A charge independent pairing interaction in $j$-$j$ coupling scheme and a charge-spin independent pairing interaction in $L$-$S$ coupling scheme are exactly solved in a degenerate model by the aid of group theory. Groups used here are independent of the number of states concerned unlike the usual shell model technique. Our method is deeply connected with the generalized Bogoliubov transformation and it is conjectured that this transformation combined with projection into the definite number and the isospin (symmetry) state is very useful for the treatment of pair-interacting many body system. The classification of states through the group used here are equivalent to the seniority scheme generalized for unlike particles and degenerate levels. Our method simplifies mathematics somewhat and derives easily some new properties of seniority scheme for unlike particles.

§ 1. Introduction

Bohr, Mottelson and Pines \(^1\) suggested the energy gap in nuclei as a result of pairing effect similar to that in superconducting states of metals. Since then many works \(^2\) \(-\) \(^4\) on pairing correlations have been done in analogy with Bardeen, Cooper and Schrieffer's theory \(^5\) of superconductivity.

One important difference is that there are two kinds of particles in nuclei, i.e. protons and neutrons, while only electrons play an essential role in metals. Hence several attempts \(^5\) \(-\) \(^{10}\) have been made to take account of neutron-proton interactions as well as proton-proton and neutron-neutron interactions simultaneously. Most of them \(^7\) \(-\) \(^9\) stressed that a simple generalization of pair correlations, i.e. the generalized Bogoliubov transformation \(^11\) is insufficient and four-body correlations are very important.

To clarify the applicability of those treatments, is desirable to solve a certain model case exactly. In the shell thory, it has long been known that a charge independent pairing interaction in a single level can be exactly solved by seniority and reduced isospin scheme \(^12\) which can, of course, be extended to the case
of degenerate levels.\textsuperscript{13}

In this paper we shall develop another exact theory for a charge independent pairing interaction in $j\cdot j$ coupling scheme (§ 2) and for a charge-spin independent pairing interaction in $L-S$ coupling scheme (§ 4). Our method is directly connected with the generalized Bogoliubov transformation and so leads us to some conjectures on the treatment of many nucleon system with pairing interactions (§ 5). Furthermore, it gives us some technical advances in shell model calculation of neutron-proton mixed system under seniority scheme. Classifications of states through the groups used in §§ 2 and 4 and their correspondence to the seniority scheme for unlike particles are interpreted in §§ 3 and 4.

§ 2. A charge independent pairing interaction in $j\cdot j$ coupling scheme

Let us consider a proton-neutron system where all nucleons interact only through a charge independent pairing interaction in $j\cdot j$ coupling scheme and single particle levels concerned are all degenerate. The Hamiltonian of the system is

$$H = -G\{E_{-1}E_{-3} + E_{-2}E_{-3} + E_{-3}E_{-3}\},$$

(2·1)

where

$$E_{a} = \sum_{j} E_{a}(j), \ (a = \pm 1, \pm 2, \pm 3)$$

(2·2a)

and

$$E_{1}(j) = \{E_{-1}(j)\}^\dagger = (1/\sqrt{2})\sum_{m}(-)^{j-m}a_{jm}^\dagger a_{j-m},$$

(2·2b)

$$E_{2}(j) = \{E_{-2}(j)\}^\dagger = \sum_{m}(-)^{j-m}a_{jm}^\dagger b_{j-m},$$

$$E_{3}(j) = \{E_{-3}(j)\}^\dagger = (1/\sqrt{2})\sum_{m}(-)^{j-m}b_{jm}^\dagger b_{j-m}.$$  

Here $a_{jm}^\dagger, a_{jm}$ ($b_{jm}^\dagger, b_{jm}$) are the creation and annihilation operators of proton (neutron) in the state $(j m)$. If the summation over $j$ is restricted to a single $j$, the Hamiltonian is proportional to the operator $Q$ defined by Edmonds and Flowers.\textsuperscript{13} Using the groups $U(2j+1)$ and $S_{p}(2j+1)$, they obtained the eigenvalues and the eigenvectors of the operator $Q$ which are represented by the seniority $v$ and the reduced isospin $t$. Helmers\textsuperscript{14} also obtained the solution by using the group $S_{p}(4)$ which is the commutator group of $S_{p}(2j+1)$. The method developed in this section is equivalent to Helmers', and a generalization of quasi-spin method for like particles.\textsuperscript{15} But we formulate it in a somewhat different way,\textsuperscript{16} since it is indispensable, for the later discussions, to know the structure and the physical meaning of the group $S_{p}(4)$, which also makes it easy to understand more complicated case of a pairing interaction in $L-S$ coupling scheme.

