Helium Flash in Less Massive Stars

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The structure of a stellar core of condensed type is generally discussed to show that physical variables at the center can be determined almost completely as functions of central entropy, if the polytropic index is known in the relatively central region. Based on this relation, the thermal runaway through the helium flash in less massive stars is investigated, treating the core mass as a parameter. Taking into account the finite thermal conductivity, a convection core is shown to develop in the degenerate region. If we assume a sufficiently large efficiency of convective heat-transport, the maximum central temperature in the course of the helium flash is too low to lead the core of $0.53 M_\odot$ to explosion, in contrast to the model sequence calculated by Schwarzschild and Härn.

The effect of the finite efficiency of convective heat-transport is investigated, using the mixing length theory of convection and treating the ratio of the mixing length to the scale-height of pressure as a parameter. Structures of cores including this effect are solved numerically. It is found that, if this ratio is smaller than a critical value which depends on the core mass, an instability occurs; the heat energy liberated by helium burning is blocked and the central temperature reaches a value, above which the effect of inertia leads the core into explosion.

§ 1. Introduction

In a star of mass less than about $2M_\odot$, helium burning begins suddenly in the electron-degenerate helium core, which results in the thermal runaway\(^1\) or the helium flash. It commences, when the core mass grows up to a certain critical value and the star reaches the tip of giant branch in the Herzsprung-Russell diagram of globular clusters.\(^2\)\(^,\)\(^3\)\(^,\) Study of the mechanism of the helium flash is important not only for itself but for problems such as the origins of planetary nebulae\(^5\) and of supernovae of Type I and the chemical mixing. These phenomena, if they occur, will have influence on the later evolution in the helium burning phase and hence on the appearance of the horizontal branch of globular clusters.\(^*\) In the present paper, we shall study the evolution of the core after the helium flash has commenced.

The critical core-mass for the helium flash has been shown by Hayashi, Hōshi and Sugimoto\(^5,\)** to depend not on the stellar mass but on the initial

\(^{\ast}\) Observationally the luminosity of the horizontal branch is nearly unique, and theoretically it is almost determined by the mass of the helium core.\(^4\)

\(^{**}\) This paper will be referred to as HHS in the following.
It is \((0.53 \pm 0.02) M_\odot\) for a chemical composition of population II and \((0.42 \pm 0.02) M_\odot\) for the population I, if the plasma-neutrino loss calculated by Ruderman et al.\(^9\) is not taken into account. This dependence on chemical composition comes mainly from the helium content or the mean molecular weight in the envelope, which is observationally unknown for population II stars. Using the semi-empirical formulae \((6D\cdot2)\) and \((6E\cdot17)\) in HHS, which estimate the above dependence, the critical core-mass is found to be about \(0.64 M_\odot\) for a star with no initial helium but with the same metal content as the population II star of HHS. The effect of the plasma-neutrino loss makes the critical core-mass larger by an amount not known with certainty. It is now under study by taking into account the energy flow toward the center and the resulting negative temperature gradient in the degenerate region of the core, which has been neglected in the recent work by Chiu.\(^6\)

Schwarzschild and Härm\(^7\),\(^8\) computed the evolution of the helium core of \(0.55 M_\odot\) through the flash dividing the core into a degenerate region and a non-degenerate region. They neglected the finite heat-conductivity in the degenerate region and thus the development of convection core in the stages preceding the maximum central temperature. The deviation in their approximate equation of state at the junction point of the two regions was reflected largely in the maximum central temperature, as will be shown in the later sections, which is important for knowing the possibility of explosion of the core. Further they made several approximations to simplify the calculations. Recently, they have extended their work to investigate the problem of mixing just after the helium flash.\(^9\) They have calculated three sequences with greater accuracy with the help of the Henyey method, but the physics have been the same as in their previous work except for the minor modifications.

In a model calculation it is difficult to estimate the influence of approximations on the solution and to know the cases for the other core-masses, because of the nonlinear character of the fundamental equations. Such a calculation should be supplemented by a more general but less quantitative investigation extracting the essential features of the physical processes. Osaki\(^9\) has discussed the variation in the central temperature through the flash using the assumptions of the homologous expansion of the core and the isothermal condition as called by him. However, both assumptions are in contradiction each other.

The aim of the present paper lies in such a general investigation of the helium flash treating the core mass as a parameter. In the next section, therefore, the structure of a stellar core is generally discussed. It is shown that the central temperature of the core can be determined almost completely as a function of central entropy or degree of electron-degeneracy, if the polytropic index is known in the relatively central region. The single star approximation of the

\(^*\) This paper will be referred to as SH in the following.
core is also formulated. In § 3 the timescale of heat conduction and the development of convection core are discussed. The variations in physical quantities in the central region through the flash are obtained in § 4 for the case of a sufficiently large efficiency of convective heat-transport. The effect of inertia, opposing the expansion of the core as a whole, is also discussed and it is found to lead a core of mass larger than \(0.70M_\odot\) to explosion.

At the peak of the flash the heat energy to be transported by convection is so enormously large that the efficiency of convective heat-transport cannot always be regarded as infinite. It is unfortunate that we have to use the mixing length theory to estimate the effect of the finite efficiency, which leaves the ratio of the mixing length to the scale-height of pressure as a parameter. According to this theory the super-adiabatic temperature gradient appears, which is related with the heat flux transported by convection. In § 5a such solutions are obtained using nondimensional variables under the single star approximation. If a value of the ratio of the mixing length to the scale-height of pressure is smaller than a critical value which depends on the core mass, no physical solution is found to exist. It means that an instability, more precisely, the blocking of heat occurs. In § 5b nondimensional structures obtained for the core in blocking of heat and it is shown that the central temperature continues to rise along an evolutionary sequence of any core, which leads the core to explosion. In the final section some examples of evolutionary sequences near the peak of the flash are obtained using physical variables, for different values of the ratio of the mixing length to the scale-height of pressure. The criterions for the explosion of the core are summarized also and briefly discussed.

§ 2. Structure of centrally condensed core

In this section we consider some general properties of the structure of centrally condensed core. We shall use the nondimensional variables defined by

\[
P = P_c \omega, \quad \rho = \rho_c \sigma, \quad M_r = M_c \varphi, \quad r = r_c \xi,
\]

where the subscript \(c\) denotes the central value and where

\[
M_c^3 = \frac{1}{4\pi G^3} \frac{P_c^3}{\rho_c^4}, \quad r_c^2 = \frac{1}{4\pi G} \frac{P_c}{\rho_c^2}.
\]

The equations of hydrostatic equilibrium of a star are given by

\[
U = 4\pi r^2 \rho = \frac{d \log \varphi}{M_r} = \frac{\xi^3 \sigma}{\varphi}, \quad V = \frac{GM_c \rho}{r P} = -\frac{d \log \omega}{d \log \xi} = \frac{\varphi \sigma}{\xi \omega}.
\]

The nondimensional density, \(\sigma\), depends not only on the stellar structure but also on the equation of state. Both dependences can be incorporated into the local polytropic index defined by

\[
1 + \frac{1}{N} = \frac{d \log \omega / d \log \xi}{d \log \sigma / d \log \xi}.
\]
The boundary conditions at the center are \( U = 3 \) and \( V = 0 \). At the stellar surface, \( U \) vanishes and \( V \) is infinity, while at the edge of the core, which is denoted by subscript 1, both \( U_1 \) and \( V_1 \) remain finite.

