Since in the case of a rotor the blade is moving relative to our coordinate system and therefore $z_1 = z_1(x, y, t)$ and $z_2 = z_2(x, y, t)$, we must complement equations (3*a*), (3*b*), and (3*c*) with this relation,

$$\int_{z_1}^{z_2} \frac{\partial P}{\partial t}\, dz = \frac{\partial}{\partial t}\int_{z_1}^{z_2} P dz - P_2 \frac{\partial z_2}{\partial t} + P_2 \frac{\partial z_1}{\partial t} \qquad (83)$$

where

$$\frac{\partial z_2}{\partial t} = \frac{\partial z_1}{\partial t} = \begin{cases} 0 \text{ for a stator} \\ U \text{ for a rotor} \end{cases} \qquad (84)$$

As before we assume that $\partial\left(\int_{z_1}^{z_2} Pdz\right)\Big/\partial t = \partial\bar{P}/\partial t = 0$. In the case of a rotor the above relations simply lead to the canceling of two terms. Thus, $\int_{z_1}^{z_2} (\partial e_0/\partial t)dz = U(e_{0_2} - e_{0_1})$ whereas $(we_0)_2 - (we_0)_1 = U(e_{0_2} - e_{0_1})$. Also noting that $(wp)_2 - (wp)_1 = U(p_2 - p_1)$ we can write

$$\rho\left\{\frac{\partial}{\partial x}(\overline{uh_0}) + \frac{\partial}{\partial y}(\overline{vh_0})\right\} = U\left(\Delta p + 2\frac{\tau_{bx}}{s}\tan\beta\right) + \frac{\partial \bar{q}_x}{\partial x} + \frac{\partial \bar{q}_y}{\partial y} + \frac{\partial}{\partial x}(\overline{\boldsymbol{V}\cdot\boldsymbol{\tau}_x}) + \frac{\partial}{\partial y}(\overline{\boldsymbol{V}\cdot\boldsymbol{\tau}_y})$$

where now the total enthalpy $h_0 = e_0 + p/\rho$ is conveniently introduced.

Finally, we decompose averages such as $\overline{uh_0} = \bar{u}\bar{h}_0 + \overline{u'h_0'}$, and invoke the boundary layer approximation. The result is

$$\rho\frac{\partial}{\partial x}(\bar{u}\bar{h}_0) + \rho\frac{\partial}{\partial y}(\bar{v}\bar{h}_0) = U\frac{\partial f_z}{\partial x} + \frac{\partial}{\partial y}(\bar{q}_y + \rho\overline{v'h'}) + \frac{\partial}{\partial y}[\overline{\boldsymbol{V}\cdot(\boldsymbol{\tau}_y + v'\boldsymbol{V}')}] \qquad (85)$$

where

$$\bar{h}_0 \equiv \bar{h} + \frac{1}{2}(\bar{u}^2 + \bar{v}^2 + \bar{w}^2 + \overline{u'^2} + \overline{v'^2} + \overline{w'^2})$$

In the above equation triple correlations such as $\overline{u'^3}$ have been neglected relative to $\bar{u}\overline{u'^2}$, for example.

The first term on the right will readily be recognized as the rotor work. The second, the heat transfer augmented by $\overline{v'h'}$ due to circumferential distortion and a (viscous plus Reynolds) stress work term again augmented by $\overline{\boldsymbol{V}\cdot v'\boldsymbol{V}'} \simeq \bar{u}\overline{v'u'} + \bar{v}\overline{v'^2} + \bar{w}\overline{v'w'}$.

We now note that equation (49) is just (85) integrated across the blade span with the help of (62) and with the assumption of adiabatic wall conditions ($\bar{q}_y + \overline{v'h'} = 0$ on inner and outer walls).

The question might rightly be posed: Why not set up an integral method leading to energy versions of equations (19) and (20)? This is, of course, the type of thing that is done in heat transfer analyses and which is a way of characterizing temperature distributions. Our feeling is that precise knowledge of temperature distributions (and entropy distributions) is definitely of second-order importance (in low speed or essentially incompressible flow it is of no importance). The principal effect of a precise knowledge of temperature distribution would be to modify the density distribution slightly.

