Here κ_1 and κ_2 are the principal curvatures in mutually perpendicular directions tangent to the surface, and R_1 and R_2 are the corresponding radii of curvature. Twice the mean curvature, $2H$, is the trace of the curvature tensor, $\mathbf{b}$ (i.e., the first invariant of the curvature tensor [7]), and the surface divergence of the field of normals, $\mathbf{N}$, to the surface [8] (the minus sign appears by convention):

$$2H \equiv \text{trace } \mathbf{b} = -\nabla_{II} \cdot \mathbf{N}. \qquad \text{(D-3)}$$

In cartesian coordinates the expression for $2H$ is *not* as in equation (7); rather, it is [6, 7]

$$2H = \nabla_2 \cdot \left[\frac{\nabla_2 z}{W}\right] = \frac{\partial}{\partial x}\left(\frac{z_x}{W}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{W}\right) \qquad \text{(D-4)}$$

i.e.,

$$2H = \frac{z_{xx}[1 + z_y{}^2] - 2z_x z_y z_{xy} + z_{yy}[1 + z_x{}^2]}{[1 + z_x{}^2 + z_y{}^2]^{3/2}} \qquad \text{(D-5)}$$

where

$$W = [1 + z_x{}^2 + z_y{}^2]^{1/2} \qquad \text{(D-6)}$$

$$\nabla_2 = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}. \qquad \text{(D-7)}$$

In polar coordinates, which may be appropriate for analyzing menisci in triangular grooves, the correct expression is

$$2H = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{rz_r}{(1 + z_r{}^2 + r^{-2}z_\theta{}^2)^{1/2}}\right] + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[\frac{z_\theta}{(1 + z_r{}^2 + r^{-2}z_\theta{}^2)^{1/2}}\right]. \qquad \text{(D-8)}$$

The contact angle is defined as the angle between respective normals to two interfaces at a point on the contact line in which they intersect [9]. If $\mathbf{n}$ is the normal to the solid surface, then the angle it makes with the normal to the interface $z(x, y)$ is given not by equation (8) but by

$$\cos\alpha = \mathbf{n} \cdot \mathbf{N} = \mathbf{n} \cdot \frac{\nabla_2 z}{W} = \frac{z_x \sin\phi - z_y \cos\phi}{(1 + z_x{}^2 + z_y{}^2)^{1/2}}. \qquad \text{(D-9)}$$

At corners, including the vertex of a triangular groove, the contact angle is undefined but may be set equal to the contact angle made by the meniscus with each of two converging walls of identical wetting character, provided the meniscus actually reaches the corner between them. It is not generally true that $z_y = -\cot\alpha$ at the vertex of a triangular groove; rather, $z_y = -\cos\alpha/(\cos^2\phi - \cos^2\alpha)^{1/2}$, if the meniscus reaches the vertex. Furthermore, in a corner of angular opening 2β, a bounded solution of the Laplace-Young equation exists if

$$\alpha + \beta \geq \frac{\pi}{2}$$

but otherwise the solution is either unbounded or fails to exist. This was pointed out by Tyupsov, according to Petrov and Chernous'ko [2], who verified the fact by numerical computation. It was proved theoretically by Concus and Finn [10]. An early study of the corner meniscus was reported by Ferguson and Vogel [11]. A detailed treatment appeared recently [12]. All of these papers are relevant to the present problem.

Collective experience with numerical solutions of the Laplace-Young equation in three dimensions is still so limited that reports of new solutions should include details of the computational procedure and error studies.

In non-isothermal systems there is of course an added complication in that surface tension varies significantly with temperature. (Contact angle also depends on temperature, although only weakly in many cases [3].) Although this can be accommodated in the Laplace-Young equation, there are temperature fields in which it is impossible to establish a fluid interface at mechanical equilibrium: Flows driven by surface-tension gradients are bound to occur [13].

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Authors' Closure

The authors agree completely with the discussers that equation (7) of the paper is mathematically not correct, and that this point needs further clarification. In the derivation of equation (7), the curvatures were basically developed from intersections of the interfaces with planes which were parallel to the z-axis. Instead, a correct derivation should have been based on planes normal to the liquid–vapor interfaces, as outlined in the discussion by Pujado and Scriven, as well as in earlier analyses of related problems.

Since the angles between the planes normal to the surface and those parallel to the z-axis were usually small in the examined cases, the actual differences between curvatures computed from equation (D-5) of the discussion and from the simulating equation (7) of the paper were small enough to be neglected for the purpose of this study. Results from numerical evaluations of equation (7) have been compared with optical observations and photographs of menisci in capillary grooves on vertical planes. In particular, a "fully-wetted" height and other characteristic points have been experimentally located and measured for different fluids and surface materials. In addition, tests were made with transparent walls, colored fluids, and solidified interfaces to facilitate a better observation. Comparison of all available data showed fair agreement between the real interfaces and those computed from equation (7) of the paper. Supplementary details have been reported in *Journal of Basic Engineering*, Trans. ASME, Series D, Vol. 93, No. 1, March 1971, pp. 87–89.