Eliminating \( \sigma \) from Eq. (2·3), we obtain

\[
d\ln \varphi = -\frac{UdV - VdU}{V(2U + V - 4)},
\]

and using Stokes' theorem, we find

\[
\int_{e} d\ln \varphi = -\int_{s} \frac{4dS}{V(2U + V - 4)^{1/2}},
\]

if the region of diverging integrand is avoided. In the \( U-V \) plane of Fig. 1 the loci of constant integrand in the right-hand side of Eq. (2·6) are illustrated by dotted lines together with the Emden solution of polytropic index 1.5 and the core just before the helium flash taken from HHS by solid curves. We can extend the integral of Eq. (2·6) to the region enclosed by two \( U-V \) curves of stars and/or cores. It usually does not contain a region where \( 2U + V - 4 = 0 \), since the \( U-V \) curves come to the upper side of the Emden solution of polytrope with index of infinity and \( V_1 \) is larger than 4. Near the center, since it is valid that

\[
3 - U = \frac{3}{5} \left( \frac{N}{N+1} \right) V, \quad \text{for } \xi \ll 1,
\]

the area for a given interval of \( U \) is proportional to \( V \) and, therefore, the integral does not diverge. For the region near the surface it does not diverge also because of the dependence of the integrand on \( V^{3} \).

For a contour as shown by the arrows in Fig. 1, which starts from the center, goes up along a polytrope of index \( N \) to the surface, down along the \( V \)-axis to \( V_1 \), to the right with constant \( V \) till the edge of the core \( U_1 \), and then goes back to the center along the \( U-V \) curve of the core, Eq. (2·6) becomes

\[
\ln \varphi_1 - \ln \varphi_1(N) = \int_{s} \frac{4dS}{V(2U + V - 4)^{1/2}} - \frac{U_1}{V_1 - 4}, \quad \text{for } 2U_1 \ll V_1 - 4.
\]

In the above equation \( \varphi_1 \) and \( \varphi_1(N) \) denote the values of \( \varphi \) of the core at its edge and of the polytrope of index \( N \) at the surface, respectively. The values of \( \varphi_1(N) \) are shown in Table I. We define the effective polytropic index of the core as the one for which Eq. (2·8) vanishes.

For a core of condensed type, i.e., \( U_1 \ll 1 \), with a jump of mean molecular weight at its edge, the second term in the right-hand side of Eq. (2·8) is small, since \( V_1 \) is about \( 4\mu_{\text{env}}/\mu_0^4 \), where \( \mu_{\text{env}} \) and \( \mu_0 \) denote the mean molecular weight of the core and of the envelope, respectively. The contribution to the integral in this equation comes mainly from relatively central region, i.e., from the region...
Table I. Values of $\varphi_1(N)$ at the surface of the Emden solution of polytrope of index $N$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>0.0</th>
<th>1.5</th>
<th>3.0</th>
<th>4.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log \varphi_1(N)$</td>
<td>0.6901</td>
<td>1.0305</td>
<td>1.2080</td>
<td>1.3506</td>
</tr>
</tbody>
</table>

Fig. 1. Dotted lines illustrate loci of constant integrand in Eq. (2·6) with its value attached. Solid lines are Emden solution of polytropic index 1.5 and the core just before the helium flash. Arrows indicate the contour of integration in obtaining Eq. (2·8).

of $V$ smaller than $(5 - N_c)/5(N_c + 1)$ as estimated using Eq. (2·7). In this region, areas for a given interval of $U$ contribute nearly equally to the integral. The contribution from the outer region is small as can be seen in Fig. 1. Therefore, the effective polytropic index is very near to the local polytropic index in the relatively central region of the core.

This situation can be understood physically as that the structure in the central region is nearly irrelevant to the outer region where pressure and density are much lower than at the center of the core of condensed type. As long as the structure in the central region is concerned, the core can be well approximated by a single star of the same mass as the core mass, which we shall call single star approximation.

In many cases the local polytropic index is nearly uniform in the relatively central region. For example, it is 1.5 in the presence of strong but nonrelativistic electron-degeneracy, and in the convection core as long as the efficiency of convective heat-transport can be considered as infinite. In such cases another approximation is also good that replaces the effective polytropic index by the local polytropic index at the center. The error due to these approximations will be discussed in the next section.

If we know the value of $\varphi_1$ or the effective polytropic index, the central temperature is expressed, substituting the equation of state from (3A-5) to (3A-9) in HHS into Eq. (2·2) and using the relation, $M_0 = M_1 / \varphi_1$, as

$$T_c = \frac{2 \pi (4\pi)^{4/3} (\mu_e H)^{8/3}}{k h^2} \frac{G^2 \varphi_1^{8/3}}{\{(2/3) F_{\gamma/3}(\varphi_1) + (\mu_e / \mu_n) F(\varphi_2)\}^3} \left( \frac{M_1}{\varphi_1} \right)^{4/3},$$

(2·9)

where $F$ and $F_{\gamma/3}$ are the Fermi-Dirac functions with argument of $\varphi$, the chemical potential of electron divided by $kT$. In the limits of strong degeneracy or of nondegeneracy, $T_c$ is proportional to $\varphi_e^{-1}$ or to $\exp(2\varphi_e/3)$, respectively.*)

*) In the limit of nondegeneracy this is an usual relation, $T_c \sim T_e$. **)
§ 3. Development of convection core in helium flash

The stability condition for convection is

$$(n + 1)_{\text{rad}} = \frac{d \log P}{d \log T} = \frac{16 \pi a c G T^4 M_r}{3 \pi P L_r} > 2.5.$$  \tag{3.1}$$

The flux of heat is given by

$$L_r = L_{n,r} - L_{a,r}, \quad L_{n,r} = \int_0^{M_r} \varepsilon_n dM_r, \quad L_{a,r} = \int_0^{M_r} \varepsilon_a dM_r,$$  \tag{3.2}$$

where $\varepsilon_n$ denotes the nuclear energy generation and $\varepsilon_a$ denotes the absorption of heat due to the change in entropy of mass element,

$$\varepsilon_a = T \left( \frac{\partial s}{\partial \rho} \right)_{M_r} = \frac{3}{2} \frac{P}{\rho} \frac{d}{d \phi} \left( \ln \frac{F_{n,\nu}(\phi) + (\mu_e/\mu_n)F(\phi)}{F_{\text{rad}}(\phi)} \right) \left( \frac{\partial \phi}{\partial t} \right)_{M_r}. \tag{3.3}$$

In what follows, $L_n$ and $L_a$ denote the total nuclear energy generation and absorption of heat, respectively.

Since the nuclear energy generation is more concentrated toward the center than the absorption of heat, $L_n$ is comparable to $L_{n,r}$ as long as the order of magnitude is concerned. Though the electron-degeneracy is strong in the central region of the core in the beginning stage of the helium flash, i.e. though $\phi_e$=20, the conduction opacity amounts to as much as $\kappa_{\text{cond}} = 5 \times 10^{-3}\text{cm}^2\text{g}^{-1}$. As estimated using Eq. (3.1), $\varepsilon_a$ above $10^6\text{erg g}^{-1}\text{sec}^{-1}$ is sufficient for convection, which corresponds to the central temperature above $0.9 \times 10^8\text{K}$.

Since the stability condition (3.1) is equivalent to

$$\frac{d \phi}{d \log P} > 0,$$  \tag{3.4}$$

$\phi$ is constant throughout the convection core, as long as the efficiency is enough to transport an appreciable fraction of the nuclear energy generation. In the radiative region, where Eqs. (3.4) and (3.1) are satisfied, the flux cannot be larger than $10^9\text{erg sec}^{-1}$ by orders of magnitude. Therefore almost all of the nuclear energy generation is absorbed to lower $\phi$ in the convection core.

