Temperature correlations in a compact hyperbolic universe

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Accepted 2000 March 7. Received 1999 December 16; in original form 1999 August 23

A B S T R A C T
The effect of a non-trivial topology on the temperature correlations of the cosmic microwave background (CMB) in a small compact hyperbolic universe with volume comparable to the cube of the curvature radius is investigated. Because the bulk of large-angle CMB fluctuations are produced at the late epoch in low-Ω₀ models, the effect of a long-wavelength cut-off owing to the periodic structure does not lead to significant suppression of large-angle power as in compact flat models. The angular power spectra are consistent with COBE data for Ω₀ ≥ 0.1.

Key words: cosmic microwave background – large-scale structure of Universe.

1 INTRODUCTION
Einstein’s equations do not specify the global structure of space–time. In other words, for a given local metric, a large number of topologically distinct models remain unspecified. In the absence of the unified theory that describes the global structure as well as the local structure, one must resort to observational methods to determine the global topology of the Universe.

Assuming that the spatial hypersurface is homogeneous, the observed high degree of isotropy in the cosmic microwave background (CMB) points to Friedmann–Robertson–Walker (FRW) models as the best candidates for cosmological models. However, if one allows the spatial hypersurface to be multiply-connected, a variety of local FRW models that are globally anisotropic and inhomogeneous may be consistent with the current observational data.

Constraints on the topological identification scales using COBE data have been obtained for some flat models with no cosmological constant (Stevens, Scott & Silk 1993; de Oliveira-Costa, Smoot & Starobinsky 1996; Levin, Scannapieco & Silk 1998) and some limited compact hyperbolic (CH) models (Levin et al. 1997; Bond, Pogosyan & Souradeep 1998). The large-angle temperature fluctuations discovered by COBE constrain the possible number of copies of the fundamental domain inside the last scattering surface to fewer than ~8 for compact flat multiply-connected models.

On the other hand, a large number of CMB anisotropies on large scales could be produced in a low-density universe owing to the decay of gravitational potential near the present epoch (Cornish, Spergel & Starkman 1998a). Therefore we expect that the constraint on the possible number of copies is less stringent for CH models. However, since the effect of the non-trivial topology becomes more and more significant as the volume of space decreases, it is very important to investigate the viability of CH models with small comoving volume.

From a theoretical point of view, the ‘smallness’ of the spatial hypersurface is an advantage for the development of a natural mechanism leading to homogeneity and isotropy. It is well known that geodesic flows on CH spaces are strongly chaotic. Therefore initial perturbations would be smoothed out because of mixing effects (Lockhart, Misra & Prigogine 1982; Gurzadyan & Kocharyan 1992; Ellis & Tavakol 1994). In inflationary scenarios, it is essential to have a physical process that homogenizes the initial region beyond the horizon scale before the onset of inflation, to accomplish sufficient smoothing of the observable universe (Goldwirth & Piran 1989; Goldwirth 1991). The chaotic mixing in CH spaces may provide a solution to the pre-inflationary initial value problem (Cornish, Spergel & Starkman 1996).

If we live in a small universe which is defined to be a locally homogeneous and isotropic space that is multiply-connected on scales comparable to or smaller than the horizon, future astronomical satellite missions such as MAP and Planck might reveal some specific features in the CMB (Cornish, Spergel, & Starkman 1998b,c; Weeks 1998).

So far, a variety of CH manifolds have been constructed by mathematicians. However, the number of known CH manifolds with small volume is relatively small. In this paper, we investigate CH models for which the spatial hypersurface is isometric to the Thurston manifold, which is the second smallest of the known CH manifolds with volume 0.981 39 times the cube of the curvature radius. The smallest manifold is the Weeks manifold with volume 96 per cent that of the Thurston manifold (see e.g. Fomenko & Kunić 1997). However, the fundamental domain (which tessellates the infinite space) of the Thurston manifold is much simpler than that of the Weeks manifold. For simplicity, we investigate the Thurston models rather than the Weeks models. The fundamental domain of the Thurston manifold is a polygon with 16 faces,
which can be constructed by appropriately identifying eight faces with the remaining eight faces (see the appendix of Inoue 1999a). It should be noted that the volume of CH manifolds must be larger than 0.16668 times the cube of the curvature radius, although no concrete examples of manifolds with such small volumes are known (Gabai, Meyerhoff & Thurston 1996).

