Perturbative solution of the motion of an asteroid in resonance with Jupiter

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ABSTRACT
Using a new technique, we derive general formulae for the rate of change of orbital parameters of an asteroid with negligible mass, perturbed by a massive planet (such as Jupiter) having a commensurable period. The relative strength of the perturbing force defines the perturbation parameter, but our results are also expanded in terms of both eccentricities (of the perturbing and the perturbed body) and the mutual inclination. This approach can serve as a valuable tool in the ongoing study of the so-called stable chaos, and may facilitate its ultimate explanation.

Key words: chaos – celestial mechanics, stellar dynamics – minor planets, asteroids.

1 INTRODUCTION
In this paper, we investigate the motion of an asteroid near the 2/1 (\( r \) in general) resonance with Jupiter. Our approach complements and extends (by allowing for non-zero inclination) the traditional treatment of this problem, as presented by Wisdom (1985) and the references cited therein.

By our choice of units, the semi-major axis \( a \) of the asteroid will equal 1 at the exact resonance, and \( G M = 1 \). In these units, the semi-major axis of Jupiter must have, to a sufficient accuracy, the value of \( r^{2/3} \), where \( M \) is the mass of Jupiter, relative to \( M_{\odot} \).

2 BASIC ALGORITHM
The quaternion formulation of the corresponding perturbed Kepler problem (Vrbik 1995) is based on the following definition of the so-called modified time \( s \), related to regular time \( t \) by

\[
t = 2 \left[ a^{3/2} \left\{ 1 + \frac{2\beta}{1 + \beta^2} \cos[2(s - s_p)] \right\} \right] ds = 2(s - s_p) + \rho(\phi + \psi - \Psi) + a^{3/2} \frac{2\beta}{1 + \beta^2} \sin[2(s - s_p)],
\]

where

\[
\Psi = \phi + \psi - 2 \frac{(a^{3/2} - 1) ds + s_p}{\rho}
\]

is the so-called resonance variable, \( 2\beta/(1 + \beta^2) \) is the eccentricity of the asteroid, \( s_p \) is a value of \( s \) at aphelion, and \( \phi, \theta \) (not appearing in the previous formula) and \( \psi \) are the three Euler angles of the orbital attitude.

Note that (1) is correct only to first-order (in \( \epsilon \)) accuracy, sufficient for the purpose of this paper.

Using \( s \) as the independent variable, one can then build formulae for the \( s \) derivative of the orbital parameters \( a, \beta, s_p, \phi, \theta \) and \( \psi \) by computing the following.

(i) The perturbing force:

\[
f = \epsilon \left( \frac{R - r}{|R - r|^3} - \frac{R}{R^3} \right).
\]

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where
\[ r = R^+ \frac{t}{1 + \beta} \exp \left[ \frac{2i(s - s_p)}{1 + \beta} \right] + \beta^2 \exp \left[ -2i(s - s_p) \right] + 2\beta R \]
(3)
is the unperturbed orbit of the asteroid with attitude
\[ R = \frac{3}{2} \left[ (\cos \tau - \gamma) + j \sqrt{1 - \gamma^2 \sin \tau} \right] \]
defines the orbit of Jupiter, placed in the x–y plane of our inertial coordinates. Here, \( \tau \) is the solution to
\[ \frac{t}{\rho} = \tau - \gamma \sin \tau \]
and \( \gamma \) is the eccentricity of Jupiter.

(ii) The expressions
\[ Q(s) = \frac{-2a \text{C}x[Rf]R^+}{1 + \beta \exp[2i(s - s_p)]} \]
(6)
and
\[ W(s) = -4ar \text{C}x[Rf]R^+ \]
(7)
where Cx[...] implies keeping the real and i (i.e. complex) components of the argument (discarding the j and k parts).

