Shock Propagation in Inhomogeneous Gases. V

Oblique Shock Propagation

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The quasi-stationary method developed by Ōno et al. is generalized to the case of oblique shock propagation, that is, when a shock propagates along directions making a finite angle with that of pressure gradient in stratifying media. Two simultaneous differential equations determining the direction and strength of shock are obtained. Its behavior is discussed in detail for polytropic gases. It is characteristic for the oblique propagation that a pattern change, i.e. transition from regular to Mach pattern, occurs ultimately, and a vortex field generates after the passage of the shock.

§ 1. Introduction

Concerning the problem of shock propagation in inhomogeneous media, one of the authors Y. Ōno and his collaborators developed a general method by extending Chisnell's to the case in which the medium is in the gravitational equilibrium, and they applied it to the problem of supernova explosion and planetary nebulae. The method, however, can be applied only to the normal shock propagation. Namely, when the normal of the shock discontinuity surface is parallel to the gradient of the initial distribution of physical quantities.

The problem of oblique shocks have almost been limited to that of reflection by a rigid wall, except that Polacheck and Seeger dealt with the reflection or refraction of a shock wave at a contact surface of gaseous media. Hitherto, no theory of oblique shock propagation in inhomogeneous media has been considered.

Meanwhile, as was pointed out by Osterblock, heating of solar corona depends on a possible mode change of waves by which the energy carried up to the lower chromosphere can be transferred into the upper-atmosphere. It may naturally be expected that this mode change might be caused by the collision of a shock wave with shocks or other discontinuities. Therefore, it is interesting to consider the oblique shock propagation in inhomogeneous media.

In this paper, using the method developed in paper I, we consider the

*) This problem is now under consideration by M. Saitō, who takes account of the effect of the oblique propagation into shock pulses.
oblique propagation of shock waves in inhomogeneous media without magnetic field. Since the fundamental idea of treating inhomogeneous medium is to regard it as stratified, any method, which solves the reflection and refraction of a shock wave between two gaseous media, can be applied to the case of shock propagation. The problem was treated by Polacheck et al., but their way of analysis is not convenient to extend to the inhomogeneous media. Therefore, in §2, we first deal with the reflection and refraction of a shock wave at an interface of two gases from a new point of view. In this way, equations are derived, which are compared with and shown to be equal to those of Polacheck et al. in Appendix. In §3, two simultaneous differential equations which determine the change of the strength and direction of shock wave with pressure or distance, are obtained in a general form by using the results of §2. Then in §4, we apply these to polytropic gases without a magnetic field and radiation. The behaviours of shock strength and direction are discussed in detail. The vorticity distribution generated after passage of a shock wave, which is characteristic of oblique propagation, is considered in §5. Astrophysical application of this method will be discussed in other papers.

§2. Interaction of a shock wave with a finite discontinuity

As was done by Polacheck et al. in their paper, we here consider a stationary shock wave traveling in a uniform gaseous medium and impinging at an arbitrary angle upon a second medium having different physical characteristics. There occur two alternative wave patterns as the result of interaction with a finite discontinuity, i.e. a triple shock-configuration and a rarefaction configuration which consists of two confluent shocks and an angular rarefaction wave. We here neglect the formulation for this latter configuration, since it is not necessary to distinguish their difference in the case of infinitesimal discontinuity, and since the object of this paper is not to rediscuss the details of refraction patterns at a gaseous interface.

In Fig. 1, we show the discontinuities generated when a shock wave of strength $z_{12} = p_2/p_1$ propagates with the direction $\mu_{12}$ in a homogeneous medium (1), and impinges on a contact discontinuity D between the gases 1 and 5, where normal unit vector is denoted by $\mathbf{n}$. A refracted shock $z_{34}$ with direction $\mu_{34}$ and a reflected wave $z_{23}$ with $\mu_{23}$ are then generated. Moreover, a tangential discontinuity between the two gases (3 and 4) is generated with direction $\mu_{34}$. Here,
and so on for other discontinuities. Other notations are the same as in the previous paper I.