2 COMPUTATION OF EIGENMODES

So far, various kinds of numerical techniques have been proposed to overcome the difficulty of computing the CMB in CH models. For several CH models, CMB fluctuations have been computed using the method of images without carrying out the mode expansion (Bond et al. 1998). These computations resulted in the COBE data strongly constraining the CH models so that the comoving volume of the fundamental domain is at least comparable to the comoving volume inside the last scattering surface. Since the method of images requires the sum of exponentially increasing images, it is difficult to obtain the distinct eigenmodes that are necessary to estimate the effect of the power spectrum with discrete peaks. Alternatively, one of us has proposed a numerical approach called the direct boundary element method for computing eigenmodes of the Laplace–Beltrami operator (Inoue 1999a). 14 eigenmodes have been computed for the Thurston manifold. It is numerically found that the expansion coefficients behave as if they are random Gaussian numbers.

In this work, we have numerically computed 36 eigenmodes in the Thurston manifold up to \( k = 13 \) (the curvature radius is normalized to one) which are approximated by quadrature shape functions which converge to the solutions faster than constant-valued shape functions. As we shall see, the contribution of the higher modes to the angular power spectra on large angular scales is relatively small for low-density models. In other words, the effect of the non-trivial topology is almost determined by the lower modes. We confirm the previously computed eigenvalues within |\( \delta k \)| ≤ 0.01.

We see from Fig. 1 that the number of eigenmodes below \( k \) is nicely fitted to Weyl’s asymptotic formula

\[
N(k) = \frac{\text{Vol}(M)(k^2 - 1)^{3/2}}{6\pi}, \quad k \gg 1, \tag{1}
\]

where Vol(\( M \)) denotes the volume of a manifold \( M \). The random Gaussian behaviour is again observed for 31 modes with 5.404 ≤ \( k \) < 13, but five degenerate states have an eigenmode that shows non-Gaussian behaviour owing to the global symmetry of the fundamental domain. It is found that the five eigenmodes have Z2 symmetry (invariant with respect to the rotation by an angle \( \pi \)) on the centre (where the injectivity radius that is equal to half of the minimum length of the periodic geodesics that lies on the point is locally maximal) of the fundamental domain. In this case, one would observe an axis around which the fluctuation is rotationally symmetric at the centre. Therefore the correlation between expansion coefficients leads to a non-Gaussian behaviour. Nevertheless, it is found that appropriate choices of the linear combination of degenerate modes recover the generic Gaussian behaviour. Furthermore, the symmetry of CH manifolds depends on the observing point. If one randomly chooses a point on the manifold, the probability of observing an exact symmetry of the manifold is very small. This result supports previous investigations of the expansion coefficients, which show Gaussian behaviour in classically chaotic systems (Aurich & Steiner 1989; Haake & Zyczkowski 1990), although the global symmetry in the system can hide the generic property (Balazs & Voros 1986).

3 TEMPERATURE FLUCTUATIONS

Perturbations in CH models can be written in terms of a linear combination of eigenmodes on the universal covering space multiplied by the expansion coefficients and the initial fluctuations plus time evolution of the perturbations. The expansion coefficients include information on the periodicity in the universal covering space. As CH models are locally homogeneous and isotropic, the time evolution of the perturbations coincides with that in open models.

The dominant physical effects producing CMB anisotropies (Hu, Sugiyama & Silk 1997) on large angular scales are the ordinary Sachs–Wolfe (OSW) effect (Sachs & Wolfe 1967), which is the gravitational redshift effect in between the last scattering surface and the present epoch, and the integrated Sachs–Wolfe (ISW) effect, which is the gravitational blueshift effect caused by the decay of gravitational potential at the curvature domination epoch, \( 1 + z \sim (1 - t')/t_0 \). For COBE scales, we can ignore the contribution from acoustic oscillations. Then the time evolution of the adiabatic growing mode of the Newtonian gravitational potential is analytically given as (see e.g. Kodama & Sasaki 1986; Mukhanov, Feldman & Brandenberger 1992)

\[
\Phi(\eta) = \Phi(0) \frac{5(\sinh^2 \eta - 3\eta \sinh \eta + 4 \cosh \eta - 4)}{(\cosh \eta - 1)^2}, \tag{2}
\]

\( \Phi(0) \) is the matter density parameter and \( \Omega_0 = 0.2 \) (left) and \( \Omega_0 = 0.4 \) (right) using expansion coefficients derived from 36 eigenmodes only (stars) and those derived using these coefficients and random Gaussian numbers for \( 13 < k < 50 \) with 100 realizations (diamonds). Eigenvalues for higher modes are approximated by Weyl’s asymptotic formula.