By the successive construction of commutators among the six operators in Eq. (2·2a), we obtain the smallest algebra including them, whose other
elements are

\[ H_i = \hat{N}_i - \Omega = \sum_{jm} a_{jm}^i a_{jm} - \Omega, \]

\[ H_z = \hat{N}_z - \Omega = \sum_{jm} b_{jm}^i b_{jm} - \Omega, \]

\[ E_a = E_{-a} = \hat{T}_{+} = \hat{T}_{-} = \sum_{jm} b_{jm}^i a_{jm}, \]

where \( \Omega = \sum (j + 1/2) \) is the total number of pairs available in the degenerate levels, and the meaning of these operators are evident. These ten operators in Eqs. (2.2a) and (2.3) satisfy the following commutation relations:

\[
\begin{align*}
[H_i, H_j] & = 0, & (i \text{ and } j = 1, 2) \\
[H_i, E_a] & = r_i(\alpha) E_a, & (\alpha = \pm 1, \pm 2, \pm 3, \pm 4) \\
[E_a, E_{-a}] & = \sum_{i} r_i(\alpha) H_i, \\
[E_a, E_p] & = \begin{cases} N_{\alpha\beta} E_{\gamma}, & \text{for } r(\alpha) + r(\beta) = r(\gamma) \\ 0, & \text{otherwise.} \end{cases}
\end{align*}
\]

Here, the roots \( r(\alpha) = (r_1(\alpha), r_2(\alpha)) \) are

\[
\begin{align*}
(1) & = -r(-1) = (2, 0) \\
(2) & = -r(-2) = (1, 1) \\
(3) & = -r(-3) = (0, 2) \\
(4) & = -r(-4) = (-1, 1)
\end{align*}
\]

while \( \Omega_{\alpha\beta} \) is given by relations

\[ \Omega_{\alpha\beta} = -\Omega_{\beta\alpha} = -N_{\alpha-\beta}, \]

and

\[ \Omega_{13} = \Omega_{14} = \Omega_{34} = \Omega_{12} = \Omega_{54} = \sqrt{2}. \]

These are just the standard commutation relations of the Lie algebra of type \( C_2, (3), (7), (8) \).

Then using the Casimir operator \( \hat{C} \) of \( C_4 \) defined as

\[ \hat{C} = \sum_i H_i^2 + \sum_{a=-\ell} E_a E_{-a}, \]

we can rewrite the Hamiltonian (2.1) as

\[
\hat{H} = -\frac{1}{2} G \left\{ \hat{C} - \sum_i H_i^2 - (E_4 E_{-4} + E_{-4} E_4) + \sum_{i=1}^{3} r_i(\alpha) H_i \right\}
\]

\[ = -\frac{1}{2} G \left\{ \hat{C} - 2 \left( \frac{\hat{N}}{2} - \Omega \right)^2 - 2 \hat{T}^2 + 6 \left( \frac{\hat{N}}{2} - \Omega \right) \right\}, \]

\( \text{a)} \) This is isomorphic to the Lie algebra of the groups of \( O(5) \) and \( S_6(4) \) (which some authors denote by \( S_{64} \)). From now on, we do not distinguish the group and its Lie algebra, and represent both by the notation \( C_2 \), so that no confusion may occur.
where $\hat{T}_z = (\hat{N}_p - \hat{N}_n)/2$ and $\hat{N} = \hat{N}_p + \hat{N}_n$. Since operators $H, \hat{C}, \hat{N}, \hat{T}_z^2$ and $\hat{T}_z$ commute with each other, we can diagonalize all these operators simultaneously on the bases of the irreducible representation of $C_2$. An irreducible representation of $C_2$ is characterized by a set of two non-negative integers $(\lambda_1, \lambda_2)$ which specifies the highest weight of the representation:

$$M = \lambda_1 M_1 + \lambda_2 M_2 ,$$  \hspace{1cm} (2.9)

where the fundamental dominant weights $M_1$ and $M_2$ are

$$M_1 = (1, 0) \text{ and } M_2 = (1, 1).$$  \hspace{1cm} (2.10)