Now, we derive the relations connecting physical quantities of the state between those discontinuities. Flow velocity $U_2$ behind the shock and the propagation velocity $U_{12}$ are expressed by the shock strength and the physical quantities of the state before the shock as follows (between 2 and 1 for example):

$$U_2 = U_1 + \phi (12 : 1) n_{12}$$

and

$$U_{12} = U_1 + \psi (12 : 1) n_{12}$$

where

$$\phi (12 : 1) = \phi (z_{12}, y_{12}, p_1, \tau_1) = \sqrt{p_1 \tau_1 (z_{12} - 1)(1 - y_{12})}$$

and

$$\psi (12 : 1) = \psi (z_{12}, y_{12}, p_1, \tau_1) = \sqrt{p_1 \tau_1 (z_{12} - 1)(1 - y_{12})}.$$  \hspace{1cm} (2.1b)

$y_{12}$ is the ratio of specific volumes $\tau_2/\tau_1$ behind and before the shock wave. Among $z_{12}$, $y_{12}$ and the physical quantities of the state 1 before the shock there exists a relation,

$$\phi (z_{12}, y_{12}, p_1, \tau_1, ...) = 0,$$  \hspace{1cm} (2.3)

which is a general form of the Rankine-Hugoniot relation.

Now, the conditions of the dynamical equilibrium must be applied between two media (no gravitation) as

$$p_1 = p_5$$  \hspace{1cm} (2.4a)

and

$$p_5 = p_4.$$  \hspace{1cm} (2.4b)

From these equations we get

$$z_{12} x_{12} = x_{14}.$$  \hspace{1cm} (2.5)

Next, the condition of the contact of two media is written as

$$U_1 = U_4$$  \hspace{1cm} (2.6a)

and

$$U_3 \cdot n_{34} = U_4 \cdot n_{41}.$$  \hspace{1cm} (2.6b)

By using (2.1a) and (2.1b), we obtain after all the following expression:

$$\phi (12 : 1) (\mu_{13} \mu_{14} + \sqrt{1 - \mu_{12}^2} \sqrt{1 - \mu_{41}^2}) + \phi (23 : 2) (\mu_{23} \mu_{24} + \sqrt{1 - \mu_{23}^2} \sqrt{1 - \mu_{43}^2})$$

$$= \phi (54 : 5) (\mu_{54} \mu_{51} + \sqrt{1 - \mu_{54}^2} \sqrt{1 - \mu_{51}^2}).$$  \hspace{1cm} (2.7)
Our problem is now to know the five quantities $z_{23}$, $\mu_{23}$, $z_{54}$, $\mu_{54}$ and $\mu_{43}$, when $z_{12}$, $\mu_{12}$ and the state 1 and 5 are given. Besides the two equations (2.5) and (2.7), we then need three more equations. These can be derived when laws of reflection and refraction are used in the following way. The propagation velocities of the three shocks and the tangential discontinuity are given respectively by

$$
\frac{(U_{12} - U_{i}) \cdot n_{12}}{\sqrt{1 - \mu_{12}^2}} = \psi(12 : 1) \frac{1}{\sqrt{1 - \mu_{12}^2}},
$$

$$
\frac{(U_{23} - U_{i}) \cdot n_{23}}{\sqrt{1 - \mu_{23}^2}} = \psi(23 : 2) + \phi(12 : 1) \frac{(n_{12} \cdot n_{23})}{\sqrt{1 - \mu_{23}^2}},
$$

$$
\frac{(U_{43} - U_{i}) \cdot n_{43}}{\sqrt{1 - \mu_{43}^2}} = \psi(54 : 5) \frac{1}{\sqrt{1 - \mu_{43}^2}},
$$

and

$$
\frac{(U_{i} - U_{i}) \cdot n_{34}}{\sqrt{1 - \mu_{34}^2}} = \psi(54 : 5) \frac{(\mu_{54} \mu_{34} + \sqrt{1 - \mu_{34}^2} \sqrt{1 - \mu_{54}^2})}{\sqrt{1 - \mu_{34}^2}}.
$$