Figure 1. Number function and the Weyl asymptotic formula.

Figure 2. \( \delta T_l/T = \sqrt{l(l+1)}C_l/(2\pi) \) for the Thurston models \( \Omega_0 = 0.2 \) (left) and \( \Omega_0 = 0.4 \) (right) using expansion coefficients derived from 36 eigenmodes only (stars) and those derived using these coefficients and random Gaussian numbers for \( 13 < k < 50 \) with 100 realizations (diamonds). Eigenvalues for higher modes are approximated by Weyl’s asymptotic formula.

Figure 3. A simulated sky map of the microwave background (convolved with the COBE DMR beam) in the Thurston model $\Omega_b = 0.2$.

where $\eta$ denotes the conformal time. The two-point temperature correlations in a CH cosmological model can be written in terms of the gravitational potential. Assuming that the initial fluctuations obey the Gaussian statistic, and neglecting the tensor-type perturbations, the angular power spectrum $C_l$ can be written as

$$(2l + 1) C_l = \sum_{m=-l}^{l} \langle |a_{lm}|^2 \rangle = \sum_{m,n} \frac{4\pi^2 P_\phi(n)}{n^2 + 1} \text{Vol}(M) |\xi_{\phi n}|^2,$$

where

$$F_{\phi l}(n) = \frac{1}{3} \Phi_l(n) X_{\phi l}(n_0 - \eta) + 2 \int_{n_0}^{n} \frac{d\Phi_l}{d\eta} X_{\phi l}(n_0 - \eta).$$

(3)

Here, $\nu = \sqrt{k^2 - 1}$, $P_\phi(n)$ is the initial power spectrum, and $n_0$ and $n_0$ are the conformal time of the last scattering and the present conformal time, respectively. $X_{\phi l}$ denotes the radial eigenfunctions in open models and $\xi_{\phi l}$ denotes the expansion coefficients. From now on we assume that the initial power spectrum is the extended (extended) Harrison–Zeldovich spectrum, i.e. $P_\phi(n) = \text{constant}$.

Although the low-lying modes make an appreciable contribution to the large-angle power, contributions of higher eigenmodes may not be completely negligible. While the computation of highly excited eigenmodes is a difficult task, we have so far succeeded in calculating the exact eigenmodes up to $k = 13$, as we mentioned before. However, we are going to assume that the $\xi_{\phi l}$ are also random Gaussian numbers for higher modes. Since the information on the periodicity in real space is lost by this approximation, we will apply the approximation only to the statistics in $k$-space which is expected to be unchanged because the periodicity is not apparent in $k$-space. As CH models are globally inhomogeneous, the expected correlation statistics depend on the point of view of the observer. Therefore one can deduce that one realization for the expansion coefficients corresponds to a certain point of view of the observer in the fundamental domain. In order to apply the random Gaussian approximation, one must also estimate the variance of the expansion coefficients. The expansion coefficients are written in terms of eigenmodes $u_\phi$ and spherical harmonics $Y_{lm}$ as

$$\xi_{\phi l}(X) = \int_{X} u_\phi(x, \theta, \phi) Y^{\ast}_{lm}(\theta, \phi) d\Omega.$$  

(5)

It should be noted that equation (5) is satisfied at arbitrary radius $X_0$. Let us consider a sphere with large radius $X_0 \gg 1$ on the Poincaré ball which is the image of the upper hyperboloid in the four-dimensional Minkowski space $(y_0, y_1, y_2, y_3)$ by a stereographic projection on to the unit ball on the $(0, y_1, y_2, y_3)$ plane using a point ($-1, 0, 0, 0$) as the base point. One can expect random behaviour of the mode functions on the sphere as the surface of the sphere which is pulled back by the discrete isometry group fills up the fundamental domain densely or ‘ergodically’. The (apparent) angular fluctuation scale $\delta \theta$ of the $k$-mode is approximated in terms of the two parameters $X_0$ and $k$ as

$$\delta \theta^2 = \frac{16\pi^2 \text{Vol}(M)}{k^2 \left[ \sinh[2(X_0 + r_{\text{ave}})] - \sinh[2(X_0 - r_{\text{ave}})] - 4r_{\text{ave}} \right]},$$