The timescale of heat conduction is given by

$$t_{\text{cond}} \approx \frac{3 \pi \rho r^4 P}{2 a c T^4},$$  \tag{3.5}$$

which is estimated using Model 30 of SH and is found to be of the order of $10^9\text{sec}$ at the edge of the helium core and $10^{14}\text{sec}$ at the edge of the degenerate region. In a region where electrons are strongly degenerate this timescale
is not so small as usually expected. Since the conduction opacity is proportional to $\psi^{-3}T^{-1}$ in the limit of degeneracy,\textsuperscript{15} we find

$$t_{\text{cond}} = 3 \times 10^5 \left( 10^5/T \right)^3 \left( UM_r/3M_\odot \right)^{1/2} \text{sec},$$

where $U$ is defined by Eq. (2·3). In order to know the change in $\psi$ in time, we must take into account the other factor appearing in Eq. (3·3). It is found that $| (\partial \psi/\partial t)_{\text{rad}} |$ is less than $10^{-10} \text{sec}^{-1}$ throughout the radiative region. After the stage, when the central temperature has reached $1.1 \times 10^8 \text{°K}$, a time to be spent is less than $10^{15} \text{sec}$ until the central temperature reaches its maximum through the helium flash. Therefore, the distribution in $\psi$ in the radiative region remains the same as the one at the stage when $T_c = 1.1 \times 10^8 \text{°K}$. Though this distribution in $\psi$ is difficult to calculate, it will be only slightly different from that at the stage immediately before the helium flash, since the central temperature is only slightly different between these stages.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{Illustration of $U-V$ curves for various fitting conditions. The solid curve labelled with E is the polytrope of index 1.5 and the dotted curve is the solution with a given distribution in $\psi$. These two curves are fitted by dashed lines under the conditions of (1) continuity of pressure and (2) continuity of $\psi$. If we discard a condition for the thermal equilibrium, the solution of the radiative zone shifts to the solid curve.}
\end{figure}

In the followings we shall examine the structure of the core, which consists of a convection core and an outer radiative zone with a fixed distribution in $\psi$. In the present discussion an example of the distribution in $\psi$ is taken from the core of $0.53M_\odot$ in HHS at the stage immediately before the helium flash.

If both the hydrostatic and the thermal equilibria are established, the usual boundary conditions (6E·14) and (4D·9) in HHS completely determine the solution of the radiative zone as well as its $U-V$ curve as illustrated in Fig. 2 by the dotted line. On the other hand the $U-V$ curve of the convection core is the Emden solution of polytropic index 1.5, which never crosses the former curve as shown by the solid line labeled with E in Fig. 2. If we fit the two
solutions, some quantities should be discontinuous at the interface of the two regions.

In the first case, if we assume the continuity of pressure, the $U$-$V$ curve jumps as illustrated in Fig. 2 by the dashed line (1), and $\psi$ is found to be larger at the internal side than at the external side. For example, $\psi_{i1} = 5.1$ and $\psi_{e1} = 0.0$, where and in what follows the subscript 2 denotes the edge of the convection core and $i$ and $e$ denote the internal and the external side, respectively. In this case the edge of the convection core is overstable, which is in contradiction. In

Table II. Development of the convection core through the helium flash and the effective polytropic index of the core.

<table>
<thead>
<tr>
<th>Stage</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\log U_{i1}$</td>
<td>2.31</td>
<td>2.25</td>
<td>2.12</td>
<td>1.94</td>
</tr>
<tr>
<td>$M_g/M_1$</td>
<td>0</td>
<td>0.78</td>
<td>0.93</td>
<td>0.97</td>
</tr>
<tr>
<td>$\psi_{i2} = \psi_{e2}$</td>
<td>19</td>
<td>5.0</td>
<td>2.6</td>
<td>1.0</td>
</tr>
<tr>
<td>${U_{1}/(V_{i1} - 4)}\log e$</td>
<td>0.0004</td>
<td>0.0004</td>
<td>0.0005</td>
<td>0.0008</td>
</tr>
<tr>
<td>$\Delta \log \psi_i$</td>
<td>0.04</td>
<td>0.005</td>
<td>0.002</td>
<td>0.001</td>
</tr>
<tr>
<td>$N_{i1}$</td>
<td>1.8</td>
<td>1.53</td>
<td>1.51</td>
<td>1.51</td>
</tr>
<tr>
<td>$\log L_{i1}/L_\odot$</td>
<td>3.26</td>
<td>2.84</td>
<td>1.91</td>
<td>0.44</td>
</tr>
<tr>
<td>$\log L_i/L_\odot$</td>
<td>3.26</td>
<td>3.22</td>
<td>3.13</td>
<td>2.98</td>
</tr>
</tbody>
</table>

the second case, if we assume the continuity of $\psi$, the $U$-$V$ curve jumps as illustrated by the dashed line (2) and the pressure at the internal side should be larger than at the external side, e.g. $P_{i1}/P_{e1} = 3.4$ for $\psi_{2} = 3.3$. It means that the convection core should expand as well as its external radiative zone.

Thus, we can discard the boundary condition $(4D \cdot 9)$ in HHS, i.e. the luminosity required in the hydrogen-rich envelope is not always supplied by the hydrogen shell-burning. Indeed the thermal equilibrium in contrast to the hydrostatic equilibrium is not necessarily established in the timescale considered at present, which is much shorter than that of the heat conduction. In this case we can get solutions for which all variables are continuous and the convection core extends to a region, where the $\psi$ has been larger than the present value of $\psi$ at the center.

In order to get idea of such a core and to estimate the uncertainty in the single star approximation, some solutions are calculated numerically for the distribution in $\psi$, which was used in the above illustrations though not exactly correct one. The results are summarized in Table II and the distribution in $\psi$
are shown in Fig. 3 and the $U$-$V$ curve of Stage 4 is shown in Fig. 2 by the solid line. Stage 1 is the solution determining the distribution in $\psi$. It is seen that the energy generation due to the hydrogen shell-burning, $L_{H}$, decays as the convection core develops. The core, however, remains of condensed type in the sense implied in the preceding section. The effective polytropic index is very close to 1.5 or in other words, the difference,

\[ \Delta \log \psi_1 = \log \psi_1 - \log \psi (N=1.5), \quad (3.7) \]

is small, especially after the convection core has grown to an appreciable fraction of the core. Even in Stage 4, $\log P_2/P_1$ amounts to 1.78, which means that the mixing of the core with the hydrogen-rich envelope cannot be expected.\(^9\) After Stage 4 the change in $\psi$ in the radiative zone must be taken into account as will be shown in the next section.

§ 4. Thermal runaway in the case of infinite efficiency of convective heat-transport

a) Variation in central temperature

The distribution in $\psi$ through the core is difficult to obtain, since in the beginning stages of the helium flash the heat-conduction is not negligible. However, the central temperature can be calculated using Eq. (2.9) without knowing the detailed distribution in $\psi$, only if we known the effective polytropic index of the core and as long as the effect of inertia, which will be discussed in subsection b), is negligible. If we assume a sufficiently large efficiency of the convective heat-transport, the local polytropic index in the convection core is 1.5. In this case the single star approximation, i.e., $N_{eq}=N_c=1.5$, is quite accurate especially near the peak of the flash as shown in the previous sections.

Variations in some physical quantities through the helium flash in the core of $0.53M_\odot$ are calculated under this approximation and are shown in Table III and in Fig. 4. For the other core-mass they can be easily calculated taking into account the dependence of the central temperature on the core-mass in Eq. (2.9). For a fixed value of $\psi$ the following relations are valid:

\[ T_c \sim M_1^{4/3}, \; \rho_c \sim M_1^3, \; P_c \sim M_1^{10/3}, \]

\[ \varepsilon_n \sim M_1^{4s/3+4}, \; L_n \sim M_1^{4s/3+5}, \; -\frac{dt}{d\psi_c} \sim M_1^{-4s/3-8/3}, \quad (4.1) \]

where $s$ denotes the power of temperature in the equation of $\varepsilon_n$ as listed in the seventh column of Table III.

The above results show that the expansion of the core is small until $\psi_c$ reaches about 4. After $\psi_c$ has become smaller than about 3, the expansion is large and it lowers the central temperature. The maximum central temperature is reached when $\psi_c=3.3$ as,
Table III. Variations in physical quantities through the helium flash in the core of 0.53 \( M_\odot \). All values are in the c.g.s. units.