Since the $t$-components (see Fig. 1) of the propagation velocities are equal to one another, we get

$$
\frac{\psi(12 : 1)}{\sqrt{1 - \mu_{12}^2}} = \frac{\psi(54 : 5)}{\sqrt{1 - \mu_{54}^2}} \quad (2.8)
$$

$$
= \frac{\psi(23 : 2) + \phi(12 : 1)(\mu_{12} \mu_{23} + \sqrt{1 - \mu_{32}^2} \sqrt{1 - \mu_{23}^2})}{\sqrt{1 - \mu_{23}^2}} \quad (2.9)
$$

$$
= \frac{\psi(54 : 5)(\mu_{54} \mu_{34} + \sqrt{1 - \mu_{34}^2} \sqrt{1 - \mu_{54}^2})}{\sqrt{1 - \mu_{34}^2}} \quad (2.10)
$$

which are the required equations.

The five equations obtained above are necessary and sufficient for solving the problem of shock reflection and refraction at a contact discontinuity.

Incidentally, it is deduced from the above equations that, in contrast to sound waves, the total reflection of shock waves cannot occur. In fact, when we put $\mu_{43} = 0$ in the right-hand member of (2.8) and (2.10), the relation

$$
\phi(54 : 5) = \psi(54 : 5)
$$

is obtained in general with $\mu_{43} = 0$. By (2.1b) and (2.2b), this relation means $y = 0$, and of course, is not permitted physically. On the other hand, for sound waves $z_{12} \rightarrow 1$, $\mu_{43}$ becomes unity, and this argument is not applied.

Polachek et al. discussed the problem from the stationary point of view of an observer traveling with triple point (which is the intersection point of the incident shock and the contact surface). We can show the equivalence of our formulae with theirs by the coordinate transformation (see Appendix).
§ 3. Shock propagation in inhomogeneous media (general formulation)

Consider the shock propagation in inhomogeneous media. The media are regarded as stratified by the homogeneous layers with infinitesimal thickness and neighboring layers are supposed to have infinitesimally different physical quantities from each other.

Then we can use the method in the preceding section, considering a fact that the corresponding physical quantities in the states 1 and 5 differ infinitesimally (the ratio of specific heats \( \gamma \) being assumed as constant). The pressure conditions (2\cdot 4a) and (2\cdot 4b), however, must be modified so as to include the effect of (gravitational) force as was done in paper I. Taking into account the inclination of the tangential discontinuity behind the shock front, we can write the condition as

\[ p_5 - p_1 = -f(x) \Delta m \] (3\cdot 1a)

and

\[ \frac{p_1 - p_5}{\mu_{34}} = -f(x) \Delta m. \] (3\cdot 1b)

The equation (3\cdot 1b) is obtained from the condition of balance between the gravitational force and the \( n \)-component of the pressure gradient of the disturbed region. From these we get

\[ z_{23} = z_{13} + \frac{\beta - \mu_{34}}{p + \Delta p} \frac{dp}{p + \Delta p} = z_{34}, \] (3\cdot 2)

instead of (2\cdot 5). Now, in this case, the difference of the strengths between incident and transmitted shocks must be infinitesimal, i.e.

\[ z_{34} = z_{13} + dz \] (3\cdot 3)

and

\[ y_{34} = y_{13} + dy. \]

Therefore, from Eq. (3\cdot 3), the strength of the reflected wave \( z_{23} \) becomes

\[ z_{23} = 1 + \left( 1 - \frac{\mu_{34}}{z} \right) \frac{dp}{p} + \frac{dz}{z}. \] (3\cdot 4)

Introducing (3\cdot 3) into (2\cdot 8), and taking only the first order terms, we have

\[ \frac{dz}{z-1} + \frac{dy}{1-y} + \frac{dp}{p} + \frac{dt}{\tau} + \frac{2\mu}{1-\mu^2} d\mu = 0. \] (3\cdot 5)

