell (1962) for a proof and De Zeeuw (1985)] for a classification of the orbits, $p_\lambda = \partial L/\partial \lambda$, $p_\mu = \partial L/\partial \mu$ and the metric is $\text{d} s^2 = \text{d} t^2 + \text{d} x^2 + \text{d} y^2 + \text{d} z^2 = \text{d} R^2 + \text{d} x^2 + \text{d} y^2$. Also $\xi^*$ and $\eta^*$ are arbitrary functions, and $\lambda$ and $\mu$ are sphericoidal coordinates which may be prolate or oblate and may degenerate into paraboloidal, cylindrical polar or spherical polar coordinates. Thus the problem appears to have been reduced to a known one. However, things are not quite so simple. $p_\phi$ is constant for each trajectory, but it will vary from one to another. In true separation the motion must separate for all orbits, so we shall need the potential $\Phi^*$ to be separable not just for some value of $p_\phi$ but for every $p_\phi$. If it separates only for special values of $p_\phi$ then the separation constant is not an integral of the motion but only a configuration invariant (Whittaker 1904; Hall 1985). There is a second problem of this type. Even if $\Phi^*$ separates for all values of $p_\phi$, it also depends on the charge/mass ratio $q'$, so electromagnetic potentials that give separable motion for electrons will not generally give separable motions for positrons, protons or helium nuclei. However, there may be certain potentials that are separable for all values of the charge-to-mass ratio; these we call super-separable.

If the square bracket in equation (1.3) is to take the form (1.4) for all values of $p_\phi$ then both $AR^{-1}$ and $\Phi_1 = \Phi + \frac{1}{2}q'A^2$ must be potentials of that separable form in the same coordinates. If both are to be of that form for all charge/mass ratios $q'$ then $AR^{-1}$, $\Phi$ and $A^2$ must be of separable form in those coordinates. Coriolis force acts on moving masses like magnetism acts on moving charges. In general relativity we meet position-dependent Coriolis forces that are even more like magnetism in that the lines of force of the field are curved and correspond to a field of zero divergence. These gravomagnetic fields (often called dragging of inertial frames) follow directly from the metric, which for stationary cases may be written

$$\text{d} s^2 = g_{00}(\text{d} t - A_\phi \text{d} w) - \gamma_{ab} \text{d} x^a \text{d} x^b,$$

where $a, b$ run over the range 1, 2, 3. The three-vector potential $A_\phi$ plays the same role in gravity as the three-vector $\mathbf{A}$ plays in electromagnetic theory, but there is one important difference. The principle of equivalence ensures that all masses couple with the same coefficient and fall the same way. Thus for gravomagnetic problems $q'$ becomes $1/c^2$ for all particles and we do not have to consider super-separability.

To reiterate, the main theorem proved above is:

The necessary and sufficient condition for separability of the motion in an axially symmetric and poloidal electromagnetic field for some value of the charge/mass ratio is that both $AR$ and $\Phi + \frac{1}{2}q'm^{-1}c^{-2}A^2$ should be of separable form in the same coordinates.

There is a similar but perhaps less interesting theorem for electromagnetic fields that are independent of $z$ with no $B_z$ component. Then $\mathbf{A}$ may be taken to be of the form $\mathbf{A} = A_\phi(x, y)\hat{\phi}$. The conserved momentum is $p_z = m\dot{z} + qAc^{-1}$, and the conditions for separable motion are that $A_\phi$ is a separable potential in some coordinates $\Gamma$ and that $\Phi(x, y) + \frac{1}{2}q'A^2$ must be of separable form in the same coordinates $\Gamma$. Super-separability would require $A_\phi$, $A^2$ and $\Phi$ all to be of separable form in $\Gamma$.

So far we have considered non-relativistic motion, but the relativistic motion of charged particles is commonly encountered, so we now...
consider it. The Lagrangian is
\[ L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + q\mathbf{c} \cdot \mathbf{A} - q\Phi, \]
from which one derives the Hamiltonian
\[ H = \mathbf{p} \cdot \mathbf{v} - L = \sqrt{(pc - qA)^2 + m^2c^4} + q\Phi. \]

The Hamilton–Jacobi equation is therefore
\[ (c\nabla S - qA)^2 + m^2c^4)^{1/2} + q\Phi = E \]
from which one finds, after squaring and putting \( \partial S / \partial \phi = p_\phi = \text{constant}, \)
\[ c^2(\nabla S)^2 - 2cqp_\phi AR^{-1} + q^2(A^2 - \Phi^2) + 2q\Phi E + m^2c^4 - E^2 = 0. \]

Since \( p_\phi \) and \( E \) are constants, the terms apart from the first may all be regarded as a potential. In its \( \nabla S \) terms this is like the classical Hamilton–Jacobi equation that we get by writing \( \nabla S \) for \( \mathbf{p} \) in (1.3), which on multiplication by \( 2mc \) takes the form \( c^2(\nabla S)^2 - 2cqp_\phi AR^{-1} + q^2A^2 + 2mc^2q\Phi - 2mcE = 0 \) in which we have regarded all terms other than the first as a potential.

Since \( E \) can take many values, we see from (1.9) that \( \Phi \) and \( \Phi^2 - A^2 \) must be potentials of separable form so we need \( A/R \) of separable form too. Notice that we can reduce this without demanding separability for all values of the charge/mass ratio; nevertheless super-separability is then fulfilled, so the distinction between separability and super-separability disappears in this relativistic case, albeit the conditions for relativistic separability are more stringent. Remarkably, the conditions for exact separability in the relativistic case do not always yield exact separability when applied non-relativistically to the classical mechanics of the motion. Relativistic separability requires \( A/R, \Phi \) and \( \Phi^2 - A^2 \) to be of separable form, while classical separability requires \( A/R \) and \( \Phi + \frac{1}{2}qA^2 \) to be. To see the classical limit of the relativistic case, expand (1.8) treating \( (mc^2)^2 \gg (c\nabla S - qA)^2 \), keeping the first two terms \( mc^2 + \frac{1}{2}m^{-1}(c\nabla S - qA)^2 = q\Phi = E \); however, our neglect of the other terms is only justified provided that the variation of \( q\Phi \) along each orbit is small compared with \( mc^2 \) so that the square of its variation may be neglected. If we neglect first and then require separability we get the classical criterion. If we do not neglect and require separability we get the relativistic criterion. The classical conditions for super-separability can be deduced from the relativistic ones, but are inexact by terms in the potential that we can neglect when the motions remain \( \ll c \). As always, the relativistic criterion is the true one physically, but each criterion is true mathematically for its own mechanics. The order of asking for exact super-separability and taking the classical limit matters because we ask for exact separability.

The third integral in the relativistic case is readily seen by rewriting (1.9) to look like (1.3):
\[ \frac{p^2}{2} + \frac{q}{c} E\Phi = \frac{q}{2c^2} (\Phi^2 - A^2) - \frac{q}{c} AR^{-1} p_\phi = \frac{1}{2}(E^2c^{-2} - m^2c^4), \]
so
\[ I = \frac{1}{2} \left[ \frac{\mu p_\lambda^2}{F^2} + \frac{\Lambda_\mu^2}{Q^2} + \frac{p_\phi^2}{R^2} (\lambda + \mu) \right] + \frac{q}{c} \xi_3 - \frac{\Lambda_\mu_3}{\lambda - \mu}, \]
where
\[ \xi_3 = \frac{E}{c^2} \xi_1 - \frac{q}{2c^2} \xi_2 - \frac{p_\phi}{c} \xi_3 \] and \( \eta_3 = \frac{E}{c^2} \eta_1 - \frac{q}{2c^2} \eta_2 - \frac{p_\phi}{c} \eta_3. \)

In quantum mechanics the Hamiltonian operator is generated from the classical Hamiltonian by the substitution \( \mathbf{p} \rightarrow -i\hbar \nabla \). Thus, from (1.3),
\[ H = -\frac{1}{2m} \hbar^2 \nabla^2 + \left[ i\hbar cq'AR^{-1} \frac{\partial}{\partial \phi} + q \left( \Phi + \frac{1}{2}qA^2 \right) \right]. \]

In all the ellipsoidal sets of coordinates \( \nabla^2 \) separates, while in axial symmetry the \( \partial / \partial \phi \) merely brings the azimuthal quantum number as a factor. Thus there is no essential difference between the classical cases considered above and the separation of the Schrödinger equation. Both separate for the same potentials and fields, and super-separation occurs under the same classical conditions. We now turn to the relativistic case of the Klein–Gordon equation first derived by Schrödinger. With a magnetic field this becomes, with \( \Box \) the four-dimensional \( \nabla \),
\[ (-i\hbar \Box - q\mathbf{A}/c^2) \psi = -m^2c^2 \psi, \]
where \( \mathbf{A} \) is the four-vector (\( \mathbf{A}, \Phi \)).