<table>
<thead>
<tr>
<th>Stage</th>
<th>( \psi_e )</th>
<th>( \log T_c )</th>
<th>( \log P_c )</th>
<th>( \log \rho_c )</th>
<th>( \log \varepsilon_u )</th>
<th>( s )</th>
<th>( \log L_u )</th>
<th>( M_u/M_1 )</th>
<th>( \log(-dt/d\psi_e) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>7.952</td>
<td>22.466</td>
<td>5.962</td>
<td>2.65</td>
<td>45.3</td>
<td>33.50</td>
<td>0</td>
<td>11.62</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>8.053</td>
<td>22.421</td>
<td>5.927</td>
<td>6.65</td>
<td>35.2</td>
<td>37.65</td>
<td>0.15</td>
<td>9.02</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>8.179</td>
<td>22.325</td>
<td>5.865</td>
<td>10.29</td>
<td>25.6</td>
<td>41.48</td>
<td>0.48</td>
<td>5.97</td>
</tr>
<tr>
<td>4</td>
<td>7.0</td>
<td>8.265</td>
<td>22.195</td>
<td>5.758</td>
<td>12.08</td>
<td>20.4</td>
<td>43.39</td>
<td>0.68</td>
<td>4.45</td>
</tr>
<tr>
<td>5</td>
<td>5.0</td>
<td>8.318</td>
<td>22.022</td>
<td>5.628</td>
<td>12.83</td>
<td>17.8</td>
<td>44.21</td>
<td>0.80</td>
<td>3.91</td>
</tr>
<tr>
<td>6</td>
<td>4.0</td>
<td>8.335</td>
<td>21.879</td>
<td>5.521</td>
<td>12.91</td>
<td>17.0</td>
<td>44.31</td>
<td>0.86</td>
<td>3.94</td>
</tr>
<tr>
<td>7</td>
<td>3.3</td>
<td>8.339</td>
<td>21.741</td>
<td>5.417</td>
<td>12.76</td>
<td>16.8</td>
<td>44.17</td>
<td>0.90</td>
<td>4.18</td>
</tr>
<tr>
<td>8</td>
<td>2.6</td>
<td>8.331</td>
<td>21.555</td>
<td>5.278</td>
<td>12.35</td>
<td>17.2</td>
<td>43.74</td>
<td>0.93</td>
<td>4.69</td>
</tr>
<tr>
<td>9</td>
<td>1.0</td>
<td>8.234</td>
<td>20.853</td>
<td>4.752</td>
<td>9.83</td>
<td>22.2</td>
<td>41.09</td>
<td>0.97</td>
<td>7.46</td>
</tr>
<tr>
<td>10</td>
<td>0.4</td>
<td>8.164</td>
<td>20.474</td>
<td>4.467</td>
<td>7.13</td>
<td>26.6</td>
<td>38.29</td>
<td>0.98</td>
<td>10.25</td>
</tr>
<tr>
<td>11</td>
<td>-0.4</td>
<td>8.023</td>
<td>19.826</td>
<td>3.981</td>
<td>1.64</td>
<td>38.0</td>
<td>32.60</td>
<td>0.99</td>
<td>15.88</td>
</tr>
</tbody>
</table>

The maximum in Eq. (4.2) is estimated using the results in the preceding section. In contrast to our result, SH have obtained \( \log T_{c,\text{max}} = 8.546 \) for the core of \( 0.55M_\odot \). In their core, the local polytropic index is not smaller than 1.5 everywhere. Therefore, the effective polytropic index is, in principle, never smaller than 1.5.*)

Their higher maximum temperature can be interpreted as due to the amplified reflection of the deviation in their approximate equation of state, which is maximum at the junction point between their nondegenerate and degenerate regions. Indeed, we can reproduce their value by using their equation of state and \( N_{\text{err}} = 1.5 \).

*) A \( U-V \) curve of such a core can easily be proved to run in the lower side of the Emden solution of polytropic index 1.5. Its effective polytropic index is never smaller than 1.5 as can be seen from Eq. (2.8), where the second term in the right-hand side is negligible.
In the core in helium flash the central temperature determines the nuclear energy generation rate, in contrast to the core in equilibrium without electron-degeneracy where the former is determined by the latter. Then, the uncertainty in the nuclear reaction rate directly influences the nuclear energy generation by the same factor. As for the $\gamma$-ray width of the 7.65MeV level of C$^{12}$, the recent two experiments by Alburger\textsuperscript{13} and by Seeger and Kavanagh\textsuperscript{14} have given almost the same values, $2.5 \times 10^{-3}$ eV and $2.4 \times 10^{-3}$ eV, respectively but the both have an uncertainty of about 60 percent. The screening factor is 1.56 at the peak of the flash, Stage 6, according to the equation for the weak screening case by Salpeter\textsuperscript{15}. Strictly speaking, the value of $\varepsilon_n$, $L_n$ and $-d\phi_e/dt$ should be multiplied by the electron-screening factor and have the same uncertainty as the nuclear reaction rate. For the values in Table III, however, the both have been neglected for the sake of convenience in transforming them to the case of other core-masses. It is to be noticed that the criterion for the explosion of the core, which will be discussed in the later sections, is only slightly affected by the above uncertainties, because the maximum value of $L_n$ through the flash depends strongly on the core-mass as shown in Eq. (4-1). The time rate of the change in $\phi_e$, i.e. a timescale of structural change is calculated by

$$L_n - L_n = 0,$$

(4-3)

together with Eqs. (3-2) and (3-3).

The mass fractions of the convection core, which are listed in the ninth column, depend on the distribution in $\phi$ as discussed in the preceding section and are thus not always exact. After Stage 10, $-dt/d\phi_e$ is larger than $10^{10}$ sec and the change in the distribution in $\phi$ due to the heat conduction will not be negligible. As Härm and Schwarzschild\textsuperscript{8} have shown, it makes the convection core shrink and the star will settle itself in the helium burning phase.

b) Effect of inertia

Neglecting the oscillation of the core, we consider the over-all effect of inertia opposing the expansion of the core. The equation of motion for the core can be written as,

$$-4\pi r^2 \frac{\partial P}{\partial M_r} = \frac{G M_r}{r^2} \left(1 + \frac{r^2}{G M_r} \frac{\partial^2 r}{\partial t^2}\right) = \frac{G M_r}{r^2} g,$$

(4-4)

in a Lagrangian scheme. The inertia term acts as if the gravitational constant is increased by a factor of $g$ defined in Eq. (4-4). Using Eq. (2-2) the value of $g$ at the center can be expressed as

$$g_c = 1 + x_c = \lim_{r \to 0} \frac{r^3}{G M_r} \frac{\partial^2 r}{\partial t^2} = \frac{1}{4\pi G \rho_e} \left\{ \frac{1}{3} \left(\frac{d \ln \rho_e}{dt}\right)^3 - \frac{d^2 \ln \rho_e}{dt^2} \right\},$$

(4-5)

In the relatively central region, which is important in determining the central temperature as discussed in § 2, the homology of structure will be approximately
conserved in time. In the case of homologous expansion, \( g \) reduces to

\[
g = 1 + x_c \frac{\varepsilon^2}{3\phi}. \tag{4.6}
\]

The nondimensional equations of hydrodynamic equilibrium can be obtained from Eq. (2.3) only multiplying the extreme right-hand side of the equation for \( V \) by \( g \). They have been solved numerically for different values of \( x_c \) assuming the polytropic index of 1.5. The results are summarized in Table IV, where \( \Delta \log \varphi_1 \) is defined by Eq. (3.7).