From (2\cdot 7), we get after a lengthy calculation the following first order equation:
\[
\frac{dy}{1-y} = \frac{\mu^2 - y(1-\mu^2)}{\mu^2 + y(1-\mu^2)} \left( \frac{dp}{p} + \frac{dz}{z-1} \right) - \frac{2\mu R}{\mu^2 + y(1-\mu^2)} \times \left\{ \frac{dz}{z} + \frac{dp}{p} \left( 1 - \frac{\mu^2 + y(1-\mu^2)}{zV\mu^2 + y^2(1-\mu^2)} \right) \right\}
\]

with

\[
R = \sqrt{\left( \frac{\phi_z}{\phi_y} \right) \frac{zy}{(z-1)(1-y)}} \times \sqrt{\mu^2 + y^2(1-\mu^2) - (1-\mu^2) \left( \frac{\phi_y}{\phi_z} \right) \frac{(1-y)yz}{z-1}}.
\]

Here, \((\phi_y)_{y=1}\) means \((\partial \phi_y / \partial y)_{y=1}\). Taking total differentiation of the general Rankine-Hugoniot relation (2.3) and multiplying this by \((1/y) \phi_y\), we obtain

\[
\frac{dz}{d \ln p} \frac{1}{1-y} \frac{\phi_z}{\phi_y} + \frac{1}{\phi_y(1-y)} \left\{ \rho \phi_p - \frac{1}{k} \tau \phi_y \right\} + \frac{1}{1-y} \frac{dy}{d \ln p} = 0,
\]

where we used the polytropic relation \(p \propto \rho^\gamma\) between the pressure and density of the undisturbed medium. Then, by using (3.6), we get after all

\[
\frac{dz}{d \ln p} \left\{ \frac{1}{z-1} \frac{\mu^2 - y(1-\mu^2)}{\mu^2 + y(1-\mu^2)} + \frac{1}{1-y} \frac{\phi_z}{\phi_y} + \frac{2\mu R}{\mu^2 + y(1-\mu^2)} \right\} \times \frac{1}{\phi_y(1-y)} \left( \rho \phi_p - \tau \phi_y \right) = \frac{1}{k} \left( \frac{d \ln p}{dz} \right).
\]

This equation is a generalization of Eq. (3.7) in reference 1) to the oblique shock propagation. In the oblique propagation, we further need the equation determining the angle of the transmitted shock. Dividing (3.5) by \(d \ln p\) and substituting (3.6) into (3.5), we get

\[
- \frac{d \mu}{d \ln p} = \frac{1}{\mu^2 + y(1-\mu^2)} \left\{ \left( \frac{\mu}{z-1} + \frac{R}{z} \right) \frac{dz}{d \ln p} + R \left( 1 - \frac{1}{k} \right) \frac{\mu^2 + y(1-\mu^2)}{zV\mu^2 + y^2(1-\mu^2)} \right\}.
\]

With Eqs. (3.8) and (3.9) thus obtained, the propagation of oblique shocks in arbitrary stratified matter can be dealt with in the first approximation. Characteristics of the media are contained in the Rankine-Hugoniot relation (2.3). When the radiative effect is important, it can also be included in that relation.*

We give here some remarks on the total reflection and refraction in inhomogeneous media. As shown in the preceding section, the total reflection

* The case of oblique magnetic shocks is rather complicated. This is now under investigation by Yamazaki et al. in our institute.
can not occur in the case of a finite discontinuity. This is also true in the case of inhomogeneous media since the same equation (2·8) and (2·9) hold. If we define the total refraction by \( z_{23} = 1 \) as Polacheck et al., equation (3·4) is reduced to

\[
\left( z - \frac{\mu^2 + y(1-\mu^2)}{\sqrt{\mu^2 + y'(1-\mu^2)}} \right) d\ln p + dz = 0. \tag{3·10}
\]

Equations (3·8) and (3·10) have a solution only if the determinant of their coefficients becomes zero, but it can be proved after some calculation that this is not the case for the propagation \( z = z_{12} > 1 \).

This proof can be applied to the total refraction through all layers. Therefore, in the inhomogeneous media the total refraction and reflection of shock waves can not occur within regular pattern.

Finally, it is noted that although here we take the "plane" shock, i.e., the shock having infinite width, it should be understood as that with finite width, for example, of the order of scale height.