Taking \( \Box \cdot \mathbf{A} = 0 \), and \( \psi \propto \exp(-iEt/\hbar) \), we get, cf. (1.9),
\[ \nabla^2 \psi - 2i(hc)^{-1} \mathbf{A} \cdot \nabla \psi = \hbar^{-2}c^{-2}[m^2c^4 + q^2(A^2 - \Phi^2) - E^2 - 2q\Phi E] \psi. \]
This separates under the conditions derived earlier for relativistic separability.

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2 RELATIVISTIC SEPARABLE SYSTEMS IN SPHEROIDAL COORDINATES

Perhaps the most interesting system we shall find is that of the Kerr electromagnetic field in which the motion separates in spheroidal coordinates. Since the other coordinate systems that allow separation are degenerate forms of those, they are the general case from which all others may be deduced by suitable limiting processes. We start with them.

Since different notations are in common use for spheroidal coordinates and parabolic coordinates, and indeed we use two different ones here, we now spell out what we mean by $\lambda, \mu$, etc. $\lambda$ and $\mu$ are the roots for $\tau$ of the quadratic equation

$$\frac{R^2}{\tau^2} + \frac{z^2}{\tau + \beta} = 1 \quad \text{with} \quad \lambda \geq \mu; \quad (2.1)$$

so $\lambda$ = constant gives a spheroid, while $\mu$ = constant gives a hyperboloid of two sheets or one sheet dependent on whether $\beta > 0$ (prolate) or $\beta < 0$ (oblate). The quadratic reads $\tau^2 - \pi(R^2 + z^2 - \beta) - \beta R^2 = 0$, so

$$\lambda + \mu = R^2 + z^2 - \beta \quad (2.2)$$

and

$$\lambda \mu = -\beta R^2. \quad (2.3)$$

This last equation shows that $\lambda$ and $\mu$ are of opposite signs in the prolate case and of the same sign in the oblate case. Indeed, $\lambda \geq 0 \geq \mu \geq -\beta$ for $\beta = b^2 > 0$ prolate spheroidal and $\lambda \geq -\beta \geq \mu \geq 0$ for $\beta = -a^2 < 0$ oblate spheroidal.

In the prolate case the spheroids $\lambda$ = constant are confocal with foci at $R = 0, z = \pm \sqrt{\beta} = \pm b$. If $r_1$ and $r_2$ are the distances from these foci then

$$\lambda = \left(\frac{r_1 + r_2}{2}\right)^2 - \beta \quad \text{and} \quad \mu = \left(\frac{r_1 - r_2}{2}\right)^2 - \beta. \quad (2.4)$$

In the oblate case there is a focal ring at $z = 0, R = a = \sqrt{-\beta}$ and, if $r_1$ and $r_2$ are the greatest and least distances of a point from this focal ring, then

$$\lambda = \left(\frac{r_1 + r_2}{2}\right)^2 \quad \text{and} \quad \mu = \left(\frac{r_1 - r_2}{2}\right)^2. \quad (2.5)$$

Notice that in both cases

$$\lambda - \mu = r_1 r_2. \quad (2.6)$$

To cover both these cases generally, we write the separable form of the potential as $\Phi = [\xi(\lambda) - \eta(\mu)]/(\lambda - \mu)$, but in each case $\lambda$ is a function of $r_1 + r_2$ and $\mu$ a function of $r_1 - r_2$, so by redefining $\xi, \eta$ we could write equally well

$$\Phi = [\xi^*(r_1 + r_2) - \eta^*(r_1 - r_2)]/(r_1 r_2); \quad (2.7)$$

actually in the prolate case this form is slightly more general since, when $r_1 - r_2$ is negative, on the lower sheet of the hyperboloid of constant $r_1 - r_2$ it allows $\Phi$ to have a different value from that on the upper sheet of the same $\mu$. Such a difference is exploited in two-centre problems when the two charges are not equal.

From (2.2) and (2.3) one deduces that the metric takes the form

$$ds^2 = dR^2 + R^2 d\phi^2 + dz^2 = \frac{\lambda - \mu}{4\lambda(\lambda + \beta)} d\lambda^2 + \frac{\lambda \mu}{(-\beta)} d\phi^2 + \frac{\mu - \lambda}{4\mu(\mu + \beta)} d\mu^2. \quad (2.8)$$

from which one may write down $\nabla^2 \Phi$ in these coordinates. In the appendix we write the expression for $\nabla^2 \Phi$.

Parabolic coordinates may be treated formally as a special case of spheroidal coordinates. Indeed, if we write $\lambda = \frac{1}{2}(z + r)$ and $\mu = \frac{1}{2}(z - r)$ then $\lambda \mu = -\frac{1}{4}R^2$, which agrees with our spheroidal formula with the formal value $\beta = \frac{1}{4}$. Furthermore, $\lambda - \mu = r$ and $\lambda + \mu = z$ and the separable potentials in parabolic coordinates are of the standard spheroidal form. The alternative form corresponding to (2.7) is

$$\Phi = [\xi^*(r + z) - \eta^*(r - z)]/r. \quad (2.9)$$

The same formulae hold as for spheroidal coordinates, but the metric is now changed to

$$ds^2 = (\lambda - \mu)(\lambda^{-1} d\lambda^2 - \mu^{-1} d\mu^2) - 4\lambda \mu d\phi^2 \quad (\mu \geq 0). \quad (2.10)$$

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We now return to general spheroidal coordinates. In Section 1 under (1.9) we showed that \( A/R, \Phi \) and \( \Phi^2 - \Lambda^2 \) must all take the separable form. We may therefore write
\[
\Phi = \left[ \zeta_1(\lambda) - \eta_1(\mu) \right]/(\lambda - \mu),
\]
\[
\Lambda = \sqrt{-\lambda \mu/\beta}[\zeta(\lambda) - \eta(\mu)]/(\lambda - \mu).
\]
(2.11)
(2.12)

We deduce that
\[
\Phi^2 - \Lambda^2 = [\lambda \mu (\zeta - \eta)^2/\beta + (\zeta_1 - \eta_1)^2](\lambda - \mu)^{-2},
\]
but separability demands that this last expression must be of the form \((\zeta_2 - \eta_2)/(\lambda - \mu)\) so this must be imposed.

Now
\[
\frac{\partial^4}{\partial \lambda^2 \partial \mu^2} (\lambda - \mu)[\zeta_2(\lambda) - \eta_2(\mu)] = 0
\]
for any \( \zeta_2 \) and \( \eta_2 \); so we deduce from (2.13) that we must have
\[
\frac{\partial^4}{\partial \lambda^2 \partial \mu^2} [\lambda \mu (\zeta - \eta)^2 + \beta (\zeta_1 - \eta_1)^2] = 0.
\]
(2.14)

However, the differential operator annihilates \( \mu(\lambda \zeta^2), \lambda(\mu \eta^2), \zeta^2 \) and \( \eta^2 \), so on division by \(-2\) we find
\[
(\lambda \zeta'')(\mu \eta^2) = -\beta \zeta'' \eta^2.
\]
(2.15)

The rest of this section deduces the results of Table 1 (see later) from this equation, each side of which is a product of functions of \( \lambda \) and of \( \mu \).