Here the weight $m = (m_1, m_2)$ is the set of eigenvalues of the operators $H = (H_1, H_2)$, and the highest weight is the weight with maximum $m_1$ and maximum $m_2$ compatible with the maximum $m_1$ in the irreducible representation. In our model, the highest weight $M = (\lambda_1 + \lambda_2, \lambda_2)$ means that the maximum proton number among the states in the irreducible representation $D(\lambda_1, \lambda_2)$ is $(\lambda_1 + \lambda_2) + \Omega$ and the maximum neutron number among the states with the maximum proton number is $\lambda_2 + \Omega$. Thus we get the restriction

$$0 \leq \lambda_1 + \lambda_2 \leq \Omega .$$  \hspace{1cm} (2.11)

Since the Casimir operator is constant for an irreducible representation, we obtain its eigenvalue,

$$C = (\lambda_1 + \lambda_2)^2 + \lambda_2^2 + 4\lambda_1 + 6\lambda_2 ,$$  \hspace{1cm} (2.12)

by operating it on the state with the highest weight. Specifying the states in the same irreducible representation by the isospin $T$, its 3rd component $T_\theta$ and the total number $N$, we obtain the eigenvalue of the Hamiltonian (2.1) as

$$E = -G \left\{ \frac{1}{2} \{(\lambda_1 + \lambda_2)^2 + \lambda_2^2\} + 2\lambda_1 + 3\lambda_2 \\
- T (T + 1) - \left( \frac{N}{2} - \Omega \right)^2 + 3\left( \frac{N}{2} - \Omega \right) \right\} ,$$  \hspace{1cm} (2.13)

This expression is completely the same as that obtained by Edmonds and Flowers\textsuperscript{12} where quantum numbers $(\lambda_1, \lambda_2)$ are replaced by their $(\nu, t)$ subject to the relations

$$\begin{align*}
\lambda_1 &= 2t , \\
\lambda_2 &= -\frac{1}{2} \nu - t + \Omega ,
\end{align*}$$  \hspace{1cm} (2.14)

and

$$\Omega = j + 1/2 .$$
§ 3. Classification of states in \( jj \) coupling scheme

In this section we present the specification of bases of an irreducible representation of \( C_2 \) and classify the states in \( jj \) coupling scheme.

Characters of the representation \( D(\lambda_1, \lambda_2) \) of \( C_2 \) are given by \(^{(3)}\)

\[
\chi(\lambda_1 \lambda_2 ; \varphi) = \xi(\lambda_1 \lambda_2 ; \varphi) / \xi(0, 0) \tag{3·1}
\]

where,

\[
\xi(\lambda_1 \lambda_2 ; \varphi) = \{ \exp(i(\lambda_1 + \lambda_2 + 2)\varphi_1) - \exp(-i(\lambda_1 + \lambda_2 + 2)\varphi_1) \} \\
\times \{ \exp(i(\lambda_1 + 1)\varphi_2) - \exp(-i(\lambda_1 + 1)\varphi_2) \} \\
- \{ \exp(i(\lambda_1 + \lambda_2 + 2)\varphi_2) - \exp(-i(\lambda_1 + \lambda_2 + 2)\varphi_2) \} \\
\times \{ \exp(i(\lambda_1 + 1)\varphi_1) - \exp(-i(\lambda_1 + 1)\varphi_1) \}, \tag{3·2}
\]

and \( \varphi = (\varphi_1, \varphi_2) \) specifies the class of the group (angles of rotation). The dimension of \( D(\lambda_1, \lambda_2) \) \(^{(3)}\)

\[
N(\lambda_1 \lambda_2) = \chi(\lambda_1, \lambda_2; \varphi_1 = \varphi_2 = 0) \\
= (1 + \lambda_1)(1 + \lambda_2)(1 + \frac{1}{2}(\lambda_1 + \lambda_2))(1 + \frac{1}{3}(\lambda_1 + 2\lambda_2)). \tag{3·3}
\]

When we want to specify the bases of the representation by the eigenvalues of \( T^a, T^m, \) and \( \hat{N} = \hat{N} - 2\Omega = H_1 + H_2 \) as in § 2, we need another quantum number \( (\beta)_T \) which is not found in the eigenvalues of the operators simply constructed from the algebra. However, if we give up to choose the eigenstates of isospin, we can find the suitable quantum numbers which characterize the subgroups of \( C_2 \), that is, two different \( SU(2)'s \), the generators of which are

(i) \( p_+ = E_1 / \sqrt{2} \), \( p_- = E_{-1} / \sqrt{2} \), \( p_0 = H_1 / 2 \), \( \beta \)