§ 4. Oblique shock propagation in ideal gases

In this section, we investigate the behavior of oblique shocks in ideal gases without the effect of radiation. For this case, the Rankine-Hugoniot relation (2·3) is given by

\[
\Phi(y, z) = y - \frac{1 + \lambda^2}{z + \lambda^2} = 0, \tag{4·1a}
\]

for adiabatic shocks, and

\[
\Phi(y, z) = y - \frac{1}{z} = 0, \tag{4·1b}
\]

for isothermal shocks. Here, we take the case of adiabatic shocks.

In the first place, putting (4·1a) and the polytropic relation in Eq. (3·5) and integrating them, we get

\[
\left( \frac{p}{p_0} \right)^{(k-1)/k} \frac{z + \lambda^2}{z_0 + \lambda^2} = \frac{1 - \mu^2}{1 - \mu_0^2}, \tag{4·2}
\]

among the pressure \( p \), shock strength \( z \) and the cosine of angle. The suffix "0" denotes their initial values. Thus, in general, the variation of angle depends on both quantities \( z \) and \( p \), except for the case of isothermal layer \( k = 1 \). In this case, the angle increases when shocks grow. In other cases, the increase or the decrease of the angle depends on whether the left-hand term of (4·2) is larger than unity or not.

The behavior of \( z \) or \( \mu \) with \( p \) is represented by the equation
\[ - \frac{dz}{d \ln p} = \left[ \frac{\mu^2 - 1 + \lambda^2 z}{z + \lambda^2} (1 - \mu^2) \right] \left( 1 - \frac{1}{k} \right) \\
+ \frac{2\mu(z + \lambda^2)R}{\mu^2(z + \lambda^2) + (1 + \lambda^2 z)(1 - \mu^2)} \left[ \frac{1 - \frac{1}{z} \sqrt{\mu^2(z + \lambda^2)^2 + (1 - \mu^2)(1 + \lambda^2 z)^2}}{z - 1 - \frac{2\mu^2(z + \lambda^2) + (1 + \lambda^2 z)(1 - \mu^2)}{(z + \lambda^2)^2}} \right] \times \left[ \frac{2\mu^2}{z - 1} - \frac{\mu^2(z + \lambda^2) + (1 + \lambda^2 z)(1 - \mu^2)}{(z + \lambda^2)^2} + 2\mu R \right]^{-1} \\
\text{(4.3)} \]

with

\[ R = \frac{1}{z - 1} \left\{ \frac{(1 + \lambda^2 z)}{(1 + \lambda^2)(z + \lambda^2)} \right\}^{1/2} \left\{ \frac{\mu^2 - (1 - \mu^2) \frac{(z - 1)}{(z + \lambda^2)^2}}{(z + \lambda^2)^2} \right\} \]

or

\[ - \frac{d\mu}{d \ln p} = \frac{(1 - \mu^2)(z + \lambda^2)}{z - 1} \left\{ \frac{\mu + R}{z} \right\} \frac{dz}{d \ln p} \\
+ R \left( 1 - \frac{1}{z} \sqrt{\mu^2(z + \lambda^2)^2 + (1 - \mu^2)(1 + \lambda^2 z)^2} \right) + \mu \left( 1 - \frac{1}{k} \right) \right\} , \quad \text{(4.4)} \]

which are obtained by (4.1a) and (3.8) or (3.9). Note that (4.2) is reduced to Eq. (3.7) of paper I for normal propagation when \( \mu = 1 \).

Now, we consider the asymptotic case of strong shock \( z \gg 1 \). The equations (4.3) and (4.4) can be written in the form

\[ - \frac{d \ln z}{d \ln p} = 2\mu \sqrt{\frac{\lambda^2}{1 + \lambda^2}} + \frac{k - 1}{k} \sqrt{\mu^2 - \lambda^2(1 - \mu^2)} / 2\mu \sqrt{\frac{\lambda^2}{1 + \lambda^2}} + \sqrt{\mu^2 - \lambda^2(1 - \mu^2)} \]

and

\[ - \frac{d\mu}{d \ln p} = (1 - \mu^2) \sqrt{\frac{\lambda^2}{1 + \lambda^2}} - \frac{1}{k} / 2\mu \sqrt{\frac{\lambda^2}{1 + \lambda^2}} + \sqrt{\mu^2 - \lambda^2(1 - \mu^2)} . \]