From (2.15),
either \((\lambda \zeta)' = -C \beta \zeta_1'' \) and \( \eta''_1 = C(\mu \eta)'\),
or \((\mu \eta)' = 0 \) and either (i) \( \eta''_1 \) or (ii) \( \zeta''_1 \) are zero too,
or \((\lambda \zeta)'' = 0 \) and either (i) \( \eta''_1 \) or (ii) \( \zeta''_1 \) are zero too.
(2.16)
(2.17)
(2.18)

Taking the general case (2.16) we find
\[
\lambda \zeta = -C \beta \zeta_1, \quad \eta_1 = C \mu \eta,
\]
(2.19)
where we have intentionally omitted linear and constant terms, since these can be absorbed by redefinition of \( \zeta, \eta, \zeta_1 \) and \( \eta_1 \) without changing \( V(A/R) \) or \( V \Phi \). Returning to \((\lambda - \mu)(\Phi^2 - \Lambda^2), (2.13), \) which must be of the form \( \zeta_2 - \eta_2 \), we find, on multiplication by \((\lambda - \mu), \mu(\beta C^2 \zeta_1'' \lambda^{-1} + \zeta_2) + \lambda(\beta^{-1} \eta_2'' \mu + \eta_2) = (\lambda \zeta_2 - \eta_2) + (\mu \eta_2 - C \mu^2 \zeta_2)\),
where the brackets are functions of \( \lambda \) and \( \mu \) respectively. On the right the function of \( \lambda \) is unadulterated by any \( \mu \) dependence, while on the left it is linear in \( \mu \) so the mixed dependences on the left must cancel, so
\[
\beta C^2 \zeta_1'' \lambda^{-1} + \zeta_2 = c_1 \lambda + c_2
\]
and
\[
\beta^{-1} \eta_2'' \mu + \eta_2 = -c_1 \mu + c_3,
\]
where the \( c_i \) are constant. Looking at what remains,
\[
\lambda \zeta_2 - \eta_2 = c_3 \lambda + c_4\]
and
\[
\mu \eta_2 - C \mu^2 \eta_2 = c_2 \mu - c_4,
\]
eliminating \( \zeta_2 \) and \( \eta_2 \) we have
\[
(1 + \beta C^2) \zeta_1'' = c_1 \lambda^2 + (c_2 - c_3) \lambda - c_4, \quad (c_2 - c_3) \lambda - c_4, \quad \beta^{-1}(1 + \beta C^2) \eta'' \mu^2 = -[c_1 \mu^2 + (c_2 - c_3) \mu - c_4].
\]

These expressions together with (2.19) give the ‘general’ solution, but to simplify it we do some renaming with \( c_2 = -\beta C^2 c_3, a_2 = c_1/(1 + \beta C^2), a_1 = -c_3, a_0 = -c_4 a_2/c_1; \)
\[
\zeta_1'' = a_2 \lambda^2 + a_1 \lambda + a_0 = s_1^2, \quad (-\beta)^{-1} \mu^2 \eta'' = a_2 \mu^2 + a_1 \mu + a_0 = s_2^2. \]

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\[ \xi = -C\beta \xi /\lambda, \]  
\( \eta_i = C\mu \eta, \)  
\( A = \left( \frac{\lambda \mu}{-\beta} \right)^{1/2} \frac{\xi - \eta}{\lambda - \mu} = \frac{(-\beta \lambda \mu)^{1/2}}{\lambda - \mu} \left( \frac{Cs\lambda}{\lambda} - s\mu \right), \)

\[ \Phi = (s\lambda + BCs\mu)/(\lambda - \mu), \]

\[ \Phi^2 - A^2 = (\xi - \eta)/(\lambda - \mu) = [(a_2\lambda - \beta C^2 a_0 \lambda^{-1}) - (-\beta C^2 a_2 \mu + a_0 \mu^{-1})]/(\lambda - \mu). \]

\( (s\mu \) can be imaginary but in the acceptable range of \( \mu \) we need \(-\beta C^2 \geq 0. s\lambda \) is real.)

At large \( \lambda, \Phi \to a_2^\lambda(-^1/2 + \beta C \mu^{-1/2} + \beta C^2 \mu^{-1}) \) and, as \( \lambda \) is of order \( r^2 \), this demonstrates that there can be no net charge unless \( a_2 = 0. \)

When that is true \( \Phi \to a_2^\lambda(-^1/2) \) so the net charge is \( a_2^{-1/2} \) and \( A \) is of order \( 1/r \), giving magnetic fields that behave as \( r^{-2} \) at large \( r \). For most applications we would want a dipolar \((\propto r^{-3})\) behaviour of the magnetic field at large \( r \), but we cannot get that from the above. We therefore turn to the special cases (2.17) and (2.18).

Taking (2.17) \( \eta = c_1 + c_2/\lambda, \) but if we write \( \xi = \xi + c_1 + c_2/\lambda, \) \( \eta = 0 \) then \( \nabla(A\Phi) \) deduced from \( \xi \) and \( \eta \) is the same as that deduced from \( \xi \) and \( \eta \), so without loss of generality we can take \( \eta = 0. \) Similarly, if \( \eta'' = 0 \) cf. (2.17) we can without loss of generality take \( \eta_1 = 0. \)

These give us

\[ A = \left( \frac{\lambda \mu}{-\beta} \right)^{1/2} \frac{\xi/\lambda}{\lambda - \mu}, \]

\[ \Phi = \xi/(\lambda - \mu). \]

We then find that without loss of generality \( \eta_2 = 0 \) and \( \Phi^2 - A^2 \) is of the separable form \( \xi^2/(\lambda - \mu) \) only if

\[ \xi_1 = \lambda \xi_2 = \lambda^2 \xi^2/(-\beta), \]

which gives us the solution for arbitrary \( \xi(\lambda) \):

\[ A = \pm (\mu/\lambda)^{1/2} \xi_1/(\lambda - \mu), \quad \Phi = \xi_1/(\lambda - \mu), \]

\[ \Phi^2 - A^2 = \xi_1^2/(\lambda - \mu). \]  

(2.21)

If in this solution we take \( \xi_1 \to q\lambda^{1/2} \) as \( \lambda \to \infty \) then the net charge will be \( q \), and furthermore \( A \) will be of order \( r^{-2} \). Particular interest attaches to those fields that come from simple sources, and we show in the appendix that when \( \xi_1 = q(\lambda + \beta^{1/2}) \) then \( \Phi \) satisfies \( \nabla^2 \Phi = 0 \) everywhere except on the disc \( R = a, z = 0, \) where \( \beta = -a^2. \) To get a separable solution we need the solution (2.21) for \( A. \) This implies

\[ AR = \pm \frac{\mu}{(\lambda - \mu)^{1/2} (-\beta)^{1/2}}. \]  

(2.22)

Calculating \( \nabla(A\Phi) \times \nabla \Phi = \text{curl} A = \mathbf{B}, \) we find after some work that

\[ \text{curl} A = -\nabla \chi, \quad \text{where} \quad \chi = \pm q\sqrt{(-\mu + \beta)/(\lambda - \mu)}. \]

(2.23)

This of course implies that \( \text{curl} \mathbf{B} = 0. \) Thus the sources for the magnetic field also lie on the disc singularity of the coordinates \( R = a, z = 0. \)

This solution is indeed the classical version of Kerr’s metric. It may be obtained from the charged Kerr–Newman solution by keeping the angular momentum per unit mass, \( a, \) fixed and the charge fixed, but letting the gravitational constant \( G \) tend to zero so the whole metric becomes flat space but the electromagnetic field remains. Indeed, the fields may be written

\[ E + i\mathbf{B} = -\nabla \Phi + i\chi \]

\[ = -\nabla \left[ q\sqrt{(\lambda + \beta \pm i\sqrt{(-\mu + \beta)})/(\lambda - \mu)} \right] \]

\[ = -\nabla \left[ q\sqrt{\lambda + \beta \pm i\sqrt{(-\mu + \beta)}} \right]/(\lambda - \mu), \]

\[ E + i\mathbf{B} = -\nabla \left[ q\sqrt{\lambda + \beta \pm i\sqrt{(-\mu + \beta)}} \right]/(\lambda - \mu), \]

where we have written \( \beta = -a^2 \) and \( a = a(0, 0, 1). \)

The final expression gives a beautifully simple way of deriving the field as follows: if \( E = -\nabla \Phi \) and \( \Phi = q/r \) then \( \nabla^2 (q/r) = 0, \) \( r \neq 0, \) hence \( \nabla[q/\sqrt{(r-b)^2}] = 0, r \neq b. \)

Now put \( ia = b = (0, 0, i\alpha) \) then, if \( \Psi = q/\sqrt{(r-ia)^2} \), \( E + i\mathbf{B} = -\nabla \Psi \) will obey \( \text{div}(E + i\mathbf{B}) = 0 \) away from singular points and cuts. It is easy to see that an appropriate cut to make the complex function analytic is on the disc \( r = a, z = 0, \) and the singular points lie on the ring that bounds it. We shall explore these fields and the orbits in them elsewhere, but we now return to the separability problem (2.17).
We have not yet discussed the possibility \((\mu \eta)'' = 0\), and \(\xi_1 = 0\), which lead without loss of generality to \(\eta = 0\) and \(\xi_1 = 0\).

\[
(\lambda - \mu)^2(\Phi^2 - A^2) = \eta_1^2 + \lambda \mu \xi_2 / \beta = \lambda \xi_2 + \mu \eta_2 - \mu \xi_2 - \lambda \eta_2,
\]

which only leads to

\[
\lambda^2 \xi_2 = -\beta (c_1 \lambda^2 + (c_2 - c_3) \lambda),
\]

\[
\eta_1^2 = c_1 \mu^2 + (c_2 - c_3) \mu,
\]

which can be formally included as a possibility in our ‘general’ case by taking \(C \to \infty\) and \(\xi_1\) and \(\eta \to 0\). Turning to the possibility of (2.18), we see that these cases are equivalent to (2.17) under the transformation

\[
\lambda \to \mu, \quad \xi \to \eta, \quad \xi_1 \to \eta_1.
\]

This completes the investigation in spheroidal coordinates, with the results recorded in Table 1 (later). In prolate spheroidal coordinates the reality conditions on the square roots are very restrictive and cannot be met at the foci. Parabolic coordinates, being a limiting case of prolate spheroidals, suffer from the same problem, so in practice there are no useful prolate spheroidal or parabolic cases unless only a very restricted patch of the coordinates is needed.