It is clear from the discussion in the principal part of the paper that the so-called "lost work" would lead to a buildup of enthalpy near the wall. This has been observed experimentally where air near the walls is measured at a higher temperature than the average. However, heat is probably transferred into the mainstream due primarily to small but finite values of $\bar{q}_y + \overline{v'h'}$ at the edge of the wall boundary layer.[8] To account for this effect would indeed be difficult. However, as previously stated it is probably not necessary.

DISCUSSION

J. H. Horlock[9] and D. Hoadley[10]

We should like to congratulate the authors on an interesting paper, which parallels some of our own work reported in References [15] and [16],[11] listed at the end of this contribution. In those papers we have described a method for calculating annulus wall boundary layers which is essentially based on the idea proposed by Railly and Howard [13], developed by Stratford [3] and implied by Mellor and Wood in equation (24)—that the "body" force term in the mainstream is the same as that in the boundary layer (if $\epsilon = O$). We subsequently derived the momentum equations (reference [15]) and the entrainment equation (reference [2]) for the "pitch-averaged" three dimensional annulus wall boundary layer.[12] Using an approach similar to reference [13],—i.e., by inserting velocity profiles for the stream-

[8] Note that in our theory we had neglected corresponding momentum transfer from the boundary layer to the main stream by assuming $\bar{\tau}_{yx}/\rho - \overline{u'v'} = \bar{\tau}_{yz}/\rho - \overline{w'v'} = 0$ at the edge of the layer. While in a compressor it is probably not zero, it is probably small relative to all other terms that appear in equations (19) and (20).

[9] Department of Mechanical Engineering, University of Cambridge, Cambridge, England.

[10] Central Electricity Generating Bond, Marchwood Engineering Laboratories, Southampton, England.

[11] Numbers 15 and 16 in brackets designate Additional References at end of discussion.

[12] It should be noted that these equations include a term involving the mean velocity of the pitch-averaged flow, $\zeta = \frac{d\bar{w}_e}{dx} = \frac{1}{\rho\bar{u}_e}\frac{\partial f_{ze}}{\partial x}$.

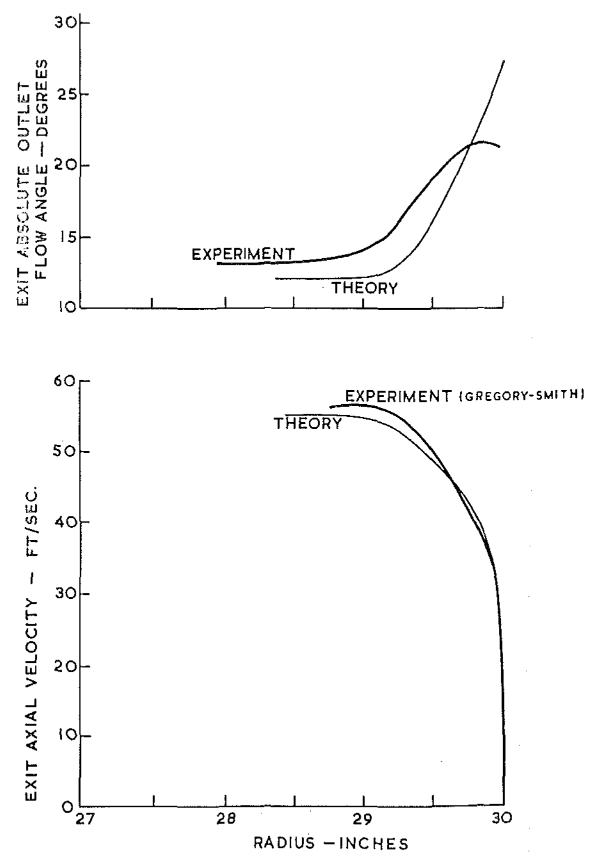

Fig. 11 Flow through rotor

wise (s) and normal (n) directions in a streamline coordinate system—we were able to obtain not only the change in boundary layer momentum thickness but also the detailed velocity profile developed in an axial flow turbomachine.