(iii) The expressions
\[ a' = \frac{2a}{N\pi} \text{Im} \int_0^{N\pi} \left[ 1 - \beta \exp[2i(s - s_p)] \right] Q(s) \, ds, \]
(8)
\[ b' = -\frac{1 + \beta^2}{4N\pi} \text{Im} \int_0^{N\pi} \left[ \beta \exp[4i(s - s_p)] + 3 \exp[2i(s - s_p)] + 3\beta + \exp[-2i(s - s_p)] \right] Q(s) \, ds, \]
(9)
\[ Z_1 = -\frac{1}{N\pi} \text{Im} \int_0^{N\pi} \left\{ \frac{\beta}{1 - \beta^2} + \frac{1}{2} + \frac{\beta^2}{1 - \beta^2} \right\} \exp[-2i(s - s_p)] W(s) \, ds, \]
(10)
\[ Z_2 = -\frac{1}{4N\pi} \text{Re} \int_0^{N\pi} \exp[-2i(s - s_p)] W(s) \, ds, \]
(11)
\[ Z_3 = \frac{1}{4N\pi\beta} \text{Re} \int_0^{N\pi} \left[ \beta(1 + \beta^2) \exp[4i(s - s_p)] + (3 - \beta^2) \exp[2i(s - s_p)] + \beta(1 - 3\beta^2) - (1 + \beta^2) \exp[-2i(s - s_p)] \right] Q(s) \, ds \]
(12)
and
\[ s'_p = \frac{Z_3}{2} - \frac{1}{4N\pi(1 + \beta^2)} \text{Re} \int_0^{N\pi} \left[ \beta^4 \exp[4i(s - s_p)] + \beta(1 - 2\beta^2) \exp[2i(s - s_p)] - (1 + \beta^2 + 3\beta^2) \right. \]
\[ - \left. \beta(2 + \beta^2) \exp[-2i(s - s_p)] \right\} Q(s) \, ds, \]
(13)
where \( N \) is the numerator of \( \rho \), and \( Z_1, Z_2 \) and \( Z_3 \) represent components of the slow rotation of the orbit, expressed in its own ‘Kepler’ frame (the integration considers the orbital elements to be constant).

(iv) Finally,
\[ \Phi' = \phi' + \phi' - 2 \frac{a^{3/2} - 1 + s'_p}{\rho}, \]
(14)
\[ \phi' = \frac{Z_1 \sin\psi + Z_2 \cos\psi}{\sin\theta}, \]
(15)
\[ \theta' = Z_1 \cos\psi - Z_2 \sin\psi \]
(16)
and
\[ \psi' = Z_3 - \phi' \cos\theta. \]
(17)

We should reiterate that the above formulae, in their current form, produce only the first-order (ε-accurate) solution, and we should also mention that the ‘averaging principle’ has been invoked. The latter implies that terms with frequencies not matching an integer multiple of the asteroid motion have been eliminated from the right-hand sides of (8)–(13).
One should note (Vrbik 1995) that it is not very difficult to modify the procedure so as to include the suppressed (oscillating) terms, and verify that their effect on the long-term development of the solution is negligible; similarly, one can extend the procedure to achieve a higher order-of-$x$ accuracy (one should realize that, in that case, it would be necessary to keep the fractional-frequency terms, as they lead to non-oscillating contributions at the $x^2$ level).

In this paper, we confine ourselves to the first-order solution.

3 SMALL-PARAMETER APPROXIMATION

There are two ways of dealing with the $ds$ integration on the right-hand sides of equations (8), (9) and (14)–(17): it can be performed either numerically (the rectangular rule being the most accurate), or analytically, by first expanding the integrands in terms of all the other (in addition to $e$) small parameters, i.e. $\beta$, $\gamma$, $\theta$ and $\alpha - 1$. The latter approach represents yet another approximation, but is very helpful in elucidating the main features of the resulting solution and clarifying the role of the individual parameters.

The rather tedious algebra of the analytical approach is easily handled by the following Mathematica program.¹

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¹ Mathematica is the product of Wolfram Research Inc., 100 Trade Center Drive, Champaign, IL 61820, USA (Europe: 10 Blenheim Office Park, Lower Road, Long Hanborough, Oxfordshire OX8 8LN; Asia: Izumi Building 8F, 3-2-15 Misaki-cho, Chido-ku, Tokyo 101, Japan).
(i) The routine ‘fix’ expands its argument in terms of $\beta$, $\gamma$, $\theta$ and $a - 1$ up to and including terms proportional to $\beta' \gamma' \theta'(a - 1)^2$, where $2j + 2k + 2l \leq 2$. It can be applied to an expression, say $x$, in one of two different ways: ‘fix[$x$]’ or ‘$x$/fix’. Similarly, ‘omega’ solves the Kepler equation $\omega - \gamma \sin \omega = t$ for $\omega$ (up to and including the $\gamma'$ term), and ‘Olnt’ performs the $\Gamma(1 \pm \xi) \, ds$ integration in (8)–(13), using a more convenient substitute variable $\xi = 2(s - p)(1 - 1/p) + \Psi$. The only integration formulae are