### 3 SYSTEMS THAT ARE RELATIVISTICALLY SEPARABLE IN SPHERICAL POLAR COORDINATES

In these coordinates the separable potentials take the form

\[
\xi(r) - \eta(\theta) r^{-2}.
\]

To see this we remember that in Section 1 we saw how to deduce the integrals for the relativistic magnetic case from the non-relativistic non-magnetic case, so we need only consider the former, writing \(h = r \times v, \; \dot{r} = r / r\):

\[
dh / dt = q m^{-1} r^{-2} r \times \nabla \eta.
\]

hence

\[
mh \cdot dh / dt = q(\dot{r} \times v) \cdot \nabla \eta = q v \cdot \nabla \eta = q d \eta / dt.
\]

So

\[
\frac{\xi}{\eta} m |h|^{-2} - q \eta(\theta) = I = \text{constant}.
\]

So \(E, \; h_{\phi}\) and \(I\) are the classical integrals of the motion, and as their mutual Poisson brackets vanish the system is separable and integrable by the theorem of Liouville (1855) [see also Whittaker (1904) or Lynden-Bell (1962)].

For relativistic separability we need

\[
A/R = \xi(r) - \eta(\theta) r^{-2},
\]

\[
\Phi = \xi(\theta) - \eta(\theta) r^{-2},
\]

and \(\Phi^2 - A^2\) to be of that same form. Thus \(r^2(\Phi^2 - A^2)\) must be the difference of a function of \(r\) and a function of \(\theta\) so the operator \(\partial^2 / \partial \theta^2\) must annihilate it. However,

\[
r^2(\Phi^2 - A^2) = -r^2 \xi_2 sin^2 \theta + 2r \xi_1 \eta sin^2 \theta - \eta^2 sin^2 \theta + r^2 \xi_1^2 - 2 \xi_1 \eta_1 + r^2 \eta_1^2,
\]

so we deduce that

\[
-(r^2 \xi_1')(sin^2 \theta)' + 2(\xi_1')(\eta sin^2 \theta)' - 2 \xi_1 \eta_1' - 2r^3(\eta_1)^2 = 0.
\]

From here on the analysis is somewhat similar to that carried out for the spherical case in Section 2. For those few who wish to check that we have covered every case, we give the analysis in the appendix. The results are tabulated in Table 2 (later).

Probably the most interesting of these electromagnetic fields are in the generalized monopole class in which

\[
AR^{-1} = -\eta(\theta) r^{-2} \text{ with } \eta \text{ arbitrary},
\]

\[
\Phi = \xi(\theta) \text{ with } \xi_1 \text{ arbitrary}.
\]

\[
\Phi^2 - A^2 = \xi_1^2 - \xi_1^2 \sin^2 \theta r^{-2}, \text{ and from (3.6) the magnetic field is given by}
\]

\[
B = Q(\theta) \dot{\theta} r^{-2},
\]

where \(Q(\theta) = -(\sin \theta)^{-2} / \partial \theta (\sin^2 \theta)\). When \(Q\) is constant, (3.7) is the magnetic field of a monopole which gives its name to the whole

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class. This special case has particular interest as, when $\xi_1$ takes the form $q/r - \frac{1}{2} Q^2 q^{-1} r^{-2}$, all bound orbits exactly close, giving planar ellipses in planes that do not lie through the origin. For general $\xi(r)$ the orbits lie on cones and precess around them (see Lynden-Bell & Nouri-Zonooz 1998).

Some other fields of this class have occurred in science previously. The split monopole $Q(\theta) = Q \text{sgn}(\cos \theta)$ has been used to model pulsar winds, and occurs naturally when the upper half of a conducting sphere carrying a magnetic dipole in a perfectly conducting force-free medium is twisted by $\pi$ relative to its lower hemisphere. See Lynden-Bell & Boily (1994), who also consider the case $Q(\theta) = Q \text{sgn}(\sin \theta/3)$ which occurs naturally when quadruples are twisted. Separable motion in such fields was considered by Hautot (1973).

Again comparing (1.9) with (1.3), with $\Phi$ given by (3.1), we see that the third integral is

$$I = \frac{1}{2} (r \times p)^2 - q \eta_1(\theta),$$

where $\eta_1$ is defined as in (1.12) and $p$ is the Cartesian momentum $m(v(1 - v^2/c^2)^{-1/2}.

## 4 SYSTEMS THAT ARE RELATIVISTICALLY SEPARABLE IN CYLINDRICAL COORDINATES

The separable form of the potential is simply $\zeta(R) - \eta(z)$. We need $A/R, \Phi$ and $A^2 - A^2$ all of that form. Evidently

$$\Phi^2 - A^2 = \hat{\xi}_1^2 + \eta_1^2 - R^2 \hat{\xi}_1^2 - 2 \hat{\xi}_1 \eta_1 + 2 R^2 \xi \eta - R^2 \eta^2.$$  

(4.1)

Operating with $\partial^2/\partial R^2$ we get

$$\zeta' \eta' = (R^2 \hat{\xi})' \eta' - R(\eta')^2'$$

(4.2)

We show in the appendix that essentially only three cases survive: those with no $\eta$ or $\eta_1$ with both $\xi(R)$ and $\xi_1(R)$ arbitrary; those with $\hat{\xi}_1$ and $\eta$ zero with both $\xi(R)$ and $\eta_1(z)$ arbitrary; and those with $\xi$ zero, $\xi_1 \propto R^2$ and $\eta^2 \propto \eta_1(z)$. These are tabulated in the second half of Table 2 (later). In these cases the third integral takes the form

$$I = \frac{1}{2} p^2 - qc^{-2}(E \eta_1 - \frac{1}{2} q \eta_2 - c \eta_2, \eta).$$

(4.3)

## 5 CLASSICALLY SUPER-SEPARABLE ELECTROMAGNETIC FIELDS

In spherical polars we need $A/R, \Phi$ and $A^2$ of the separable form $\zeta(r) - \eta(\theta)r^{-2}$.

Evidently

$$r^2 A^2 = r^4 \xi^2 \sin^2 \theta - 2r^2 \xi \sin \theta \eta + \eta^2 \sin^2 \theta$$

(5.1)

must be the difference of a function of $r$ and one of $\theta$, so $(r^4 \xi^2)'(\sin^2 \theta)' = 2(r^2 \xi)'(\sin^2 \theta)'$, momentarily taking $\theta$ to be constant. $(r^4 \xi^2)' = 2C(r^2 \xi)'$ which leads to $\xi \propto 1/r^2$. That can be absorbed, however, by changing $\eta$, so without loss of generality $\xi = 0$. It follows that $\eta(\theta)$ is arbitrary and we may combine it with any $\Phi$ of the form $\zeta_1(r) - \eta_1(\theta)r^{-2}$ with $\zeta_1$ and $\eta_1$ arbitrary. The integral is

$$I = \frac{1}{2} (r \times p)^2 - q \eta_1,$$

In cylindrical polars $\Lambda R^{-1} = \zeta(R) - \eta(z)$,

$A^2 = R^2 \xi^2 - 2R^2 \xi \eta + \eta^2$

has to be a difference function. Without loss of generality we then find $\eta = 0$ and $\zeta(R)$ can be arbitrary. Thus

$$B = B(R)\hat{\xi},$$

where $B(R) = (R^2 \xi')/R$,

so we have a straight magnetic field dependent on $R$. We can combine this with $\zeta_1$ and $\eta_1$ arbitrary in the electrical potential $\Phi = \zeta_1(R) - \eta_1(z)$. Among these magnetic fields is the important case $\xi = \frac{1}{2} B_0 = \text{constant}$, which gives a uniform field $B_0$ along the $z$-axis.