(ii) \( n_+ = E_3 / \sqrt{2} \), \( n_- = E_{-3} / \sqrt{2} \), \( n_0 = H_2 / 2 \), \( \beta \)

we shall call these operators (i) and (ii) of (3·4) the proton and neutron quasi-spin respectively, and denote their magnitudes by \( S_p \) and \( S_n \). Since \( p^2, n^2, H_1 \) and \( H_2 \) commute with each other, we can specify the wave functions by two schemes:

(i) \(|(\lambda_1, \lambda_2)(T, \beta) \mathcal{J}, T_0; \alpha J \rangle \),

(ii) \(|(\lambda_1, \lambda_2)(S_p, S_n)m_1, m_2; \alpha J \rangle \), \tag{3·5}

where \( J \) is the total spin which can, of course, be chosen as a quantum number, since the generators of the above \( C_1 \) are all scalars in total spin space, and \( \alpha \) denotes the other additional quantum numbers. \( J \) and \( \alpha \) distinguish between the states with the same values of \( (\lambda_1, \lambda_2) \), \( (S_p, S_n) \) and \( (m_1, m_2) \) (i.e. those which belong to the equivalent but different irreducible representation of \( C_2 \)).

The set of bases of \( D(\lambda_1, \lambda_2) \) is shown graphically by the corresponding set of points on \( m = (m_1, m_2) \) plane which is called a weight diagram. As an
example, the weight diagram of $D(2,0)$ is given in Fig. 1, where the highest weight is shown by the suffix $h$, and operations $E_\alpha (\alpha = 1, 2, 3$ and 4) are shown by arrows which are root vectors $r(\alpha)$ in Eq. (2·5).

Decomposition of the set of points according to the scheme (i) and (ii) of Eq. (3·5) are illustrated in Fig. 2. From their physical meaning, it is understood that the seniority and the reduced isospin are the nucleon number and isospin of the state with the lowest weight which is shown by the suffix $l$ in Fig. 1. Therefore, the relation (2·13) is derived.

We obtain the multiplicity of each point (the number of independent vectors with the same weight $m$) by rewriting the character (3·1) into the form

$$x(\lambda_1, \lambda_2; \varphi) = \sum_m r_m \exp(i m \varphi),$$

(3·6)

where the coefficient $r_m$ gives the multiplicity of the weight $m$ from the definition of character.

From the Eqs. (3·1) and (3·6), and the geometry of the weight diagram, we obtain the following theorems:

(I) States in the representation $D(\lambda_1, 0)$, i.e. states with $(v=2\Omega-2t, t=\lambda_1/2)$ are uniquely specified only by $(T, T_0, \mathcal{H})$, and states with $\mathcal{H}=\pm 2(t-k)$ consist of those with $T=t, t-1, \cdots, t-k$, where $k$ is integer and $t \geq k \geq 0$.

(II) States in the representation $D(0, \lambda_2)$, i.e. states with $(v=2\Omega-2\lambda_2, t=0)$ are uniquely specified only by $(T, T_0, \mathcal{H})$, and states with $\mathcal{H}=\pm 2(2k-\lambda_2)$ and states with $\mathcal{H}=\pm 2(2k+1-\lambda_2)$ consist of those with $T=0, 2, \cdots, 2k$ and those with $T=1, 3, \cdots, 2k+1$, respectively, where $k$ is integer and $0 \leq k \leq \lambda_2/2$.

(III) States in the representation $D(1, \lambda_2)$, i.e. states with $(v=2\Omega-2\lambda_2-1, t=1/2)$ are uniquely specified only by $(T, T_0, \mathcal{H})$, and states with $\mathcal{H}=\pm (2k+1)$ consist of those with $T=1/2, 3/2, \cdots, 1/2+(\lambda-k)$, where $k=0, 1, 2, \cdots, \lambda$.

The proof of these theorems will be given in Appendix A.
of the representations $D(\lambda_1, \lambda_2)$ specified by $(J, T)$ and $(S_p, S_n)$ are tabulated in Tables I and II in Appendix B up to $\lambda_1 + \lambda_2 \leq 6$.