<table>
<thead>
<tr>
<th>( x_c )</th>
<th>( \log g_x )</th>
<th>( \Delta \log \varphi_1 )</th>
<th>( \log \tilde{g} )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.004</td>
<td>−0.011</td>
<td>0.007</td>
<td>1.7</td>
</tr>
<tr>
<td>0.03</td>
<td>0.013</td>
<td>−0.033</td>
<td>0.022</td>
<td>1.7</td>
</tr>
<tr>
<td>0.1</td>
<td>0.041</td>
<td>−0.105</td>
<td>0.070</td>
<td>1.7</td>
</tr>
</tbody>
</table>

It is found that \( \varphi_1 \) decreases as the effect of the inertia term, i.e. \( x_c \) increases. According to Eq. (2.9) this decrease in \( \varphi_1 \) gives rise to an increase in the central temperature for a fixed value of \( \psi_1 \). In order to get the same increased central temperature in Eq. (2.9), this decrease in \( \varphi_1 \) can be understood in another way; opposing the expansion, the gravitational constant is effectively increased by a factor,

\[
\tilde{g}(x_c) = \left( \frac{\varphi_1(N=1.5)}{\varphi_1(x_c)} \right)^{2/3}, \tag{4.7}
\]

while the value of \( \varphi_1 \) remains the same as in the case of negligible effect of inertia. To a good approximation we can express this factor as

\[
\tilde{g}(x_c) = g_c^p, \quad p = 1.7, \tag{4.8}
\]

as shown in Table IV.

Using the evolutionary sequence in which the effect of inertia is not taken into account, we shall consider to obtain an evolutionary sequence in which this effect is thoroughly taken into account. Quantities calculated in the former sequence will be denoted by a superscript (0). For a stage with a fixed value of \( \psi_1 \), the following relations hold:

\[
\frac{T_c}{T_c^{(0)}} = \tilde{g}^2, \quad \frac{L_n}{L_n^{(0)}} = \tilde{g}^{2(x+3/2)}. \tag{4.9}
\]

In order to satisfy the second member of Eq. (4.5), another relation should hold,

\[
\frac{x_c}{x_c^{(0)}} = (\frac{L_n}{L_n^{(0)}})^2 = \tilde{g}^{2(x+3)}. \tag{4.10}
\]

Strictly speaking, the first equality in Eq. (4.10) is valid only in the case for \( x_c \ll 1 \), but the following discussions are not altered as long as the order of
magnitude is concerned. From Eqs. (4·10) and (4·8), we find

$$\bar{g} = (1 + x_c^{(0)})^{-\frac{4p(s+3)}{4p(s+3)-1}}, \quad (4·11)$$

which has a solution for $\bar{g}$ only in the case,

$$x_c^{(0)} = \frac{1}{4p(s+3)} \left( 1 + \frac{1}{4p(s+3)-1} \right)^{-\frac{4p(s+3)+1}{4p(s+3)}}, \quad \approx \frac{1}{4ep(s+3)}, \quad (4·12)$$

This can be understood in the following way. A rise in the central temperature due to the effect of inertia accelerates the evolution and makes the effect of inertia larger, which results in another rise in the central temperature. If Eq. (4·12) is not satisfied, an instability occurs in a sense that the above feedbacks diverge.

In the real case the homologous expansion is not exactly realized, since the value of $\varphi$ changes and it partly compensates the increase in $x_c$. However, the central temperature will attain such a high value as to lead the core to explosion, if Eq. (4·12) is not satisfied in the order of magnitude. For the core of $0.53M_\odot$ in Table III $x_c^{(0)}$ is only $3 \times 10^{-9}$ at the peak of the flash. Using Eq. (4·1) a core more massive than $0.70M_\odot$ is found to explode, if the effect due to the finite efficiency of the convective heat-transport is neglected.

c) **Carbon flash**

Before closing this section we should mention as to the carbon flash. It has been shown by HHS to commence when the carbon-oxygen core grows up to about $0.72M_\odot$, if the neutrino loss due to the universal Fermi interaction is not taken into account. Since the electron-degeneracy is as weak as $\psi_e=6$ at the time of the commencement, the maximum central temperature at $\psi_e=2.3$ is as low as

$$\log T_c,\text{max} = \log T_c(\text{at } \psi_e=6) + 0.125, \quad (4·13)$$

according to Eq. (2·9), and the explosion of the core can not be expected.

5. **Effect of finite efficiency of convective heat-transport**

In the preceding section we have neglected the effect of the finite efficiency of the convective heat-transport. It may, however, be important especially near the peak of the helium flash, since the energy flow in the convection core reaches as much as $10^{44}$ erg sec$^{-1}$.

For a fully developed turbulent convection in a stellar core, the exchange of heat between the large eddies by the microturbulence$^{16}$ will be important in determining the heat transport, which stands for the exchange of heat by radiation in a stellar surface convection zone.$^{17}$ Herring$^{18}$ has made detailed calculations of convection in a thin layer taking into account the nonlinear coupling between
the fluctuations and the mean flow. However, similar calculation for the present problem has never been made. Moreover, the compressibility of fluid will be important as well as the spherical geometry, since the scale-height of pressure is smaller than or comparable to the distance from the center. Therefore, the Boussinesq approximation, which is often used to study the convection in a thin layer, may fail in our problem. It is unfortunate that we have to use the mixing length theory to estimate the effect of the finite efficiency of convective heat-transport, which leaves one parameter corresponding to this efficiency undetermined.

a) Structure of the core with super-adiabatic temperature gradient

In this sub-section we shall discuss the condition for the existence of solution consistent with respect to the super-adiabatic temperature gradient and the nuclear energy generation. The evolutionary sequences of cores will be discussed in § 6.

According to the mixing length theory the flux of heat transported by convection is written as

\[ L_r = 4\pi r^2 \epsilon_p \rho \left( \frac{G M_r}{T r^4} \right)^{1/2} (\Delta T)^{3/2} \frac{r}{4}, \]  

(5.1)

where \( \Delta T \) denotes the super-adiabatic temperature gradient,

\[ \Delta T = \left( 1 - \frac{1}{T} \right) \frac{T}{P} \frac{dP}{dr} = \frac{dT}{dr}, \]  

(5.2)

and \( l \) denotes the mixing length. We assume the mixing length is equal to \( \alpha \) times the scale-height of pressure, i.e. using Eq. (2.3)

\[ l = -\alpha \frac{dr}{d \ln P} = \alpha \frac{r}{V}. \]  

(5.3)

Another possibility is to assume the mixing length to be proportional to \( r \). However, in this case the super-adiabatic temperature gradient would be infinitely large in the very central region, in order to transport the flux which is proportional to \( r^3 \).

We approximate the nuclear energy generation and the absorption of heat, Eq. (3.3), by

\[ \epsilon_n = \epsilon_{n,e} \sigma^3 \theta^4, \quad \epsilon_a = \epsilon_{a,e} \sigma^p \theta^n, \]  

(5.4)

respectively, where \( \theta \) is defined by

\[ \theta = T/T_e, \]  

(5.5)

and where \( s \) is taken as 15 in this section. The flux of heat to be transported, Eq. (3.2), can be written as

\[ L_r = f L_n = \epsilon_{n,e} M_0 \left\{ F_n(\xi) - \frac{\epsilon_{n,e}}{\epsilon_{n,e}} F_n(\xi) \right\}, \]  

(5.6)
where

\[
F_n(\xi) = \int_0^\xi \sigma^3 \theta^2 \xi^3 d\xi, \quad F_u(\xi) = \int_0^\xi \sigma^\nu \theta \xi^3 d\xi.
\] (5·7)

Using the equation of state and using Eqs. (2·1) ~ (2·4), (5·2) and (5·3), the equations of the convective heat-transport (5·1) and (5·6) are transformed to the equation of the local polytropic index as,

\[
\frac{5}{3} \frac{N}{N+1} - 1 = -C \left( \frac{f^2 \sigma}{\xi^2 \omega} \right)^{1/3},
\] (5·8)

where \(C\) is a constant defined by

\[
C = \left( \frac{8}{3} \right)^{1/3} \left( \frac{G}{3} \right)^{1/3} \left( \frac{\rho_e}{P_e} \right)^{1/3} \left( \frac{L_n}{\alpha^2} \right)^{1/3}.
\] (5·9)

For the purpose to know the central temperature or the effective polytropic index, the core can be well approximated by single star solution as shown in § 2. In the present case Eqs. (2·3) and (2·4) must be numerically integrated to obtain such single star solutions, since the local polytropic index, Eq. (5·8), changes by much in the relatively central region, in contrast to the case of the preceding section.