(6)

where $r_{\text{ave}}$ denotes the averaged radius of the ‘in-radius’ and ‘out-radius’ of the fundamental domain. One can approximate $u'_{\phi l}(X_0) \approx u_{\phi l}(X_0)$ by choosing an appropriate radius $X_0$ that satisfies $k^{-2} \exp(-2X_0) = k^{-2} \exp(-2X_0)$. Averaging equation (5) over $l$ and $m$, one obtains

$$\langle |\xi_{\phi l}|^2 \rangle \sim \frac{\exp(-2X_0)}{\exp(-2X_0)} \langle |\xi_{\phi l}|^2 \rangle,$$

(7)

which gives $\langle |\xi_{\phi l}|^2 \rangle \sim c^{-2}$. We have found that the computed variances of the $\xi_{\phi l}$ for $2 \leq l \leq 20$, $-l \leq m \leq l$ are in remarkably good agreement with the analytical estimate.

From Fig. 2, one can see that the uncertainty in the Gaussian approximation is very small. Remarkably, each realization gives almost the same value so that 100 points for given $l$ are plotted as a tiny speck. The contribution of higher modes becomes significant as $\Omega_b$ is increased because the curvature-dominant era is shifted to late times so that the OSW effect becomes dominant over the ISW effect. It is found that the contributions of the modes $k > 13$ to $C_l$ for $2 \leq l \leq 20$ are approximately 7 and 10 per cent for $\Omega_b = 0.2$ and 0.4, respectively. Thus the contribution of modes $k > 13$ for which we employ a Gaussian approximation is almost negligible on large angular scales, especially in low-$\Omega_b$ models. One realization (for the initial fluctuation) of a typical CMB

$^1$In-radius is the radius of the largest sphere inscrutable inside the fundamental domain and out-radius is the radius of the smallest sphere circumscribable around the fundamental domain.
fluctuation as seen by COBE is plotted in Fig. 3 for \( \Omega_0 = 0.2 \). In the simulation, we used only 36 ‘exact’ eigenmodes. We have chosen a point where the injectivity radius is locally maximal at the centre (belonging to the ‘thick’ part of the manifold). One can see that the structure arising from the periodic boundary conditions is not apparent. However, the approximate number of copies of the fundamental domain inside the last scattering surface is \( \sim 500 \) for the Thurston model with \( \Omega_0 = 0.2 \). Therefore the effect of the non-trivial topology is expected to be significant.

The mode cut-off at \( k = 5.404 \) which corresponds to the largest wavelength inside the fundamental domain causes suppression of the angular power on large angular scales as in compact flat models. However, the decay of the Newtonian potential in the curvature-dominant era makes a difference. Since the bulk of the large-angle power comes from the decay of the potential well after the last scattering time, the large-angle power does not suffer significant suppression. We see from Fig. 4 that the slope of the large-angle power is not steep even for the model with \( \Omega_0 = 0.2 \), in contrast to the compact flat models without cosmological constant. The two peaks in the power spectrum for the CH model are important in understanding the effect of the non-trivial topology. The angular scale that gives the first peak is equivalent to the angular fluctuation scale of the lowest eigenmode (\( k = 5.404 \)) on the last scattering surface. Substituting the comoving radius of the last scattering surface in units of the curvature radius \( R_{\text{curv}} \),

\[
R_{\text{LSS}} = R_{\text{curv}} \cosh^{-1}(2/\Omega_0 - 1),
\]

into equation (6) gives the angular scales \( l = 17 \) for \( \Omega_0 = 0.2 \) and \( l = 7.4 \) for \( \Omega_0 = 0.4 \). Beyond this scale, the OSW contribution is strongly suppressed as in compact flat models. However, eigenmodes with angular scales below the given scale at the last scattering surface can have large angular scales after the last scattering. Therefore, in the presence of the ISW effect, the suppression of power beyond the scale that corresponds to the first peak is very weak, in contrast to flat models. The angular scale that gives the second peak corresponds to the scale of the projected lowest eigenmode at the last scattering. Below this scale, the angular power asymptotically converges to that of open models because the effect of the modes with wavelength larger than the cut-off wavelength is negligible. Since we have ignored the effects of subhorizon perturbations at the last scattering surface, such as the so-called ‘early’ ISW effect during the