These equations can be integrated in explicit form in terms of \( \mu \) as follows:

\[ \left( \frac{p}{p_0} \right)^{1/k} = f_1(\mu) \]

and

\[ \left( \frac{z}{z_0} \right)^{-1/1} = f_2(\mu) \]

where \( f_1(\mu) \) and \( f_2(\mu) \) are given by

\[ f_1(\mu) = \frac{1}{1 - \mu^2} \left( \mu \sqrt{1 + \lambda^2} - \sqrt{1 - \lambda^2} (1 - \mu^2) \right)^{(1 + \lambda^2)/2Y^2} \]

\[ \times \left( \mu + \sqrt{1 - \lambda^2} (1 - \mu^2) \right)^{Y(1 + \lambda^2)/2Y^2} \]

\[ f_2(\mu) = \frac{1}{1 - \mu^2} \left( \mu \sqrt{1 + \lambda^2} - \sqrt{1 - \lambda^2} (1 - \mu^2) \right)^{(1 + \lambda^2)/2Y^2} \]

\[ \times \left( \mu - \sqrt{1 - \lambda^2} (1 - \mu^2) \right)^{Y(1 + \lambda^2)/2Y^2} \]
and

$$f_z(\mu) = \{ f_z(\mu) \}^{(k-1)/k} (1 - \mu^2)^{-1/k}$$

respectively. Note that the angle always increases as $p$ decreases because the right-hand term of $(4 \cdot 4a)$ is positive irrespective of $k$.

Next, when the strength of the shock is infinitesimal, i.e. $z$ nearly equals to unity, Equations (4 \cdot 3) and (4 \cdot 4) can be integrated in the explicit forms

$$z - 1 \propto p^{-\{(k-1)/k\}} \left\{ 1 - \left(1 - \mu_0^2 \right) \left( \frac{p}{p_0} \right)^{(k-1)/k} \right\}^{-1/4}$$

and

$$1 - \mu^2 \propto p^{(\theta-1)/k}.$$ (4 \cdot 8)

It is seen that the angle of the weak shock decreases with the decrease of pressure, which corresponds to the well-known sound propagation.

In Figs. 2 and 3 we show the behavior of shocks as the result of the numerical integration for the case $k=1$ and 5/3. As for $\gamma$, we take the value 5/3. The initial strength is taken as $z_0=2$ (mediate shock) and $z_0=10$ (strong shock). As for the initial angle we take $\theta_0=0$ (normal), 15° and 30°. As is seen from the Figures, when the shocks start with the same initial strength from the same
height (pressure), one with larger initial angle grows more fast. On the other hand, the angle increases more rapidly, when the initial strength is larger. It reaches soon the extreme angle $\sim 64^\circ$, over which there is no physical solution. This is one of the characteristic feature of oblique shocks. This property is similar to that of the reflection of shock at a wall. Above this extreme angle, the regular pattern can not be permitted and the Mach-pattern may appear. Since the latter is not stable in general cases, we can imagine that the shock wave after this extreme angle is destroyed into many discontinuities.

§ 5. Vortex distribution behind oblique shocks

We investigate here the vortex distribution behind shock fronts. The existence of vortex is a remarkable feature of oblique shocks. Since there are no stabilizing factors such as magnetic field, the infinitesimal tangential discontinuities behind the shock fronts are absolutely unstable and will generate turbulent fields.

The vorticity $\tilde{\omega}$ is expressed by

$$\tilde{\omega} = \text{rot } \mathbf{v}. \quad (5 \cdot 1)$$

In the $\xi - \eta$ coordinate system (see Fig. 4), where $\xi$-axis is taken to be parallel to the unit vector $\mathbf{n}_{43}$ and $\eta$-axis to $\mathbf{t}_{43}$, the above equation is reduced to $-\Delta \mathbf{v}_x / \Delta \xi$ as a result of $\Delta \mathbf{v}_x = 0$. Since $\Delta \mathbf{v}_x$ and $\Delta \xi$ are written as

$$\Delta \mathbf{v}_x = (U_4 - U_3) \cdot \mathbf{t}_{44}$$

and

$$\Delta \xi = \Delta r \cdot \left( \frac{\mathbf{n}_{43} \cdot \mathbf{t}_{44}}{\mathbf{n} \cdot \mathbf{t}_{44}} \right)$$

respectively, we get

$$\tilde{\omega} = - \left\{ \sqrt{1 - \mu^2} / (\mu_{43} \sqrt{1 - \mu_{43}^2} - \mu_{13} \sqrt{1 - \mu_{13}^2}) \Delta r \right\} \{ \phi (54 : 5) \left( \mathbf{n}_{44} \cdot \mathbf{t}_{44} \right) \}

