On the Mass of the Pseudoscalar Meson in Nambu's Model

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The mass of the pseudoscalar meson in Nambu's model is calculated. The principle of the approximation is that the arguments concerning to the mass of the meson must include the possible effective four boson interactions. It is shown that a pseudoscalar meson of non-zero mass is generated dynamically in the model which has been thought to be massless.

In a previous note,\(^1\) it was shown that a pseudoscalar field without bare mass, \(A\), can have non-zero physical mass when an interaction of the type \(-\lambda A^4\) is taken into account, if \(\lambda > 0\) and the mass of nucleon is not zero, although single nucleon-meson interaction cannot give meson field any finite mass. In other words, it is essential to include possible four boson interaction when we discuss the boson mass. On the other hand, Y. Nambu and G. Jona-Lasinio\(^2\) showed in their model that a massless pseudoscalar meson is generated as a counter contribution of non-zero physical mass of the nucleon. However, their argument based on the conserved current is very questionable. Their second argument is that the homogeneous Bethe-Salpeter equation for a particle-anti-particle system has a solution corresponding to a massless pseudoscalar field if it is approximated by the ladder diagram of nucleon bubbles. They also showed, by the same procedure, that there are a scalar meson and a scalar deuteron. The purpose of this note is to clarify the discrepancy between the treatment by Nambu–Jona Lasinio and our previous results.

If bosons are interpreted as bound states, there may be the effective four bosons interactions as shown in Fig. 1, where wavy-, dashed- and double-lines represent the pseudoscalar meson, scalar one and the scalar deuteron respectively. So, by the previous arguments, to discuss the mass of the pseudoscalar meson
we must include higher order processes which may possess the effective interactions as shown in Fig. 1. (The analysis based on the simple bubble diagrams corresponds to take only the effective nucleon-meson interaction.) Therefore we must calculate the contributions to the nucleon-nucleon scattering amplitude from a chain of nucleon bubbles at least with or without an inner bubble as shown in Fig. 2.

\[ \text{Fig. 2.} \]

The sum of these diagrams can be written simply as

\[ 2g i \mathcal{F}_s \frac{1}{1 - J_p(q) - J_{p'}(q) - J_{p''}(q)} \]

where \( J_p(q) = (g/4\pi) \int \frac{d^2 \kappa}{(\kappa^2 + \kappa^2/\Lambda^2)} \frac{(1 - 4m^2/\kappa^2)^2/(q^2 + \kappa^2)}{d\kappa^2} \) (\( \Lambda^2 \) is a large cutoff momentum squared) and it corresponds to the diagram (a) of Fig. 3, and \( J_{p'}(q) \) and \( J_{p''}(q) \) come from diagrams (b) and (c) respectively. To see the meaning of \( J_{p'}(q) \) we separate this as

\[ J_{p'}(q) = J_{p'}^{(1)}(q) + J_{p'}^{(3)}(q) \]

where \( J_{p'}^{(1)}(q) \) and \( J_{p'}^{(3)}(q) \) are contributions from the diagrams of Figs. (4a) and (4b) respectively.

\[ \text{Fig. 3.} \]

\[ \text{Fig. 4a.} \]

\[ \text{Fig. 4b.} \]

That is, Fig. (4a) contains the radiative corrections by the composite mesons and Fig. (4b) includes those by the scalar deuteron. Then \( J_{p'}^{(1)}(q) \) becomes
\[ J_p^{(1)}(q) = \frac{i g^2}{(2\pi)^2} \int d^4k \int d^4t \int d^4l \frac{2 \times 32m^2}{A} \left[ f(m^2, k, t, l) + h(m^2, k, t, l, q) \right] \]

where

\[
A = (k^2 + m^2 - i\varepsilon) ((k-t)^2 + m^2 - i\varepsilon) ((k-t-q)^2 + m^2 - i\varepsilon) ((k-q)^2 + m^2 - i\varepsilon) \\
\times ((l^2 + m^2 - i\varepsilon) ((l-t)^2 + m^2 - i\varepsilon),
\]

\[ f = k^2 (k-t)^2 + m^2 (k-t)^2 + m^4 (k^2 + m^2) \]

and

\[
h = q^2 \left\{ 2 (l-k)^2 - (t+l) \cdot (t+k) - k \cdot l \right\} + q \cdot l (q \cdot (k-l)) - (k^2 + m^2) q \cdot (2k-t) \\
- k \cdot l (q \cdot (k-t) - 2k \cdot t + t^2).\]

It is obvious that the contribution from \( h \)-term is proportional to \( q^2 \). But, the highest divergent term of \( J_p^{(1)}(q) \) is \( O(g^3 m^2 A^4) \) which comes from the \( f \)-part, and does not depend on \( q^2 \). The expected highest divergent term \( O(g^3 A^6) \) disappears owing to the cancellation between the contribution from scalar meson and the one from the pseudoscalar meson. Next, \( J_p^{(2)}(q) \) becomes

\[ J_p^{(2)}(q) = \frac{i g^2}{(2\pi)^2} \int d^4k \int d^4t \int d^4l \frac{32}{A} \left\{ F(m^2, k, t, l) + H(m^2, k, t, l, q) \right\} , \]

where

\[ F = ((t-k)^2 + m^2) (k^2 + m^2) l \cdot (t-l) + 2m^2 (k-t)^2 + m^4 (k^2 + m^2) + 2m^2 k^2 (k-t)^2 , \]

while

\[ H = - q^2 \left\{ (k \cdot l) (t-k) \cdot (t-l) - k \cdot (t-k) l \cdot (t-l) + k \cdot (t-l) l \cdot (t-k) + m^4 \right\} \\
+ (k^2 + m^2) \left\{ (q \cdot l) t \cdot (k-t) - (q \cdot t) l \cdot (l-k) + (q \cdot k) l \cdot (t-l) \right\} \\
+ ((t-k)^2 + m^2) \left\{ F(k \cdot q) - (k \cdot l) (k \cdot t) + (t \cdot k) (q \cdot l) - (k \cdot q) (t \cdot l) \right\} \\
+ 2q \cdot (k \cdot t) \left\{ 2 (k \cdot l) (q \cdot l) - (k \cdot q) l^2 - (l \cdot q) k^2 \right\} \\
+ 2m^2 \left\{ - k \cdot q (k-t)^2 - k \cdot (k-q) q \cdot (k-t) + l \cdot q q \cdot (l-t) - m^2 q \cdot (2q-t) \right\} . \]