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**Figure 4.** \( \delta T_i/T = \sqrt{(l+1)C_l/(2\pi)} \) for the Thurston models at centre (diamonds), and ensemble-averaged values (squares) and open models (stars) with \( \Omega_0 = 0.2 \) (left) and \( \Omega_0 = 0.4 \) (right). Because of the global inhomogeneity, the \( \delta T_i/T \) have a dependence on the observing points inside the fundamental domain that causes the uncertainty in \( \delta T_i/T \). 2\( \sigma \) ‘geometric variance’ is shown as vertical error bars which have been obtained from 500 realizations for the expansion coefficients.

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**Figure 5.** \( \delta T_i/T = \sqrt{(l+1)C_l/(2\pi)} \) in \( \mu \)K for the Thurston model with \( \Omega_0 = 0.1 \) (top left), \( \Omega_0 = 0.4 \) (top right), \( \Omega_0 = 0.2 \) (bottom left) and \( \Omega_0 = 0.6 \) (bottom right). The light grey band corresponds to the 1\( \sigma \) cosmic variance in the centre, and the dark grey band corresponds to the sum of the 1\( \sigma \) cosmic and the 1\( \sigma \) ‘geometric’ variances. The COBE DMR measurements analysed by Gorski and Tegmark are plotted as diamonds and squares respectively.
matter–radiation equality epoch and the Doppler effect arising from the acoustic velocity, the angular power on large to intermediate scales must be slightly boosted. However, these effects are irrelevant to the global effect of the non-trivial topology inasmuch as one considers the typical topological identification scale which is not significantly smaller than the present horizon.

In Fig. 5, the angular power spectra for low-$\Omega_0$ models are plotted with the COBE data (Górski et al. 1996) (diamonds). They have been calculated using 36 eigenmodes and the Gaussian approximation, taking account of $\sim 10$ per cent contributions from higher eigenmodes. The slope of the power becomes steep as $\Omega_0$ is lowered, since the ISW contribution transfers to large scales.

We have performed a simple $\chi^2$ fitting analysis to the COBE DMR-band power measurements (Tegmark 1997) (squares in Fig. 5), which are uncorrelated. We have adjusted the normalization of the initial power to minimize the value of $\chi^2$. As shown in Table 1, the angular power for a model with $\Omega_0 = 0.1$ is still within the acceptable range. The apparent primordial spectral index is approximately $n = 1.6$ for $\Omega_0 = 0.1$.

### 4 CONCLUSIONS

We have found that the Thurston models with $\Omega_0 \gg 0.1$ are not constrained by the angular power spectrum from the COBE data, which confirms the preliminary result by Inoue (1999b). The peak at $l \sim 4$ in the COBE data may be merely a coincidence resulting from the large cosmic variance, but it is interesting that a model with $\Omega_0 \sim 0.6$ has the first peak in this scale. Consequently, the Thurston models agree better with the COBE data than do any FRW models. A similar conclusion, that the constraints $\Omega_0 \geq 0.3$ for an ‘orbifold’ model with volume 0.7173068$k^3$, has been obtained by Aurich (1999). Although orbifolds have singular points, the behaviour of eigenmodes for orbifolds is expected to be similar to that of manifolds. Therefore the result for an orbifold model supports our conclusion.

### ACKNOWLEDGMENTS

We thank Dr Jeff Weeks and the Geometry Center at the University of Minnesota for providing us with the data on CH spaces, and Dr Neil J. Cornish for useful comments. The numerical computation in this work was carried out by the VPP 800 at the Data Processing Center of Kyoto University. KTI is supported by JSPS Research Fellowships for Young Scientists, and this work was supported partially by the Grant-in-Aid for Scientific Research Fund (Nos. 9809834 and 11640235).

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