- \phi (12 : 1) \left( \mathbf{n}_{43} \cdot \mathbf{t}_{44} \right) - \phi (23 : 2) \left( \mathbf{n}_{43} \cdot \mathbf{t}_{44} \right) \}.$$
By taking the requisit expressions for $\phi(54:5), etc., this becomes

$$\frac{\partial}{\sqrt{p\sigma(d\ln p/dr)}} = -\frac{V}{2y} \left\{ \frac{1 - \mu^2}{z - 1} \left( 1 - \frac{\Phi_x}{\Phi_y} \right) \frac{dz}{d\ln p} + \left( 1 - \frac{1}{k} \right)(1 + y) \right\}
- \frac{2yz}{z - 1} \left( 1 - \frac{1}{z} \frac{\mu^2 + y(1 - \mu^2)}{\sqrt{\mu^2 + y^2(1 - \mu^2)}} \right).$$ (5.2)

This is the general expression of the vorticities.

When the incident shock is weak, the generated vortices are also weak, depending on the first order of the strength. On the other hand, strong shock waves generate strong vortices, which depend on the square root of $z$. In Fig. 5, the variations of vorticity with pressure are shown for isothermal gas layer. Strong vortices can be generated by strong shocks having large incident angles.

The turbulent field caused by these vortices and its dissipation may be a very interesting problem. In order to treat these it will be needed to develop the theory of pulse shock taking into account viscosity.

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Appendix

Proof of the agreement of our formulae with those of Polacheck and Seeger

The latter authors formulated the problem from the stationary point of view of an observer traveling with the triple point 0.

Our notations differ from theirs, which are as follows:

$\eta$: the density ratio across the incident shock I (normalized respect to the region behind I).

$\sigma$: the material velocity in front of I and normal to it (normalized with respect to the acoustic speed behind I) and $u$: the material speed observed from the stationary point 0.

The quantities used in our theory are expressed by their notations as follows:
for velocities.

\[
\begin{align*}
(n_{12} \cdot t) &= \sin \omega, \\
(n_{12} \cdot n_{23}) &= \cos \beta, \\
(n_{25} \cdot t) &= \sin(\beta + \omega), \\
(n_{54} \cdot t) &= \sin(\phi - \omega), \\
(n_{54} \cdot t) &= \sin(\delta + \phi - \omega)
\end{align*}
\]

and

\[
\delta + \phi + \beta + \alpha = 2\pi \quad (A \cdot 1)
\]

for angles.

Now, Eq. (2.8) becomes

\[
\frac{\tau}{\sin \omega} = \frac{\sigma''}{\sin(\phi - \omega)}, \quad (A \cdot 2)
\]

which corresponds to Eq. (9) expressing the law of refraction in their paper. The equations (2.7), (2.9) and (2.10) are respectively reduced to

\[
\tan \alpha = \frac{\tau'}{-t'}, \quad (A \cdot 3)
\]

\[
\sigma' = t \sin \beta + \sigma \cos \beta \quad (A \cdot 4)
\]

and

\[
\tan \delta = \frac{\tau''}{-t''} \quad (A \cdot 5)
\]

with their notations. By putting (A \cdot 4) and (A \cdot 5) in (A \cdot 1) we get

\[
\tan^{-1} \frac{\tau'}{-t'} = 2\pi - \left(\phi + \beta + \tan^{-1} \frac{\tau''}{-t''}\right),
\]

which shows that the flow behind the transmitted and refracted shock is tangential. This supposition is in our theory as a boundary condition at the disturbed medium. Thus, our theory is the same as theirs, but as seen from the above transformation their procedure is more cumbersome.
References