In this subsection we shall treat only the case where the super-adiabatic temperature gradient is small compared with the adiabatic one and we shall be concerned with structures near the peak of the helium flash only. Then, \(P\) and \(q\) in Eq. (5·7) can be well approximated by 0 and 1, respectively, since \(\psi\) is nearly constant in the convection core. To calculate the nuclear energy generation we must know the distribution of temperature in the core. For simplicity, the relation between the local polytropic index and the temperature gradient is approximated by one at \(\psi = 4\), i.e.

\[
\frac{d \log \theta}{d \log \omega} = \frac{1}{n+1} = \frac{2}{5} \left( 1.251 - \frac{5}{3} \frac{1}{N+1} \right) 0.251.
\] (5·10)

Since the extent of the convection core is about 80 percent of the core near the peak of the flash as shown in the preceding section, \(f\) defined by Eq. (5·6) is set to vanish at \(\psi/\psi_1 = 0.8\) adjusting \(\varepsilon_{a,e}/\varepsilon_{n,e}\). In the outer region, where \(V\) is larger than about 10, the local polytropic index is not important in determining \(\psi\), and then it is put equal to 1.5 for the sake of convenience. The boundary conditions are usual ones for a single star.

We solve the structures for different values of \(C\) defined by Eq. (5·9) in the increasing order, beginning with \(C = 0\) which gives the Emden solution of polytropic index 1.5. As the heat flux \(f(\xi)\) for the \(i\)-th solution is not known, we use \(f(\xi)\) calculated from the \((i-1)\)-th solution to obtain the first approximation of the \(i\)-th solution. Then, the second approximation of the \(i\)-th solution is obtained by using \(f(\xi)\) calculated from the first approximation. The difference
Table V. Nondimensional structures of core with the super-adiabatic temperature gradient.

<table>
<thead>
<tr>
<th>(i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C)</td>
<td>0</td>
<td>0.03</td>
<td>0.06</td>
<td>0.09</td>
<td>0.12</td>
<td>0.15</td>
<td>0.18</td>
<td>0.21</td>
<td>0.24</td>
</tr>
<tr>
<td>(\log \varphi_1)</td>
<td>1.0305</td>
<td>1.0225</td>
<td>1.0145</td>
<td>1.0063</td>
<td>0.9981</td>
<td>0.9898</td>
<td>0.9812</td>
<td>0.9726</td>
<td>0.9638</td>
</tr>
<tr>
<td>(-\Delta \log \varphi_1)</td>
<td>0</td>
<td>0.0080</td>
<td>0.0159</td>
<td>0.0242</td>
<td>0.0324</td>
<td>0.0407</td>
<td>0.0493</td>
<td>0.0578</td>
<td>0.0667</td>
</tr>
<tr>
<td>(\Delta \log T_c)</td>
<td>0</td>
<td>0.0106</td>
<td>0.0213</td>
<td>0.0322</td>
<td>0.0431</td>
<td>0.0542</td>
<td>0.0657</td>
<td>0.0771</td>
<td>0.0889</td>
</tr>
<tr>
<td>(\Delta \log L_n)</td>
<td>0</td>
<td>0.163</td>
<td>0.325</td>
<td>0.493</td>
<td>0.660</td>
<td>0.829</td>
<td>1.005</td>
<td>1.180</td>
<td>1.361</td>
</tr>
<tr>
<td>(-\log F_n)</td>
<td>0.530</td>
<td>0.566</td>
<td>0.603</td>
<td>0.641</td>
<td>0.679</td>
<td>0.718</td>
<td>0.757</td>
<td>0.796</td>
<td>0.836</td>
</tr>
<tr>
<td>(\log F_a)</td>
<td>0.721</td>
<td>0.703</td>
<td>0.685</td>
<td>0.667</td>
<td>0.649</td>
<td>0.630</td>
<td>0.611</td>
<td>0.592</td>
<td>0.572</td>
</tr>
<tr>
<td>(-\log (\alpha/\mu^{1/3})_1)</td>
<td>0.000</td>
<td>0.014</td>
<td>0.028</td>
<td>0.042</td>
<td>0.056</td>
<td>0.070</td>
<td>0.085</td>
<td>0.099</td>
<td>0.113</td>
</tr>
<tr>
<td>(N_{\min})</td>
<td>1.500</td>
<td>1.420</td>
<td>1.339</td>
<td>1.258</td>
<td>1.177</td>
<td>1.096</td>
<td>1.015</td>
<td>0.935</td>
<td>0.855</td>
</tr>
<tr>
<td>(N_{\text{eff}})</td>
<td>1.5</td>
<td>1.447</td>
<td>1.396</td>
<td>1.344</td>
<td>1.293</td>
<td>1.243</td>
<td>1.192</td>
<td>1.142</td>
<td>1.092</td>
</tr>
<tr>
<td>(\log \gamma)</td>
<td>(\infty)</td>
<td>1.210</td>
<td>1.052</td>
<td>0.991</td>
<td>0.967</td>
<td>0.965</td>
<td>0.979</td>
<td>1.002</td>
<td>1.034</td>
</tr>
</tbody>
</table>

Fig. 5. Structures of the cores of \(i=1\) and \(5\) in Table V (solid lines) and of \(\tau=0.05\) in Table VI (dotted line):

(a) \(U-V\) curves.
(b) distributions of the heat flux, \(f\).

between the values of \(\log \varphi_1\) obtained from the second and the first approximations is found to be less than 0.0005 in our calculations. It means that the detailed functional form of \(f(\xi)\) is much less important than the absolute value of \(C\) and that the approximations in calculating \(f(\xi)\) do not introduce an appreciable error.

The results of the above calculation with an electronic computer IBM 7090 are summarized in Table V, where \(\Delta\) means the difference from the case for \(C=0\), i.e. the case of infinite efficiency of the convective heat-transport, but for a fixed \(\varphi_x\). The values of the integrals (5-7) at the surface, \(F_n\) and \(F_a\), are shown to see the central concentrations of energy generation and absorption, which have been taken into account to calculate \(\Delta \log L_n\) from \(\Delta \log T_c\). The minimum local polytropic index, \(N_{\min}\), is found to occur near \(V=0.2\) and it is
smaller than the effective polytropic index. For some solutions the \( U-V \) curves and the heat fluxes \( f \) are illustrated in Fig. 5 to visualize the effect of the superadiabatic temperature gradient.

Using a structure, for which the effect of the finite efficiency of convective heat-transport is neglected, we consider to obtain a structure, for which this effect is taken into account but \( \psi_e \) is unchanged. Quantities of the former structure will be denoted with the superscript \((0)\). Both structures are related by Eq. (5·9), which is rewritten as,

\[
\alpha = \left( \frac{320}{27} G^2 \right)^{1/4} \left( \frac{\rho_e^{(0)}}{P_e^{(0)}} \right)^{5/4} L_n^{(0)1/2} \gamma. \tag{5·11}
\]

In the above equation \( \gamma \) is defined by

\[
\log \gamma = \frac{1}{2} \Delta \log L_n - \frac{5}{4} \Delta \log T_e - \frac{3}{4} \log C, \tag{5·12}
\]

and its values are shown in the bottom of Table V. Since \( \gamma \) is minimum where \( C \) is about 0.15, a corresponding lower limit of \( \alpha \) exists. Quantities corresponding to this minimum value of \( \gamma \) will be called as critical ones. If we assume \( \alpha \) smaller than this critical value, no consistent solution exists. In other words, if the convective heat-transport is less efficient than what this critical value gives, an instability occurs in the sense that an increase in the nuclear energy generation, due to the superadiabatic temperature gradient and the subsequent structural change, always overcomes an increase of the convective heat-transport due to this superadiabatic gradient. The solutions for \( C>0.15 \) are physically meaningless.