An example of such calculations is given in Fig. 12, which is taken from reference [15]. This shows the axial velocity and flow angle predicted at exit from an isolated rotor row, in comparison with measurements made by Mr. D. Gregory-Smith. We should point out that we are always able to work in absolute coordinates, as the body force does not enter directly into the equations we use. Instead the body force is represented by a vorticity term which is dependent upon the mainstream conditions at the edge of the boundary layer. The only input required to our computer program, besides the initial conditions, is the absolute flow angle and the axial velocity at the edge of the boundary layer, as a function of the axial distance through the machine.

The limitations of our method are:

(i) No allowance is at present made for tip clearance.
(ii) We are limited in the pitch-averaged equations to small deflections.
(iii) The cross-flow velocity is not well represented by Mager's profile. The Mellor-Wood proposal here is a great improvement.
(iv) We cannot cope with large accelerations because the s profile (we have used Coles') loses its validity. We are doubtful whether any method using conventional boundary layer assumptions is valid in such cases.

We consider the most useful points for us in the Mellor and Wood paper are:

(i) The cross-flow proposal to take account of secondary flow.
(ii) The jump condition equations.

Additional References

15 Horlock, J. H., "Boundary Layer Problems in Axial Turbomachines," Proceedings of Symposium on Flow Research on Blading, Dzung, L. S., ed., Elsevier Pub. Co., 1970, pp. 355–357.
16 Horlock, J. H., and Hoadley, D., "Calculation of the Annulus Wall Boundary Layers in Axial Flow Turbomachines," CUED/A-Turbo/TR 16, 1970, Aeronautical Research Council Report No. 31955.

J. W. Railly[13]

The authors have made a significant contribution to the theory of the end-wall boundary layer by introducing a blade force "defect" term which is retained as an additional dependent variable. By virtue of the introduction of this variable it is possible to make restrictive assumptions about the form of the flow at outlet from each blade row in accordance with ideas about secondary flow and tip clearance flow. That it is reasonable to expect a blade force "defect" may be seen by an order of magnitude argument as follows: since δp over the boundary layer thickness is $O(\delta)$, where δ is small compared with the blade chord, therefore the pressure difference across a blade in the boundary layer is $O(\delta)$-$O(\delta)$ which is also $O(\delta)$. The integrated blade force term is then $O(\delta^2)$ which is negligible when the boundary layer thickness is much smaller than the blade chord but significant when δ becomes comparable with chord. This argument could explain the good agreement with theory obtained by the Cambridge workers (cf., Gregory-Smith[14]) for the boundary layer growth in an inlet guide vane row where the boundary layer is relatively thin.

In the writer's earlier work (reference [13] of the paper) the divergence between that theory and experiment was most marked in the failure to obtain the correct flow direction through the boundary layer at blade row exit. In that work, the two variables, boundary layer thickness and boundary layer deflection angle, were completely determined by the two equations (also equations (19) and (20) of the paper). If some restriction be placed upon the value of deflection angle at blade outlet (as, for example, that suggested by the author for closely-pitched blades that the flow at outlet be collateral) then another variable must be introduced. This could be a parameter describing some feature of the stream-wise velocity profile. However, this could not take care of the important consideration raised by the authors that the boundary layer should tend to an asymptotic thickness distribution with a large number of stages; only the presence of a negative blade force "defect" as suggested would make this possible. It is therefore of some importance to the theory that such a boundary layer development should be established experimentally.

13 Senior Lecturer in Mechanical Engineering, Department of Mechanical Engineering, University of Birmingham, Birmingham, England.

14 Gregory-Smith, D. G., "An Investigation of Annulus Wall Boundary Layers in Axial Flow Turbomachines," Paper No. 70-GT-92.

Leroy H. Smith, Jr.[15]

This discusser is glad to see that the authors have recognized that blade force defects are important in the establishment of end-wall boundary layers. However, it is believed that their assumption that the blade force angle does not vary significantly through the boundary layer is somewhat of an over-simplification. In Fig. 12, velocity and force vectors are shown with angles and magnitudes that are believed to be of typical proportions for a rotor operating in the repeating-stage environment. This figure can be compared with Fig. 6. The orientations of the relative velocity vectors in the two figures are not inconsistent, and in both cases the axial component of force is greater in the boundary layer than in the main stream to account for wall shear stress and clearance effects. The tangential force component, however, is different; in Fig. 12 it is seen to be somewhat smaller in the boundary layer than in the main stream. This does not mean that the rotor work input per unit mass is lower in the boundary layer; the stagnation enthalpy rise is, in fact, larger in the boundary layer because the change in tangential velocity across the rotor is larger there.[16] The tangential force is lower because the mass flux is lower in the boundary layer due to the reduced axial velocity there.