Here also the contribution from \( H \)-term is proportional to \( q^2 \). The highest divergent term of \( J_p^{(2)}(q) \), \( O(g^3 A^6) \), is \( q^4 \)-independent and involved in \( F \)-term. Performing the integrations, \( J_p^{(2)}(0) \) becomes

\[ J_p^{(2)}(0) = \frac{i g^2}{2\pi^2} \int d^4k \int d^4t \int d^4l \frac{1}{A} \left( (t-k)^2 + m^2 \right) (k^2 + m^2) l \cdot (t-l) \]

\[ = \left( \frac{g A^3}{2\pi^2} \right)^3 \frac{1}{16} \left\{ \prod_{i=1}^{3} d\alpha_i \cdot \frac{\alpha_i \alpha_i \alpha_i \delta (\sum \alpha_i - 1)}{\alpha_i \alpha_i \alpha_i + \alpha_i \alpha_i \alpha_i + \alpha_i \alpha_i \alpha_i + \alpha_i \alpha_i \alpha_i} \right\} \times \frac{m^2 + A^2}{m^2 + \alpha_i A^2} \left( 1 + \frac{m^2}{A^2} \right)^3 . \]
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The main term of $J_p'(q)$ is $J_p'(0)$ which is negative, and we obtain

$$0 > J_p'(q) \approx J_p'(0) > - \left( \frac{gA^2}{2\pi^2} \right)^3 \frac{1}{32},$$

as the lower bound of the highest divergent term.

For $J_p''(q)$, in the same way as for $J_p'(q)$, we obtain

$$0 > J_p''(q) \approx J_p''(0) \geq - \left( \frac{gA^2}{2\pi^2} \right)^3 \frac{1}{32}.$$

So that $J_p'(q) + J_p''(q)$ is bounded as

$$0 > J_p'(q) + J_p''(q) \approx J_p'(0) + J_p''(0) \geq - \left( \frac{gA^2}{2\pi^2} \right)^3 \frac{1}{16}.$$

Because $J_p(q)$ is a decreasing function of $q^2$, $1 - J_p(q) - J_p'(q) - J_p''(q) = 0$ can have a root at some $q^2 = -\mu^2$, and we have

$$0 = 1 - J_p(0) + (J_p(0) - J_p(-\mu^2)) - (J_p'(0) + J_p''(0)).$$

The highest divergent term of $J_p(0) - J_p(\mu^2)$ is $- (\mu^2/4\pi^2) \log (A^2/m^2)$. To compare this with the lower bound of $J_p'(\mu^2) + J_p''(\mu^2) \approx J_p'(0) + J_p''(0)$, we use again the self-consistent equation $1 = J_p(0)$, then

$$J_p(0) - J_p(-\mu^2) \approx - \frac{g\mu^2}{4\pi^2} \log \frac{A^2}{m^2} \approx - \frac{g\mu^2}{16\pi^2} \left( \frac{gA^2}{2\pi^2} \right)^3 \log \frac{A^2}{m^2}.$$

So that the final result comes out to be

$$\frac{g\mu^2}{16\pi^2} \left( \frac{gA^2}{2\pi^2} \right)^3 \log \frac{A^2}{m^2} < \frac{1}{16} \left( \frac{gA^2}{2\pi^2} \right)^3,$$

or

$$0 < \mu^2 < \frac{1}{2} \log (A^2/m^2).$$

If we take $A^2 < 50m^2$, then $\mu^2 < 4m^2$. This shows that, for appropriate magnitude of $A^2$, there actually exists a pseudoscalar meson with non-zero physical mass.

We will remark that, in the previous note, the mass of the meson became

$$\mu^2 = 3\lambda_0 \frac{1}{2(2\pi)^3} \left[ A\sqrt{A^2 + \mu^2 - \mu^2 \log \frac{A + \sqrt{A^2 + \mu^2}}{\mu}} \right] \approx 3\lambda_0 A^2,$$

$\lambda_0$'s for various four boson coupling, in Nambu's model, can be calculated from each corresponding diagrams, for instance Fig. 5. That is,

$$\lambda_0 \propto G^4 \times \text{(nucleon loop integral)}.$$

Following Y. Nambu and G. Jona-Lasinio, each $G^2$ is the order of $(\log (A^2/m^2))^{-1}$, and the loop integral is the order of $\log (A^2/m^2)$, so $\lambda_0 = O((\log (A^2/m^2))^{-1})$. This indicates that the meson mass $\mu^2$ is the order of $A^2/\log (A^2/m^2)$ which is consistent with the value of the mass of the meson calculated above.

In conclusion we can say that the pseudoscalar meson with non-zero physical
mass is generated dynamically and there is no zero mass meson, and that the zero-mass boson problem does not exist also in the model.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{fig5.png}
\caption{Fig. 5.}
\end{figure}

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References

1) M. Yamada, Prog. Theor. Phys. 31 (1964), 713.