In an evolutionary sequence, \( (\rho_e^{(0)}/P_e^{(0)})^{5/4}L_n^{(0)1/2} \) is increasing until the stage of \( \psi_e=4 \), as shown in the preceding section. If the assumed \( \alpha \) is larger than the critical value for the stage of \( \psi_e=4 \), the core safely passes through the peak of the flash. On the other hand, if it is smaller, the value of \( \gamma \) reaches its minimum before the core reaches the peak of the flash. For example this critical value of \( \alpha \) is 0.22 for the core of 0.53\( M_\odot \). Strictly speaking, the powers \( p \) and \( q \) of the absorption of heat in Eq. (5·4) increase, because of the appearance of the gradient in \( \psi \) in the convection core as seen from the values of \( \log (\sigma/\sigma^{(0)})_1 \) in Table V. This results in a decrease of \( f \), in an increase of the critical value of \( C \) and in a decrease of the critical value of \( \alpha \). In the limit of complete blocking of heat, i.e. in the limit that the nuclear energy generation is equal to the absorption of heat everywhere in the core, \( f \) vanishes and the critical value of \( C \) is infinitely large, which corresponds to the vanishing \( \alpha \). However, until this limit is precisely realized, the decrease of \( f \) will be small and the critical value of \( \alpha \) is not much different from one calculated above. Then, after the inequality,

\[
C > 0.12, \tag{5·13}
\]
b) Structure of the core in blocking of heat

Along an evolutionary sequence, the structures of the core are calculated using Eq. (2·3) with the condition for the blocking of heat, \( \varepsilon_n = \varepsilon_a \). This condition is rewritten by using the nondimensional variables and replacing the differential by the difference equation as,

\[
\log \left( \frac{\omega}{\sigma^{5/3}} \right)_{r+\delta r} = \log \left( \frac{\omega}{\sigma^{5/3}} \right)_r + \left( \frac{\sigma^2 \theta^2}{\sigma} - 1 \right) \delta \tau ,
\]

(5·14)

where change along an evolutionary sequence is denoted by \( \delta \). In this equation the nondimensional time interval between the two successive stages is defined by

\[
\delta \tau = \frac{2}{3} \log e \frac{\rho_e}{P_e} \varepsilon_n \delta t = \delta \log \frac{P_e}{\rho_e^{5/3}} = - \delta \log \left( \frac{\omega}{\sigma^{5/3}} \right)_1 ,
\]

(5·15)

and is taken as 0.01 in the numerical calculation. The equation of state, which is required to calculate Eq. (5·14), is approximated by

\[
\theta^{5/3} = \omega \left( \frac{\sigma}{\sigma^{5/3}} \right)^{4.23} ,
\]

(5·16)

where the power, 4.23, is the mean one between \( \psi_e = 4 \) and 3.3. The initial distribution in \( \omega/\sigma^{5/3} \) at \( \tau = 0 \) is taken from the solution for \( C = 0.12 \) in Table V.

The results of calculation with IBM 7090 are summarized in Table VI and the \( U-V \) curve for the solution, \( \tau = 0.05 \), is illustrated in Fig. 5(a) by dotted line. In Table VI \( \Delta \) means the difference from the first solution in Table V for fixed \( \psi_e \), which extracts the change due to the blocking of heat only. In obtaining a change along an evolutionary sequence, the change due to the decrease in \( \psi_e \) must be added to the above change. According to Eqs. (2·9) and (5·14), the change in the central temperature can be written as,

\[
\delta \log T_e = \delta \log T_e \left[ -2 \frac{d \log \left( \frac{(2/3)F_{\psi/3}(\psi)}{(1/2)F(\psi)} \right)}{d \log \left( \frac{(2/3)F_{\psi/3}(\psi)}{F(\psi)} \right)} \right] \delta \tau .
\]

(5·17)

The distributions in \( \omega/\sigma^{5/3} \) through the core are quite different between the solutions in Table VI and ones in Table V as will be illustrated in Fig. 7. However, \( \Delta \log T_e \) as a function of \( \omega/\sigma^{5/3} \), \( _1 \) are found to be only slightly different between these two cases. After \( \psi_e \) has become smaller than 3, Eqs. (5·16) and (5·14) overestimate \( -\delta (\omega/\sigma^{5/3}) \) and the distribution in \( \omega/\sigma^{5/3} \) should be corrected. According to the above comparison, however, the relation between \( \Delta \log T_e \) and \( -\log (\omega/\sigma^{5/3}) \) can be applicable to a good approximation even after the degeneracy has dissolved at the center of the core.

In the limit of nondegeneracy, the decrease in the central temperature due to the second term in the right-hand side of Eq. (5·17) is maximum and it
§ 6. Criterion for explosion of the core and summary of the results

It is easy to calculate an evolutionary sequence of a core with physical variables, using the solutions in the previous sections, if the core mass and the ratio of the mixing length to the scale-height of pressure, $\alpha$, are fixed. In this section the core mass is taken as $0.53M_\odot$ for numerical examples as in § 4.

If the efficiency of the convective heat-transport can be considered as infinite, i.e. if $\alpha=\infty$, the results have already been obtained in § 4. The maximum central temperature through the flash is given by Eq. (4·2) and the explosion occurs only in a core of mass larger than $0.70M_\odot$. As long as the plasma-neutrino loss is neglected in the commencement of the helium flash, such a large core-mass can not be expected for any reasonable initial chemical composition.

If $\alpha$ is finite, the super-adiabatic temperature gradient appears in the core, which gives a higher central temperature than the adiabatic one. However, if Eq. (5·13) is not satisfied throughout the flash, which can be rewritten taking into account the dependences on the core mass as

$$\alpha < 0.22 \left( M_\ast/M_\odot \right)^8, \quad \text{for} \quad M_\ast \leq 0.68M_\odot, \quad (6\cdot1)$$

the blocking of heat does not occur. Table V gives the nondimensional solutions without the blocking. In this case the maximum central temperature to be attained is larger than Eq. (4·2) only by 0.043 logarithmically at most, as seen from the fifth solution in Table V. It reduces the lower limit of the core mass for explosion to $0.68M_\odot$ at most. Evolutionary sequences near the peak of the flash are summarized in Table VII for two cases of $\alpha=0.5$ and 0.25 and the run of the central temperature is illustrated in Fig. 6.

If the condition (6·1) is satisfied, the blocking of heat occurs. It is an instability in the sense that an increase in the nuclear energy generation due to the super-adiabatic temperature gradient overcomes the increased heat-transport due to this gradient. The solutions in Table VI are the continuation of the fifth
D. Sugimoto

Table VII. Evolutionary sequences near the peak of the helium flash of the core of 0.53M\(_{\odot}\), in the case of finite efficiency of the convective heat-transport but without the blocking of heat.

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>Stage</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>(\phi_e)</td>
<td>0.006</td>
<td>0.018</td>
<td>0.023</td>
<td>0.021</td>
<td>0.014</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>(\log T_e)</td>
<td>8.267</td>
<td>8.324</td>
<td>8.343</td>
<td>8.347</td>
<td>8.336</td>
<td>8.234</td>
</tr>
<tr>
<td></td>
<td>(\log L_n)</td>
<td>43.42</td>
<td>44.31</td>
<td>44.44</td>
<td>44.28</td>
<td>43.82</td>
<td>41.09</td>
</tr>
<tr>
<td></td>
<td>(\phi_1)</td>
<td>7.2</td>
<td>5.3</td>
<td>4.3</td>
<td>3.5</td>
<td>2.7</td>
<td>1.0</td>
</tr>
<tr>
<td>0.25</td>
<td>(\phi_e)</td>
<td>0.012</td>
<td>0.045</td>
<td>0.078</td>
<td>0.064</td>
<td>0.032</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td>(\log T_e)</td>
<td>8.269</td>
<td>8.334</td>
<td>8.363</td>
<td>8.362</td>
<td>8.342</td>
<td>8.235</td>
</tr>
<tr>
<td></td>
<td>(\log L_n)</td>
<td>43.46</td>
<td>44.45</td>
<td>44.73</td>
<td>44.51</td>
<td>43.92</td>
<td>41.11</td>
</tr>
<tr>
<td></td>
<td>(\phi_1)</td>
<td>7.4</td>
<td>5.8</td>
<td>5.0</td>
<td>3.9</td>
<td>2.8</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Fig. 6. Runs of the central temperatures near the peak of the helium flash of the core of 0.53M\(_{\odot}\) for the cases of \(\alpha=\infty\), 0.25 and 0.22. In the last one the blocking of heat occurs after the circle and the effect of inertia leads the core to explosion after the cross on this curve.