An Analytical Investigation of Free Convection Heat Transfer to Supercritical Water[1]

R. J. Simoneau[2] and R. C. Hendricks.[2] The discussers agree that the authors' work is a logical extension of the work of Fritsch and Grosh [5] and should be done. The authors are to be com-

[1] By E. S. Nowak and A. K. Konanur, published in the August, 1970, issue of the Journal of Heat Transfer, Trans. ASME, Series C, Vol. 92, No. 3, pp. 345–350.

[2] Aerospace Research Engineer, NASA–Lewis Research Center, Cleveland, Ohio.

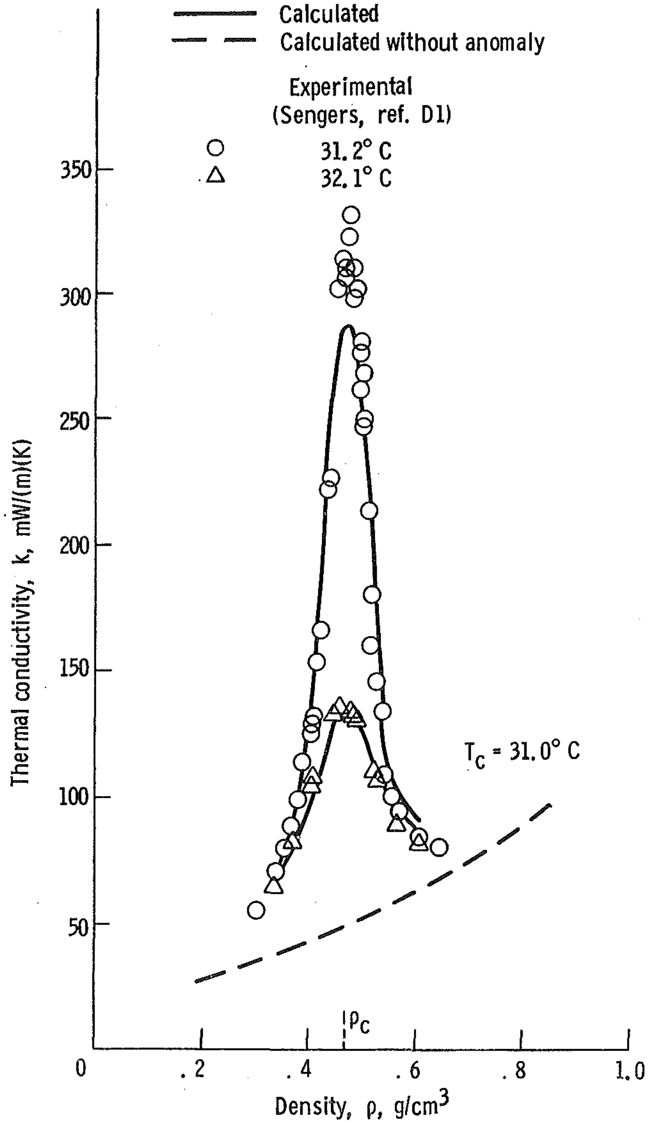

Fig. 1 Calculated and experimental thermal conductivities of carbon dioxide, from reference [18].

mended for basing their conclusions on the complete variable property results even though the partial variable property computations, using a free stream reference temperature, came closer to the data. The discussers agree, in general, with the authors' conclusions; however, they feel the paper could benefit from an expanded discussion in some areas. Specifically they would like to raise two questions.

First, have the authors made any attempt to take into account the anomalous spike that occurs in thermal conductivity near the critical point? The measurements of Sengers [13] in carbon dioxide show convincingly that the thermal conductivity peaks sharply in the near critical region. Subsequent measurements in argon [14], ammonia [15], methane [16], and hydrogen [17] have pretty well established the general occurrence of the phenomenon. While it has not to our knowledge been shown, there is no reason to believe the phenomenon does not occur in water. The absence of data, however, requires that the spike be computed—an admittedly difficult task. Brokaw [18] has successfully computed Sengers' data by treating the fluid as a dissociating polymer and the results are shown in Fig. 1. The computation requires an equation of state with accurate derivatives. For several fluids a recent equation of state by Bender [19] has been successfully employed in computing the spike [20], using Brokaw's theory. The discussers have no experience with water.

Very recently Sengers and Keyes [21] have published a scaling relationship for excess conductivity using Sengers' CO_2 data [13].

In the opinion of the discussers, if the influence of the anomalous spike is going to be important, it will be most important in free convection. Here the fluid dynamics are governed by the thermodynamics and the system is free to adjust to changes in thermal properties.

The second question is whether any computations were carried out with the free stream temperature below the transposed critical temperature? Since Fritsch and Grosh [3] report data with free stream temperature below the transposed critical temperature, it would be interesting to see the results of the partial variable property computations under these circumstances. The influence of the reference temperature might become clearer.

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Authors' Closure

The senior author (E.S.N.) would like to thank the discussers for their review and thought-provoking remarks.

The authors have made no attempt to take into account the so-called anomalous spike in the thermal conductivity near the critical point, the reason for this being that as of this writing the spike in thermal conductivity has not been conclusively demonstrated for the case of steam. However, if the spike in fact exists for steam then the authors concur with the discussers that it could have a significant effect on free convection heat transfer.

The authors are initiating a study in which the plate temperature is higher than the transposed critical temperature and the free stream temperature is lower than the transposed critical temperature. The discussers are correct in stating that the influence of the reference temperature might become clearer.