The reasons for concluding that the tangential force defect

15 Aircraft Engine Technical Division, General Electric Co., Cincinnati, Ohio.

16 A small allowance for tangential shear stress on the stationary casing is not sufficient to change this statement.

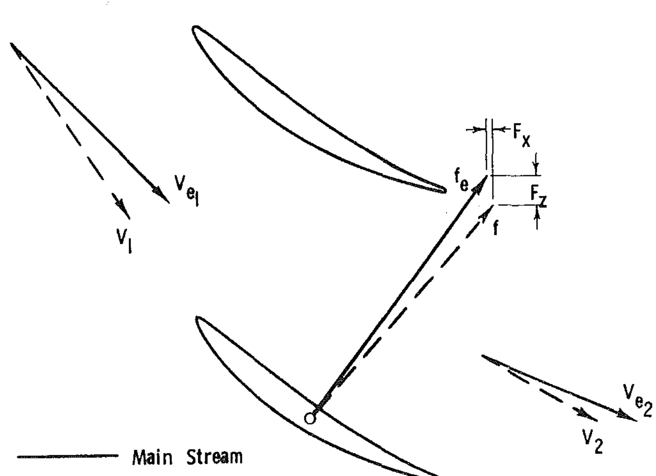

Fig. 12 Velocity and force vectors in the repeating-stage environment

should be opposite in sign to that proposed by the authors is based on the measurements and performance synthesis work described in reference [14]. It was found from measurements of tangential momentum flux before and after blade rows operating in the repeating-stage environment that the tangential force component falls off somewhat through an end-wall boundary layer. (Although end-wall tangential shear stresses were neglected when computing the tangential force from the traverse probe measurements, it is believed that this does not affect the conclusion because tangential force defects were found at both the inner and outer walls where the shear stresses act in opposite directions.) Furthermore, it turned out to be necessary to employ a tangential force defect of roughly the same magnitude as that measured in order to calculate the efficiency trends that were measured in a series of tests in which blade aspect ratio was varied from 2 to 5. The low aspect ratio blades, with their thicker wall boundary layers in the repeating-stage environment, could not have been as efficient as they were if there had not been a tangential force defect near the walls.

In the section of the paper "Model Compressor Calculations" the authors found that their theory is insensitive to variations in the parameter ϵ. Although this is true for their formulation, it should be understood that ϵ, as used in equation (40), deals with the orientation of the force *defect* vector. Since the defect force magnitude is made much less than the complete blade force magnitude, $\epsilon = 0.2$ represents only a very minute variation in blade force angle through the boundary layer. If changes as large as those indicated in Fig. 12 (still rather small in an absolute sense) were employed, an increase in efficiency large enough to be noticeable should result.

Authors' Closure

The authors are pleased to respond to the comments of such distinguished colleagues.

It appears that behind Professor Horlock's and Dr. Hoadley's opening comments is a misunderstanding. If the blade defect forces vanish, it is true the theory reduces to that of Railly and Howard and Stratford, and there is then no need to consider the circumferential momentum integral equation. However, in our Introduction we have argued against this situation and, throughout the paper, argued for the necessity of retaining blade defect forces. Setting $\epsilon = 0$ in equation (24) exactly defines the (constant) direction of blade "body" force and does not imply that its scalar magnitude is constant.

In our paper we have made only a small reference to the vorticity-velocity description of the flow field—for which Professor Horlock is noted—in favor of the pressure-velocity description. It is probable that the ideas and concepts associated with both modes of thinking should be considered side by side. A result, for example, might be an improvement in equation (38) or its derivation.

The authors appreciate Professor Railly's written comments and also his oral comments presented at the Brussels meeting. These indicate that we share very similar views.