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(\xi + \delta) \, d\xi}{(a^2 + 2a - 2a \cos \xi)\alpha} = a^\alpha \cos \delta \left( A^{n + m} \alpha \Gamma(\alpha, n) \right) \left( \alpha + n; n + 1; \frac{a^2}{A} \right),$$

$$\int_0^{2\pi} \frac{\cos \left( \frac{n}{m} \xi + \delta \right) \, d\xi}{(a^2 + 2a - 2a \cos \xi)\alpha} = 0 \tag{18}$$

and

$$\int_0^{2\pi} \cos \left( \frac{n}{m} \xi + \delta \right) \, d\xi = 0,$$

where $n$ and $m$ are relative primes (Gradshteyn & Ryzhik 1980). Finally, ‘clean’ evaluates the resulting coefficients numerically.

(ii) Quaternion quantities are represented in the following manner: ‘{', ‘}', ‘*’ and ‘/’ represent $\mathbf{R}$, $\mathbf{r}$ and $f$ converted to the Kepler frame of the perturbed body; ‘$r_{mag}$’ stands for $\mathbf{r}$, $\mathbf{Q}$ and $\mathbf{W}$ are now called ‘$Q$’ and ‘$W$’, and $a'$, $\beta'$, $s_{p'}$, $\phi'$, $\theta'$ and $\psi'$ have become ‘$Da'$’, ‘$D\beta'$’, ‘$D\theta'$’, ‘$D\phi'$’ and ‘$D\psi'$’.

(iii) The notations of the program follow quite closely the notation of our paper, with the following modifications: ‘$R_{Q}$’, ‘$r'$’ and ‘$f$’ represent $\mathbf{R}$, $\mathbf{r}$ and $f$ converted to the Kepler frame of the perturbed body; ‘$r_{mag}$’ stands for $\mathbf{r}$, $\mathbf{Q}$ and $\mathbf{W}$ are now called ‘$Q$’ and ‘$W$’, and $a'$, $\beta'$, $s_{p'}$, $\phi'$, $\theta'$ and $\psi'$ have become ‘$Da'$’, ‘$D\beta'$’, ‘$D\theta'$’, ‘$D\phi'$’ and ‘$D\psi'$’.

(iv) Note that two orders of the $\lambda$-accuracy are lost when computing $s_{p'}$ and $\phi'$ (owing to dividing by $\beta$ in line (26)); the program makes the corresponding adjustment in line (33), multiplying the affected quantities by $\lambda^2$. Similarly, one order of $\lambda$ is lost when computing $\phi'$ owing to the sin $\theta$ division in line (29).

### 4 RESULTING EQUATIONS

To simplify the presentation, we quote only the first few terms of the resulting expansions, first for the 2/1 case:

$$a' = -6.00e\beta \sin(2\Psi) - 1.079e\gamma \sin(2\Psi - \phi - \psi) - 1.033e\beta' \sin(4\Psi - 2\psi) + \ldots, \tag{19}$$

$$\beta' = 0.75e\gamma \sin(2\Psi) + 4.273e\beta \sin(4\Psi) + 2.826e(a - 1) \sin(2\Psi) + e\gamma [0.363 \sin(\phi + \psi) + 3.129 \sin(4\Psi - \phi - \psi)] + e\beta' [0.497 \sin(6\Psi - 2\phi) - 0.551 \sin(2\Psi - 2\psi) - 1.704 \sin(2\Psi)] + \ldots, \tag{20}$$

$$\phi' = \frac{0.75e\cos(2\Psi)}{\beta} + 1.954e + 4.273e \cos(4\Psi) - 0.516e \cos(4\Psi - 2\phi) + e\gamma [0.363 \cos(\phi + \psi) + 3.129 \cos(4\Psi - \phi - \psi)] + 2.826e(a - 1) + e\beta' [0.497 \cos(6\Psi - 2\phi) + 0.551 \cos(2\Psi - 2\psi) - 1.704 \cos(2\Psi)] + \ldots, \tag{21}$$

$$2\psi' \equiv \psi' - 1.197e + 0.516e \cos(4\Psi - 2\phi) - 2(a^{3/2} - 1) + \ldots, \tag{22}$$

$$\phi' + \phi' \equiv \phi' + 0.516e \cos(4\Psi - 2\phi) - 0.977e + \ldots, \tag{23}$$

and

$$\theta' \equiv \frac{0.516e}{\beta} \sin(4\Psi - 2\phi) + \ldots \tag{24}$$

In their current form, equations (19)–(24) are useful mainly for exploring qualitative features of the corresponding solution; to achieve high-accuracy results, one would have to include higher order terms in ($\beta$, $\gamma$, $\theta$, and $a - 1$), as the corresponding series converge rather slowly (especially in $\beta$ and $\gamma$). An interested reader can easily modify our MATHEMATICA program, to obtain such an extension of the results presented here.