### 5.1 Others

There are no classically super-separable cases with non-zero magnetic fields in spheroidal or parabolic coordinates. One shows this by writing

$$A = \sqrt{\frac{\lambda \mu}{\beta} \frac{\zeta(\lambda) - \eta(\mu)}}$$

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and imposing the demand that
\[
\frac{\partial^2}{\partial \alpha \partial \mu} \left[ (\lambda - \mu) \lambda^2 \right] = 0 = \frac{\partial^2}{\partial \lambda^{-1} \partial \mu^{-1}} \left[ \frac{q}{\beta} \left( \frac{\xi - \eta}{\lambda^{-1} - \mu^{-1}} \right) \right].
\]
We do not give further details.

6 CLASSICALLY SEPARABLE ELECTROMAGNETIC POTENTIALS

The only requirements are that
\[
AR^{-1} \text{ and } \Phi_1 = \Phi + \frac{q}{2m} A^2 c^{-2}
\]
should both be of separable form \([\xi(\lambda) - \eta(\mu)]/(\lambda - \mu)\). For a single charge/mass ratio this is easy to arrange. We choose \(\xi\) and \(\eta\) arbitrarily. Then we know \(A\). We then take \(\Phi\) of the form
\[
(\xi_1 - \eta_1)/(\lambda - \mu) - \frac{q}{2m} A^2 c^{-2},
\]
where \(\xi_1\) and \(\eta_1(\mu)\) are arbitrary. This clearly gives \(\Phi_1\) of separable form.

As a simple example illustrating how this is done, consider the uniform field \(A = \frac{1}{4}B_0 R\). Then the motion in the potential
\[
(\xi_1 - \eta_1)(\lambda - \mu) - \frac{q}{8m} B_0^2 R^2 c^{-2}
\]
will be separable. This may seem an odd result until one remembers Larmor’s theorem. Larmor showed that, to first order in \(B_0\), the effect of a uniform magnetic field could be eliminated by looking at the problem in axes that rotate at the rate \(\Omega = qB_0/(2mc)\), although the cancellation is not exact because the \(O(B_0^2)\) centrifugal force is not cancelled. However, the extra electrical potential that we have included in (6.1) is precisely that which is needed to cancel the effect of the centrifugal force in Larmor’s rotating axes. Thus by going to those axes we shall now recover exactly the motion we would have had without Coriolis force, without magnetic field, without the extra bit in the potential and without centrifugal force. Thus if the potential is doctored by this extra piece, Larmor’s theorem becomes exact.

7 TABLES OF CARTER SEPARABLE ELECTROMAGNETIC FIELDS

Whereas former sections explain the theory behind the separability, the aim of this section is to provide the reader with a quick and accessible list of the fields in which the motion of a charged particle separates. If a reader finds a listed electromagnetic field of interest, our aim here is to provide an outline of how the classical, relativistic or quantum motions of the particle may be found. So that the relevant formulae are easily found beside the tables a number of them are recalled in the table notes and below.

Table I gives the axially symmetrical electromagnetic fields that lie in planes through the axis of symmetry (with \(B \neq 0\)) for which both the relativistic Hamilton–Jacobi equation (1.9) and the Klein–Gordon equation (1.13) separate in spheroidal coordinates (\(\lambda, \mu, \phi\)). If \(R^2 = x^2 + y^2\) these coordinates are defined as the roots \(\lambda \geq \mu \geq \lambda \geq \mu \geq 0\) of the equation
\[
\frac{R^2}{\tau} + \frac{\tau}{\tau + \beta} = 1,
\]
that is,
\[
\tau^2 - \pi R^2 + \xi^2 - \beta - \beta R^2 = 0,
\]
where \(\beta\) is a constant that gives the semi-distance between foci of the spheroids. Writing \(r^2 = x^2 + y^2 + z^2 = R^2 + \xi^2\), we see from the quadratic that \(\lambda + \mu = \tau^2 + \beta\) and \(\lambda \mu = -\beta R^2\), which imply \(\xi^2 = \beta^{-1}(\lambda + \mu)(\mu + \beta)\). The metric is given by \(ds^2 = dR^2 + dx^2 + dy^2 + dR^2 + dz^2\). The metric is given by \(ds^2 = dR^2 + dx^2 + dy^2 + dz^2\), where
\[
p^2 = \frac{\lambda - \mu}{4\lambda(\lambda + \beta)}, \quad \sigma^2 = \frac{\mu - \lambda}{4\mu(\mu + \beta)}.
\]
For prolate spheroids, \(\beta = b^2 > 0\), \(\lambda \geq 0 \geq \mu \geq -b^2\). For oblate spheroids, \(\beta = -a^2 < 0\), \(\lambda \geq a^2 \geq \mu \geq 0\).

In these coordinates the Hamilton–Jacobi equation (1.9) takes the form
\[
\frac{1}{2} \left[ p^2 - \frac{\left( \partial S_\lambda / \partial \lambda \right)^2}{\lambda} + Q^2 - \frac{\left( \partial S_\mu / \partial \mu \right)^2}{\lambda} + R^2 - \frac{\partial \phi}{\partial \phi} \right] + \frac{q}{\lambda - \mu} \left[ \xi(\lambda) - \eta(\mu) \beta_\mu - \mu + 1 \right] (m^2 c^4 - E^2) = 0,
\]
which separates on multiplication by \((\lambda - \mu)\), giving
\[
2\lambda(\lambda + \beta)(\partial S_\lambda / \partial \lambda)^2 + \beta p_\phi^2 + q\xi(\lambda) + \frac{1}{2}(m^2 c^4 + E^2)\lambda = 2\mu(\mu + \beta)(\partial S_\mu / \partial \mu)^2 + \beta p_\phi^2 + q\eta(\mu) + \frac{1}{2}(m^2 c^4 + E^2)\mu,
\]
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where we have written $S = S_3(\lambda) + S_\mu(\mu) + p_\phi \phi$. Since the left-hand side is a function of $\lambda$ only and the right-hand side a function of $\mu$, both must be constant and one derives expressions for $S_3$ and $S_\mu$ as quadratures.

A very similar separation occurs for the Klein–Gordon equation (1.13) in which $\psi$ separates in the product form $\psi = \psi_\lambda(\lambda)\psi_\mu(\mu) e^{in\phi}$ to give

$$\frac{4}{\beta} \sqrt{\lambda + \beta} \frac{d}{d\lambda} \left( \lambda \frac{d}{d\lambda} \psi_\lambda + \frac{m^2}{\lambda} \psi_\lambda + [\hbar^2 q \zeta_3(\lambda) - \hbar^2 (m^2 c^2 - E^2 c^{-2}) \lambda] \psi_\lambda \right)$$

$$= \frac{4}{\beta} \sqrt{-\mu - \beta} \frac{d}{d\mu} \left( \mu \frac{d}{d\mu} \psi_\mu + \frac{m^2}{\mu} \psi_\mu + [\hbar^2 q \eta_3 - \hbar^2 (m^2 c^2 - E^2 c^{-2}) \mu] \psi_\mu, \right)$$

so the solution reduces to ordinary differential equations.

Table 2 gives the potentials in which relativistic motion separates in spherical polar or cylindrical polar coordinates, assuming that the fields are axially symmetric and poloidal. Under those same restrictions, Table 3 lists the potentials in which classical non-relativistic motion separates exactly.

<table>
<thead>
<tr>
<th>Case</th>
<th>Potential functions</th>
<th>Notes</th>
</tr>
</thead>
</table>
| 1.1 ‘General’ see (2.16) | \[
\begin{aligned}
- \frac{C_\beta s_1}{\lambda} & \quad s_1 \\
(\beta)^{1/2} s_2/\mu & \quad \alpha_1 \lambda - \beta c^2 a_0 \lambda^{-1}
\end{aligned}
\] | If $\lambda = \mu$ is accessible then only oblate coordinates. $\xi_1(\lambda)$ arbitrary, oblate only, $\xi_1 = q_1(\lambda + \beta)^{1/2}$ for Kerr. |
| 1.2 see (2.17i) | \[
\begin{aligned}
(\beta)^{1/2} s_1(\lambda)/\lambda & \quad \xi_1(\lambda) \\
0 & \quad \xi_2(\lambda) = \xi_1(\lambda)^2/\lambda
\end{aligned}
\] | $\eta_1(\mu)$ arbitrary, oblate only, $\eta_1 = D(\beta - \mu)^{1/2}$ has $\nabla^2 \Phi = 0$. |
| 1.3 see (2.17ii) | \[
\begin{aligned}
(\beta)^{-1/2} \eta_1(\mu)/\mu & \quad \eta_1(\mu) \\
0 & \quad \eta_2(\mu) = \eta_1(\mu)^2/\mu
\end{aligned}
\] | $\eta_1$ and $\eta_2$ are listed in pairs of rows above. The $a_i$ are arbitrary constants. |

The formulae for Table 2 are as follows.