According to the mixing length theory, the velocity of the convection current \(v\) and the turbulent pressure \(P_t\) can be written as solution in Table V. A critical case of \(\alpha=0.22\) is shown in Table VIII, in which Stages later than 6 are in blocking. The run of the central temperature and the distributions in \(\phi\) in the core are illustrated in Figs. 6 and 7, respectively. After Stage 9, the value of \(x_e^{(v)}\) is larger than the critical value given by Eq. (4.12). Then, the effect of inertia, opposing the expansion of the core, gives higher values of the central temperature than ones given in Table VIII, and it leads the core to explosion, as discussed in § 4b.

If the blocking of heat occurs, the central temperature continues to rise even if the electron degeneracy has been dissolved, as noticed in § 5b. Then the condition (6·1) is also the criterion for explosion of the core of mass smaller than 0.68M\(_{\odot}\). For the core of mass above 0.68M\(_{\odot}\), explosion occurs without the blocking of heat.
Table VIII. Evolutionary sequence of the core of 0.53 \( M_\odot \) for \( \alpha = 0.22 \). The core is led to explosion after Stage 9.

<table>
<thead>
<tr>
<th>Stage</th>
<th>( \phi_e )</th>
<th>( \log T_e )</th>
<th>( \log P_e )</th>
<th>( \log \rho_e )</th>
<th>( \log \epsilon_{\text{av}} )</th>
<th>( \log L_n )</th>
<th>( \phi_1 )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stages with the convective heat-transport</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>7.00</td>
<td>8.270</td>
<td>22.217</td>
<td>5.765</td>
<td>12.17</td>
<td>43.46</td>
<td>7.5</td>
<td>0.014</td>
</tr>
<tr>
<td>2</td>
<td>5.75</td>
<td>8.311</td>
<td>22.129</td>
<td>5.703</td>
<td>12.81</td>
<td>44.14</td>
<td>6.4</td>
<td>0.030</td>
</tr>
<tr>
<td>3</td>
<td>4.91</td>
<td>8.341</td>
<td>22.064</td>
<td>5.652</td>
<td>13.22</td>
<td>44.55</td>
<td>6.0</td>
<td>0.060</td>
</tr>
<tr>
<td>4</td>
<td>4.42</td>
<td>8.361</td>
<td>22.030</td>
<td>5.621</td>
<td>13.48</td>
<td>44.79</td>
<td>5.9</td>
<td>0.090</td>
</tr>
<tr>
<td>5</td>
<td>4.00</td>
<td>8.378</td>
<td>21.987</td>
<td>5.586</td>
<td>13.68</td>
<td>44.97</td>
<td>5.8</td>
<td>0.120</td>
</tr>
<tr>
<td>Stages in blocking of heat</td>
<td>( t(\text{sec}) )</td>
<td>( x_e^{(0)} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3.78</td>
<td>8.394</td>
<td>21.982</td>
<td>5.577</td>
<td>13.91</td>
<td>45.06</td>
<td>-22.1</td>
<td>2.5 \times 10^{-4}</td>
</tr>
<tr>
<td>8</td>
<td>3.39</td>
<td>8.424</td>
<td>21.974</td>
<td>5.560</td>
<td>14.33</td>
<td>45.19</td>
<td>-8.4</td>
<td>1.6 \times 10^{-3}</td>
</tr>
<tr>
<td>9</td>
<td>3.21</td>
<td>8.439</td>
<td>21.970</td>
<td>5.552</td>
<td>14.52</td>
<td>45.22</td>
<td>-5.0</td>
<td>3.6 \times 10^{-3}</td>
</tr>
<tr>
<td>10</td>
<td>3.05</td>
<td>8.452</td>
<td>21.967</td>
<td>5.544</td>
<td>14.71</td>
<td>45.24</td>
<td>-2.8</td>
<td>7.7 \times 10^{-3}</td>
</tr>
<tr>
<td>11</td>
<td>2.90</td>
<td>8.466</td>
<td>21.964</td>
<td>5.536</td>
<td>14.89</td>
<td>45.23</td>
<td>-1.0</td>
<td>1.7 \times 10^{-2}</td>
</tr>
<tr>
<td>12</td>
<td>2.75</td>
<td>8.478</td>
<td>21.961</td>
<td>5.528</td>
<td>15.06</td>
<td>45.22</td>
<td>0.0</td>
<td>3.7 \times 10^{-2}</td>
</tr>
</tbody>
</table>

Fig. 7. Distributions in entropy through the core of Stages 2, 5 and 10 in Table VIII.

until the solution for \( \tau = 0.07 \) in Table VI or Stage 12 in Table VIII. Even in the later stages it is confined in a very central region of 1/50 of the core mass. In the case of 0.53\( M_\odot \) the effect of inertia becomes important first, but in a much less massive core the supersonic flow will possibly appear before the explosion as a bulk occurs.

Opik\(^{(6)}\) has shown that the convective heat-transport in a thin layer is about a thirtieth of the value which is expected assuming the mixing length equal to the depth of the layer, since the microturbulences exchange heat between the layers.

\[ \frac{v^2}{a^2} = \frac{9}{5} \frac{P_t}{P_g} = \frac{9 \alpha^2}{100} \left( \frac{5 P_g}{2 c_p \rho T} \right)^{7/5} \times \left( 1 - \frac{5}{3} \frac{N}{N+1} \right), \text{(6.2)} \]

where \( a \) and \( P_g \) denote the velocity of sound and the gas pressure, respectively. Since \( 5P_g/2c_p \rho T \) is of the order of unity, the supersonic flow and the turbulent pressure become important only for \( N+1 \ll 1 \). Such a region does not appear until the solution for \( \tau = 0.07 \) in Table VI or Stage 12 in Table VIII. Even in the later stages it is confined in a very central region of 1/50 of the core mass. In the case of 0.53\( M_\odot \) the effect of inertia becomes important first, but in a much less massive core the supersonic flow and the resulting shock wave will possibly appear before the explosion as a bulk occurs.

Opik\(^{(6)}\) has shown that the convective heat-transport in a thin layer is about a thirtieth of the value which is expected assuming the mixing length equal to the depth of the layer, since the microturbulences exchange heat between the layers.\[ \frac{v^2}{a^2} = \frac{9}{5} \frac{P_t}{P_g} = \frac{9 \alpha^2}{100} \left( \frac{5 P_g}{2 c_p \rho T} \right)^{7/5} \times \left( 1 - \frac{5}{3} \frac{N}{N+1} \right), \text{(6.2)} \]
large eddies. Then, a possibility of such a value of $\alpha$, as considered in our calculation to lead the core of $0.53M_\odot$ to explosion, can not be excluded.

Therefore, it is important to investigate the convection in the stellar core quantitatively as noticed in § 5. It is also important to determine the critical core-mass for helium flash in the presence of plasma-neutrino loss as noticed in § 1. These will give us some information related with the problems such as the stellar explosion in helium flash and as the luminosity of horizontal branch in HR diagrams of globular clusters. If the explosion occurs, the star will be related with super-nova of Type I, the energy release of which is of the same order of magnitude as the potential energy of the core, i.e. $10^{48}\sim 10^{49}$ erg. Even if the explosion as a bulk does not occur, it is to be noticed that small explosions in the outer region of the envelope may be expected, since the timescale of free-fall in this region, about a month, is much longer than the timescale of the helium flash in the core.

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