One can show both analytically (Vrbik 1996) and by the corresponding numerical simulation that, when $\gamma$ and $\theta$ are equal to zero, all solutions to the above differential equations are periodic (this remains true to any order-of-$\beta$ accuracy). This means that, by finding an accurate solution for a single cycle, we have solved the problem for all times.

The situation changes dramatically as soon as either $\gamma$ or $\theta$ becomes non-zero (we know this to be the actual case – the eccentricity of Jupiter equals 0.0485). Each solution then becomes rather irregular, and highly sensitive to initial conditions. For some initial conditions, the solution even qualifies as chaotic, even though of a somehow weaker variety (the so-called stable chaos of Milani, Nobili & Knezevic 1997). We hope that some further study of the above (and similar) equations can eventually lead to a proper explanation of this interesting phenomenon.

By changing the value of $\rho$ in line (16) of our MATHEMATICA program, one can similarly generate the corresponding set of differential
equations for a resonance of any order. Since there is no need to clutter this article with a long list of these, we just mention that the results of any \( r/(r - 1) \) resonance are quite similar to those obtained in the 2/1 case – only numerical values of the individual coefficients differ; similarly, all \( r/(r - 2) \) resonances are alike, etc.

For illustration, we quote the leading terms obtained for the 5/2 resonance (note that it was necessary to modify line [16] of our MATHEMATICA program correspondingly, and to increase the order of its approximation – lines [2] and [33]):

\[
\frac{a'}{e} = -39.362 \beta^3 \sin(5\Psi) - 98.78 \gamma \beta^2 \sin(5\Psi - \phi) - 2.799 \beta \theta^2 \sin(5\Psi - 2\psi) - 2.703 \gamma \theta^2 \sin(5\Psi - 3\psi - \phi) + \ldots,
\]

\[
\frac{b'}{e} = 0.151 \gamma \sin(\phi + \psi) + 7.380 \beta^2 \sin(5\Psi) + 12.347 \gamma \beta \sin(5\Psi - \phi - \psi) + 0.175 \theta^2 \sin(5\Psi - 2\psi) + \ldots,
\]

\[
\frac{\psi'}{e} = 0.928 + 0.151 \gamma \frac{\cos(\phi + \psi)}{\beta} + 7.380 \beta \cos(5\Psi) - 1.399 \beta \cos(5\Psi - 2\psi) + 4.258(a - 1) + 0.175 \theta^2 \frac{\cos(5\Psi - 2\psi)}{\beta} + \ldots,
\]

\[
\frac{(\phi + \psi')}{e} = 0.464 + 0.151 \gamma \frac{\cos(\phi + \psi)}{\beta} + 7.380 \beta \cos(5\Psi) + 2.129(a - 1) + 0.175 \theta^2 \frac{\cos(5\Psi - 2\psi)}{\beta} + \ldots,
\]

\[
5\Psi' = 1.156e + 0.453e\gamma \frac{\cos(\phi + \psi)}{\beta} + 22.141e\beta \cos(5\Psi) + 5.459e(a - 1) + 0.525e \theta^2 \frac{\cos(5\Psi - 2\psi)}{\beta} - 4(a^{1/2} - 1) + \ldots,
\]

\[
\frac{\theta'}{e} = 1.399 \theta \beta \sin(5\Psi - 2\psi) + 1.352 \theta \gamma \sin(5\Psi - 3\psi - \phi) + \ldots.
\]

Note that, in this case, the \( \gamma \)-proportional 'chaotic' terms dominate over the regular part of the solution [this is a general trend: for an \( r/(r - q) \) resonance, exponents of the leading \( \beta \) terms will increase with \( q \)]. One is thus led to uncover many classes of interesting dynamical systems, the systematic investigation of which goes beyond the aim of this paper.

REFERENCES

Wisdom J., 1985, Icarus, 63, 272

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