Definitions:

$$r^2 = x^2 + y^2 + z^2; \quad R^2 = x^2 + y^2; \quad ds^2 = dR^2 + R^2 d\phi^2 + dz^2;$$

$$\phi_\delta = R \phi; \quad p_\phi = \frac{\partial L}{\partial \phi} = (E - q \phi) R \phi_\delta + q A^{-1} = \text{constant};$$

$$E = -\nabla \Phi, \quad B = \nabla(AR) \times \nabla \phi;$$

$$\zeta_3 = e^{-2}(E \zeta_1 - \frac{1}{2} q \zeta_2 - cp_\phi \eta); \quad \eta_3 = e^{-2}(E \eta_1 - \frac{1}{2} q \eta_2 - cp_\phi \eta).$$

For spherical coordinates:

$$\Phi = \zeta_1(r) - \eta_1(\theta) r^{-2};$$

$$AR^{-1} = \zeta_1 - \eta_1(\frac{1}{r^2}) r^{-2};$$

$$\Phi^2 - A^2 = \zeta_2(r) - \eta_2(\theta) r^{-2};$$

$$H = E \sqrt{m^2 c^2 + c^2 (p_\theta^2 + r^{-2} p_\phi^2) + (p_\phi c R^{-1} - q \lambda)^2} + q \Phi;$$
\[ I = \frac{1}{2}(r \times p)^2 - q\eta_3(\theta). \]

For cylindrical coordinates:

\[ \Phi = \zeta_1(R) - \eta_1(z); \]
\[ AR^{-1} = \zeta(R) - \eta(z); \]
\[ \Phi^2 - A^2 = \zeta_2(R) - \eta_2(z). \]

\[ H = E = \sqrt{m^2c^4 + c^2(p_R^2 + p_z^2) + (p_\phi c R^{-1} - qA)^2 + q\Phi}; \]
\[ I = \frac{1}{2}p_z^2 - q\eta_3(z) \]

(see 4.3).

In both cases, \( H, I \) and \( p_\phi \) are constants of the motion. \( a_i, b_i \) and \( c_i \) are constants.

**Table 2.** Electromagnetic potentials giving relativistic motion separable in spherical/cylindrical coordinates.

<table>
<thead>
<tr>
<th>Case</th>
<th>Potential functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical</td>
<td></td>
</tr>
<tr>
<td>2.1 ( \pm r^{-2}\sqrt{a_2 + a_3r^{-2}} )</td>
<td>( C\zeta r^2 )</td>
</tr>
<tr>
<td></td>
<td>( C^2a_2 + b_2r^{-4} )</td>
</tr>
<tr>
<td></td>
<td>( C^{-1}\eta\sin^2 \theta )</td>
</tr>
<tr>
<td></td>
<td>( a_2\sin^2 \theta - (C^2b_2/\sin^2 \theta) )</td>
</tr>
<tr>
<td>(Notes: ( \zeta ) and ( \eta ) in column 1.)</td>
<td></td>
</tr>
<tr>
<td>2.2 ( \pm r^{-2}\sqrt{a_2 + a_3r^{-2}} )</td>
<td>( C\zeta r^2 )</td>
</tr>
<tr>
<td></td>
<td>( C^2\zeta r^4 = 2C\sqrt{b_2}\zeta + b_2r^{-4} )</td>
</tr>
<tr>
<td></td>
<td>( \pm \sqrt{b_2 + a_3\sin^2 \theta - \sqrt{2}/\sin^2 \theta} )</td>
</tr>
<tr>
<td></td>
<td>( \sin^2 \theta(\eta^2 + a_2) )</td>
</tr>
<tr>
<td>(Notes: ( b_2 \cong 0, \zeta ) and ( \eta ) in column 1.)</td>
<td></td>
</tr>
<tr>
<td>Generalized monopole</td>
<td></td>
</tr>
<tr>
<td>2.3 { 0 }</td>
<td>( \zeta_1(r) )</td>
</tr>
<tr>
<td>( \eta(\theta) )</td>
<td>( \zeta_1^2 )</td>
</tr>
<tr>
<td>( \eta^2\sin^2 \theta )</td>
<td>( \eta^2\sin^2 \theta )</td>
</tr>
<tr>
<td>(Notes: ( \zeta_1 ) arbitrary, ( \eta ) arbitrary.)</td>
<td></td>
</tr>
<tr>
<td>2.4 { \frac{-c_2r^4}{c^2} }</td>
<td>( \frac{1}{2}c_5^2r^{-4} + \alpha_7r^{-2} )</td>
</tr>
<tr>
<td>( \frac{c_5}{c^2}(a_1 + \frac{1}{2}c_5\sin^2 \theta) )</td>
<td>( \zeta_1 )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( \eta^2\sin^2 \theta )</td>
</tr>
<tr>
<td>(Notes: ( \zeta_1 ) in column 2, ( \eta ) in column 1.)</td>
<td></td>
</tr>
<tr>
<td>Cylindrical</td>
<td></td>
</tr>
<tr>
<td>2.5 { \zeta_1(R) }</td>
<td>( \zeta_1(R) )</td>
</tr>
<tr>
<td>( \zeta_1^2 - R^2\zeta^2 )</td>
<td>( \zeta_1^2 )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>(Notes: ( \zeta, \zeta_1 ) arbitrary.)</td>
<td></td>
</tr>
<tr>
<td>2.6 { \zeta_1(R) }</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( -R^2\zeta^2 )</td>
</tr>
<tr>
<td>(Notes: ( \zeta ) arbitrary, ( \eta_1 ) arbitrary.)</td>
<td></td>
</tr>
<tr>
<td>2.7 { \eta(z) }</td>
<td>( c_5^2R^2 )</td>
</tr>
<tr>
<td>( -\frac{1}{2}\eta^2/c_2 )</td>
<td>( -\frac{1}{4}\eta^2/c_2 )</td>
</tr>
<tr>
<td>(Notes: ( \eta ) arbitrary.)</td>
<td></td>
</tr>
</tbody>
</table>
There are no classically super-separable cases with magnetic fields in spheroidal or paraboloidal coordinates. As explained in Section 6, there are many potentials in which the classical motion separates exactly for one chosen value of the charge/mass ratio.

As stated in the Introduction, we have carried out the allied investigation for systems that are not axially symmetric but are independent of $z$ with $E_z = B_z = 0$. While this was done as thoroughly as the axially symmetric case, the analyses leading to Table 4 are somewhat easier, so we do not give the proofs that we have for all cases. The elliptic cylindrical coordinates $\lambda, \mu$ are defined by the roots for $\tau$ with $\lambda \geq \mu$ of

$$\frac{x^2}{\tau} + \frac{y^2}{\tau + \beta} = 1; \quad \tau^2 - \pi(x^2 + y^2 - \beta) - \beta x^2 = 0,$$

$$dx^2 = P^2 d\lambda^2 + Q^2 d\mu^2 + dz^2,$$

with $P$ and $Q$ the same functions of $\lambda$ and $\mu$ as for spheroidal coordinates. The integrals are $H, p_\xi$ and $I$, where $I$ is readily constructed by analogy with those discussed for axial symmetry. Likewise the separations are straightforward.