Our view of the role of the normal pressure gradient may be in agreement with that of Professor Railly, but we are not certain. It seems to us that the *mean* normal pressure gradient, $\partial\bar{p}/\partial y$, may be neglected based on the usual scaling arguments, but if one wished to tackle the full, three dimensional, blade-wall corner flow problem, then the local $\partial p/\partial y$ is certainly not negligible and, in fact, cannot be so if a blade defect force exists. In the local, corner flow problem derivatives like $\partial(\quad)/\partial z$ are of the same order as $\partial(\quad)/\partial y$ whereas in the mean they are not. Thus, we might argue, using what we believe to be a correct interpretation of Professor Railly's symbolism, that δp locally is $0(1)$ whereas the mean $\delta\bar{p}$ is $0(1) - 0(1) = 0(\delta)$.

Before proceeding to Dr. Smith's comments it might be useful to summarize the fact that our theory is essentially comprised of equations (19), (20), and (44*a*, *b*)—plus some analysis following the latter equations—which *are* relatively precise, physical statements, and, we believe, are not likely to be substantively amended. On the other hand, the theory also includes equations (24), (25*a*, *b*) and (38). These latter equations *represent* physical statements that must be included in a complete theory; however, the current statements are based on assumptions that are liable to substantive amendment. Dr. Smith is proposing such an amendment which we believe worthy of further development. Let us represent the span-wise variation of blade force in the boundary layer according to

$$f_x = \frac{f_{ex}}{f_{ez}}\,[1 - \phi(y)]f_z$$

where $\phi(y) \sim 0$ for large y. If $\phi \ll 1$ and $|\mathfrak{f}_e - \mathfrak{f}| \ll |\mathfrak{f}_e|$, then we can write

$$f_{ez}\left[\int_0^\delta (f_{ex} - f_x)dy - \frac{f_{ex}}{2}\int_0^\delta \phi dy\right] = f_{ex}\left[\int_0^\delta (f_{ez} - f_z)dy + \frac{f_{ez}}{2}\int_0^\delta \phi dy\right]$$

Analogous to equation (24) or its integral, equation (40), we have

$$f_{ez}\left[F_x - \frac{f_{ex}}{\rho \bar{V}_e^2}\,\delta_\phi\right] - f_{ex}\left[F_z + \frac{f_{ez}}{\rho \bar{V}_e^2}\,\delta_\phi\right] = 0 \qquad (40')$$

where $\delta_\phi \equiv \int_0^\delta \phi dy$ and $f_{ex}/f_{ez} \equiv -(1 - \epsilon)(\bar{w}_e/\bar{u}_e)$. According to our calculations, the results should not be sensitive to ϵ. (Dr. Smith is apparently mistaken. $\epsilon = 0.2$ *radians* is a fairly large angle; however, it represents *no variation* in blade force angle through the boundary layer. δ_ϕ now represents such a variation.)

Thus, (40′) is an amended defect blade force equation which, however, requires a determination of δ_ϕ.

In the case of reasonably flat compressor blades, a small value of $\bar{\phi} \equiv \delta_\phi/\delta^*$—to define a nondimensional, average, blade force, angle deviation in the boundary layer—would, it seems to us, require a relatively large redistribution of the chordwise blade pressure distribution within the boundary layer. In other words, we are suggesting that if we let $|\mathfrak{f}_e - \mathfrak{f}| = |\mathfrak{f}_e|0(\epsilon_f)$ and ϵ_b be the order of the angle formed by the blade surface and the chord line, the terms represented by F_x and F_z are $0(\epsilon_f)$, whereas the terms in question are $0(\epsilon_f \cdot \epsilon_b)$. The existence of a blunt nose and the classical potential flow, leading edge effect or, on the other hand, consideration of blade viscous stresses may, but properly will not, invalidate this argument.

Nevertheless Dr. Smith refers to data which, unfortunately, since this is being written on short notice by the first author while living in Moscow, is not readily at hand. However, in reference [14] it is recalled that the data was analyzed neglecting wall friction, whereas we point out here that in a repeating stage, wall friction and blade defect forces are of comparable magnitude. And, with $\delta_\phi = 0$, we have seen that it is indeed possible for the calculated, tangential blade force to be smaller in the boundary layer than in the main stream. In Fig. 8(*b*) (corrected somewhat from the original preprint) this is the case wherever F_x is positive.

In all of this our arguments are by no means conclusive, and the authors are indebted to Dr. Smith for raising this point.