### Table 4. Electromagnetic potentials giving separable classical motion for all $q/m$.  

<table>
<thead>
<tr>
<th>Case</th>
<th>$\xi$</th>
<th>$\xi_1$</th>
<th>$\xi_2$</th>
<th>Functions in the potentials</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical polar coordinates</td>
<td>$\zeta (\theta)$</td>
<td>$\eta (\theta)$</td>
<td>0</td>
<td>$\Phi = \zeta (\theta) - \eta (\theta) r^{-1}$</td>
</tr>
<tr>
<td>3.1</td>
<td>$\zeta (\theta)$</td>
<td>$\eta (\theta)$</td>
<td>$-\eta^2 \sin^2 \theta$</td>
<td>$A = -\eta (\theta) \sin \theta r^{-1}$</td>
</tr>
<tr>
<td>3.2</td>
<td>$\zeta (R)$</td>
<td>$\eta (\zeta)$</td>
<td>$-R^2 \zeta^2$</td>
<td>$A = R [\zeta (R) - \eta (\zeta)]$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Separation Coordinates</th>
<th>$\xi$</th>
<th>$\xi_1$</th>
<th>$\xi_2$</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relativistically separable</td>
<td>$\zeta (x)$</td>
<td>$\eta (y)$</td>
<td>$C^{-1} \eta (y)$</td>
<td>$\zeta (x)$ and $\eta (\mu)$ arbitrary</td>
</tr>
<tr>
<td>4.1 Cartesian</td>
<td>$\zeta (y)$</td>
<td>$C^{-1} \eta (y)$</td>
<td>$(C^2 - 1) \xi^2$</td>
<td>$\zeta (x)$ and $\eta (\mu)$ arbitrary</td>
</tr>
<tr>
<td>4.2 Cartesian</td>
<td>$\zeta (y)$</td>
<td>$\eta (y)$</td>
<td>$0$</td>
<td>$\zeta (x)$ and $\eta (\mu)$ arbitrary</td>
</tr>
<tr>
<td>4.3 Cartesian</td>
<td>$\zeta (y)$</td>
<td>$\eta (y)$</td>
<td>$0$</td>
<td>$\eta^2 - \eta^2$</td>
</tr>
<tr>
<td>4.4 Elliptic or parabolic cylindrical</td>
<td>$\sqrt{a_2 \lambda^2 + a_1 \lambda + a_0}$</td>
<td>$C \zeta$</td>
<td>$(C^2 - 1) \xi^2$</td>
<td>$\zeta$, $\eta$ and $\eta_1$ as in columns 1 and 2</td>
</tr>
<tr>
<td>4.5 As above</td>
<td>$\sqrt{a_2 \lambda^2 + a_1 \lambda + a_0}$</td>
<td>$C^{-1} \eta$</td>
<td>$(C^2 - 1) \eta^2$</td>
<td>$\zeta$, $\eta$ and $\eta_1$ as in columns 1 and 2</td>
</tr>
<tr>
<td>4.6 Cylindrical polar</td>
<td>$\zeta (r)$</td>
<td>$\eta (\phi)$</td>
<td>$0$</td>
<td>$\zeta (r)$ and $\eta (\phi)$ arbitrary</td>
</tr>
<tr>
<td>4.7 As above</td>
<td>$\zeta (r)$</td>
<td>$\eta (\phi)$</td>
<td>$0$</td>
<td>$\zeta (r)$ and $\eta (\phi)$ arbitrary</td>
</tr>
<tr>
<td>Classical super-separable (for all $q/m$)</td>
<td>$\zeta (x)$</td>
<td>$\eta (y)$</td>
<td>$\zeta (x)$</td>
<td>$\zeta (x)$, $\xi_1 (x)$ and $\eta_1 (y)$ arbitrary</td>
</tr>
<tr>
<td>4.8 Cartesian</td>
<td>$\eta (y)$</td>
<td>$\zeta (x)$</td>
<td>$\eta (y)$</td>
<td>$\zeta (x)$, $\eta (y)$ and $\eta (\mu)$ arbitrary</td>
</tr>
<tr>
<td>4.9 Cartesian</td>
<td>$\eta (y)$</td>
<td>$\zeta (x)$</td>
<td>$\eta (y)$</td>
<td>$\zeta (x)$, $\eta (y)$ and $\eta (\mu)$ arbitrary</td>
</tr>
<tr>
<td>4.10 Cylindrical polar</td>
<td>$\zeta (r)$</td>
<td>$\eta (\phi)$</td>
<td>$\zeta (r)$</td>
<td>$\zeta (r)$, $\xi_1 (r)$ and $\eta_1 (\phi)$ arbitrary</td>
</tr>
</tbody>
</table>

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APPENDIX A: EXPRESSIONS FOR \(\nabla\) AND CURL CURL USEFUL FOR DERIVING THE CHARGES AND CURRENTS THAT GENERATE THE SEPARABLE POTENTIALS

\[
\nabla^2 \left( \frac{\xi_\lambda(\lambda) - \eta_\mu(\mu)}{\lambda - \mu} \right) = \frac{4}{(\lambda - \mu)^2} \left\{ (\lambda + \mu) \left[ (\lambda + \beta)^2 \xi_\mu'' + \frac{1}{2} \xi_\mu' - \frac{1}{2} \xi_\mu' \right] - (\lambda + \mu) \left[ (\lambda + \beta) \xi_\mu' - \frac{1}{2} \xi_\mu' \right] \right\} + \text{conjugate,}
\]

where the conjugate may be obtained by writing \(\eta\) for \(\xi\) and exchanging \(\lambda\) and \(\mu\) in the expression written.

\[
\text{Curl curl}(A \hat{\phi}) = \text{curl} B \text{ and we write } A/R = A.
\]

Then

\[
\text{curl } B = -R^2 \nabla \phi [\nabla^2 A + 4 \partial_i A \partial_i R^2]
\]

\[
= \text{Curl curl}(A R^2 \nabla \phi)
\]

\[
= \frac{4 \lambda \mu \nabla \phi}{(\lambda - \mu)^2} \left\{ (\lambda + \mu) \left[ (\lambda + \beta)^2 (\lambda \xi_\mu'' + \xi_\mu') + \frac{1}{4} (\lambda + \beta) \xi_\mu' - \lambda \beta^{-1} \xi_\mu \right] - (\lambda + \mu) \left[ (\lambda + \beta) \xi_\mu' + \frac{1}{4} \xi_\mu' + \lambda \beta^{-1} \xi_\mu \right] \right\} + \text{conjugate.}
\]

The \(\nabla^2\) may be checked by seeing that \(\xi_\lambda = (\lambda + \beta)^{1/2}\) causes each square bracket to vanish, while the expression for curl \(B\) may be checked by noting that the two square brackets vanish when \(\zeta = \lambda^{-1}(\lambda + \beta)^{1/2}\).

APPENDIX B: PROOFS THAT THE POTENTIALS TABULATED ARE THE ONLY ONES

B1 Cylindrical coordinates

For relativistic separability we need \(\Phi^2 - A^2\) to be of the form \(\xi_\zeta(R) - \eta_\zeta(\zeta)\). Hence \(\partial^2/\partial \zeta \partial \zeta (\Phi^2 - A^2) = 0\). From (4.2) this gives

\[
-\xi_\lambda' \eta_\lambda + (R^2 \xi_\lambda' \eta_\lambda' - R \eta_\lambda'') = 0.
\]

Momentarily putting \(R = \text{constant} \neq 0\) we see that

\[
(\eta_\lambda')' = 2c_1 \eta_\lambda' - 2c_2 \eta_\lambda' + \frac{1}{2} \eta_\lambda';
\]

\[
\eta_\lambda', \quad \xi_\lambda';
\]

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Since $\eta$ depends on $z$ alone this must hold at all $R$. Putting (B2) back into (B1), we deduce the $f(R)$ or $g(z)$ relationship

$$\frac{d^2\eta'}{dR^2} - 2c_1R\eta' = (\xi_1' - 2c_2R)\eta_1'. \quad (B3)$$

Hence $\eta = 0$ (a constant being absorbable into $\xi$), or $\xi_1 = c_2 R^2$, case (2.7) (a constant gives no $E$), or $\eta_1 = C \eta$ [but that merely leads to the dull case $\eta = \text{constant from (B2)}$]; so we are left with just the two cases $\eta = 0$ and $\xi_1 = c_2 R^2$.

$\eta = 0$ gives a suitable result provided that either $\xi_1 = c_2 R^2$ or $\eta_1$ is zero (a constant is dull). Hence

$$A = Rf(R) \quad \text{with} \quad \Phi = \xi_1(R) \quad \text{case (2.5)}$$

or with $\Phi = \eta_1(z) \quad \text{case (2.6)}$

or case (2.7), $\xi_1 = c_2 R^2$, which leads to $\eta = 0$ again or $\xi = c_1 + c_2 R^{-2}$. The second is equivalent to $\xi = 0$ since $c_1 R^{-2}$ gives $B = 0$ and $c_1$ is dull, so we are left with $\xi_1 = c_2 R^2$, $\xi = 0$ and $\eta_1 = -2c_2 \eta_1$ which give

$$A = -R\eta_1(z) \Phi, \quad \Phi = c_2 R^2 + \frac{1}{2}\eta_1^2/c_2,$$

$$\Phi^2 - A^2 = c_2^2 R^4 + \frac{1}{4}c_2^2 \eta_1^4 = \xi_2(R) - \eta_1(z).$$

These form entries (2.5), (2.6) and (2.7) of Table 2.

### B2 Spherical coordinates

The condition that $A/R$, $\Phi$ and $\Phi^2 - A^2$ are all of the separable form $\xi(r) - \eta(\theta)r^{-2}$ reduces to (3.5). Momentarily fixing $r$ at some non-zero value and denoting constants by $c_0$,

$$(\eta_1')' = c_1(\sin^2 \theta) + c_2(\sin^2 \theta)' + c_3 \eta_1'. \quad (B4)$$

Since $\eta_1$ is a function of $\theta$ alone, (B4) holds always. Inserting it back into (3.5),

$$(r^2 \xi_1')' + 2c_1 r^2 = [2(\xi_1')' - 2c_2 r^{-3}][\eta(\sin^2 \theta)'] + (2\xi_1' + c_3 r^{-3})\eta_1' = 0. \quad (B5)$$

Momentarily fixing $\theta$ at some value $\neq 0$ or $\pi$ we find

$$(r^2 \xi_1')' + 2c_1 r^{-3} = 2c_4[(\xi_1')' - 2c_2 r^{-3}] + c_5(\xi_1' + c_3 r^{-3}); \quad (B6)$$

since all terms are functions of $r$ alone this holds for all $\theta$. Hence, from (B5),

$$[(\xi_1')' - 2c_2 r^{-3}][\eta(\sin^2 \theta)'] - c_4(\sin^2 \theta)' = [(\xi_1' + c_3 r^{-3})\eta_1' + c_5(\sin^2 \theta)']. \quad (B7)$$

Each side is a product of a function of $r$ and a function of $\theta$, so provided that

$$\eta_1' = c_3(\sin^2 \theta)' \neq 0 \quad \text{and} \quad (\xi_1')' - 2c_2 r^{-3} \neq 0$$

we may deduce that

$$\eta(\sin^2 \theta)' = C\eta_1 + (c_4 + C c_5) \sin^2 \theta + c_6$$

and

$$\xi_1 = C \xi_1^2 + (C c_5 + \frac{1}{2}c_3) r^{-2} + c_7. \quad (B8)$$

An $r^{-2}$ in $\xi_1$ may be eliminated by adding a constant to $\eta_1$, while an $r^{-2}$ in $\xi$ may be eliminated in favour of a constant added to $\eta$. So we can take $c_3 = -2C c_5$ and $c_7 = 0$, which leaves $\xi_1 = C \xi_1^2$.

Inserting these into (B6) and integrating, we find

$$(\xi_1')^2 - 2(\xi_1') (C c_5 + c_4) = c_8 + [c_1 + 2C c_6 + C c_5]) r^{-2};$$

hence $\xi' = a_1 \pm \sqrt{a_2 + a_3 r^{-2}}$, where $a_1 = C c_5 + c_4$, $a_2 = c_8 + a_1^2$ and $a_3 = c_1 + 2C c_6 + C c_5$. Similarly inserting (B8) into (B4) we obtain

$$\eta_1' = (c_1 + c_2 a_1) \sin^2 \theta - c_2 C \eta_1 + c_2 c_6 + c_9,$$
\[ \eta_i = b_1 \pm \sqrt{b_2 + b_3 \sin^2 \theta}, \]

where \( b_1 = -\frac{1}{2} c_2 b_2 + c_0 + c_2 c_6 + b_1^2 \) and \( b_3 = c_1 + c_3 a_1 \).

Putting these results back into (3.4) we find the required separation if \( a_3 = b_3 \) and \( b_1 = 0 \). Thus our solution for case 1 is

\[ \xi_i = C^2 r^2, \quad \zeta = \pm r^{-2} \sqrt{a_2 + a_3 r^{-2}}, \]

\[ \eta_i = \pm \sqrt{b_2 + a_3 \sin^2 \theta}, \quad \eta = (C \eta_1 + c_6)/\sin^2 \theta, \]

which is recorded as entry (2.3) in Table 2.

We have left out the \( a_1 \) terms since they cancel in \( \zeta - \eta r^{-2} \), and we could have put \( c_6 = 0 \) since it leads to no \( B \) field, but see below.

Our potentials are

\[ A = \{r \zeta \sin \theta - r^{-1} (C \eta_1 + c_6)/\sin \theta) \}, \]

\[ \Phi = \xi_i - r^{-2} \eta_i, \]

where \( \xi_i \) and \( \eta_i \) are the expressions from (B9):

\[ \Phi^2 - A^2 = \xi_2(r) - r^{-2} \eta_2(\theta), \]

where \( \xi_2 = \xi_1^2 + 2c_6 \xi_1 + b_2 r^{-4} \) and \( \eta_2 = \sin^2 \theta \eta^2 + a_2 \).

The lines of magnetic force are given by \( \zeta^2 \sin^2 \theta - C \eta_1 = k \) = constant which may be solved for \( r \):

\[ r^2 = a_3/\{ (k + C \eta_1)/\sin^2 \theta \} - a_2 \].

If we demand potentials that are regular as \( \sin \theta \to 0 \) then we need \( c_6 = -\pm C \sqrt{b_2} \) which gives entry (2.2) of Table 2:

\[ \eta = \pm C a_3/(\sqrt{b_2} + a_3 \sin^2 \theta + \sqrt{b_2}). \]

This completes case 1, so we now return to case 2 when either \( \eta_i = -c_5 \sin^2 \theta \) or \( \zeta^2 = -c_2 r^{-2} \), which is left aside under (B7). From (B7) we have, taking the first alternative, \( \eta \sin^2 \theta = c_4 \sin^2 \theta + a_6 \) with \( \eta_i = -c_5 \sin^2 \theta \), but the \( a_6 \) term may be set to zero since it leads to no \( B \) field, and a constant \( \eta \) term may be absorbed into \( \zeta \) so we may take \( \eta = 0 \), \( \zeta = \zeta(r) \) and \( \eta_i = -c_5 \sin^2 \theta \).

Then

\[ \Phi^2 - A^2 = -r^2 \xi_1^2 \sin^2 \theta + \xi_1^2 + 2c_5 \xi_1 r^{-2} \sin^2 \theta + c_5^2 r^{-4} \sin^4 \theta, \]

which is only separable in the dull case \( c_5 = 0 = \eta_i = \eta = \zeta \) which has no \( B \) field. There remains only the second alternative \( \zeta = -c_2 r^{-4} \):

\[ r^2(\Phi^2 - A^2) = -c_2^2 r^{-4} \sin^4 \theta - 2c_5 r^{-2} \eta \sin^2 \theta - \eta^2 \sin^2 \theta + r^2 \xi_1^2 + 2\xi_1 c_5 \sin^2 \theta + r^{-2} c_5^2 \sin^4 \theta. \]

If \( c_5 = 0 \) then we must have \( c_5 = 0 \), and then \( \eta \) and \( \zeta \) can be chosen arbitrarily.

This solution is of the generalized monopole class

\[ A = -r^{-1} \sin \theta \eta(\theta) \phi, \quad \Phi = \xi_i(r), \]

\[ B = Q(\theta) \phi \]

where \( Q = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} [\sin^2 \theta \eta(\theta)] \),

recorded as entry (2.3) in Table 2.

Alternatively, if \( c_2 \neq 0 \) then \( \xi_i = \frac{1}{2} c_2 c_4^{-1} r^{-4} - a \eta r^{-2} \) and \( \eta = (a \eta + \frac{1}{2} c_5 \sin^2 \theta) c_3 / c_2 \), so for entry (2.4) of Table 2 we have

\[ A = -[c_2 r^{-3} + r^{-1} \eta] \sin \theta \phi, \]

\[ \Phi = \xi_i + c_5 r^{-2} \sin^2 \theta, \]

\[ \Phi^2 - A^2 = \xi_2 - r^{-2} \eta_2, \]

with \( \xi_2 = \xi_1^2 \) and \( \eta_2 = \eta^2 \sin^2 \theta \). These are tabulated in Section 4 and Table 2.

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