On the other hand, the allowed values of total spin $J$ and $\alpha$ for a given representation $D(\lambda_1, \lambda_2)$ depend on levels under consideration. In order to find these values, we have only to consider the states with the lowest weight, i.e. the $v$-particle states with the seniority $v$ and reduced isospin $t$. Returning to the usual shell model method, where $(v, t)$ specify the irreducible representation of $S_p(2\Omega)$, we can reduce this representation into the irreducible representations of the subgroup $O(3)$ which is the rotation of space. This generalization of $S_p(2j + 1)$ to $S_p(2\Omega)$, which is recently carried out by A. Arima and H. Kawarada, gives rise to some new situations, e.g. the representation $(v=2, t=0)$ of $S_p(2\Omega)$ includes states with $J=0$ unless $\Omega$ consists of single $j$.

It is worthwhile to devote the final part of this section to the following remarks. It seems to the author that the usual shell model technique for unlike particles is not yet established. One of the important obstacles is that we can not easily obtain the coefficients of fractional parentage of the type

$$[j^\alpha(\beta \alpha \nu \nu T N J) j^\alpha(10) T J] j^{\alpha+\beta \alpha \nu \nu T J]^*$$

(3.7)

which play an essential role in the seniority scheme.

According to our formulation of seniority scheme developed in this paper, we may obtain a somewhat advanced method to treat unlike particle system. For example, the above three theorems show that $\beta_i$ and $\beta$ are unnecessary in the c.f.p. for special $(v, t)$ which situation makes the problem very easy. Moreover, since our method does not depend on $j$ explicitly, we can get the results available for any $j$ states. This formalism will be applied to the usual c.f.p. techniques, especially to the calculation of c.f.p. of the type (3.7) in a subsequent paper.

§ 4. A charge-spin independent pairing interaction in $L-S$ coupling scheme

Next, let us consider a pairing interaction studied by Flowers and Vujčić, which is invariant under any unitary transformation in charge-spin space. We again restrict ourselves to a strong coupling limit. The Hamiltonian is then written as

$$H = -G \sum_{pq} \sum_{l,m} (-)^{l-m} a^\dagger_{l m} a^\dagger_{l-m} \sum_{l',m'} (-)^{l'-m'} a_{l-m'q} a_{l' m' p}. \quad (4.1)$$

Here, $a^\dagger_{l m}$ and $a_{l m}$ are creation and annihilation operators of nucleon with orbital angular momentum $l$, its $z$-component $m$ and charge-spin component $p$. The notations $p(=1, 2, 3, 4)$ refer to the four charge-spin states $(\tau, \sigma)$

$^a$ Notations follow the definition in the textbook by de-Shalit and Talmi, and they are the same as those in Eq. (3.5) in this paper.
Such a Hamiltonian is a generalization of the $Q$ operator defined by Racah\(^{20}\) which can be diagonalized by the aid of the group $U(2l+1)$ and $O(2l+1).^{*}\)

Now we seek for the exact solution of this Hamiltonian by a similar method as developed in § 2. First we define the following operators:

$$E(p, q) = E^\dagger(\bar{p}, \bar{q}) = -E(q, p) = -E^\dagger(\bar{q}, \bar{p})$$

$$= \sum_l \sum_m (-1)^{-m} a^+_l m a^+_l \ldots$$

$$(4.2)$$

and

$$H_p = \sum_l a^+_l m a^+_l m - \Omega,$$

where

$$\Omega = \sum_l \left( l + \frac{1}{2} \right).$$

Then their commutation relations are given by

$$[H_p, H_q] = 0,$$

$$[H_p, E(\alpha)] = r_p(\alpha) E(\alpha),$$

$$[E(\alpha), E(\bar{\alpha})] = \sum_p r_p(\alpha) H_p$$

$$\{ [E(\alpha), E(\beta)] = N_{a\beta} E(\gamma). \}$$

Here $\alpha$ and $\bar{\alpha}$ denote the sets of parameters $(p, q)$, $(\bar{p}, \bar{q})$ and $(p, q)$, $(\bar{p}, \bar{q})$ $p\neq q$, and their hermitian conjugate sets, i.e. $\bar{(p, q)} = (\bar{p}, \bar{q})$, $\bar{(\bar{p}, \bar{q})} = (\bar{p}, \bar{q})$, and $(p, q) = (q, \bar{p})$. The roots $\mathbf{r}(\alpha) = (r_1(\alpha), r_2(\alpha), r_4(\alpha))$ are

$$\begin{cases} r_p(\alpha, q) = -r_p(\bar{p}, \bar{q}) = (\delta_{pp^*} + \delta_{q}\gamma), \\ r_p(\bar{p}, \bar{q}) = -r_p(q, \bar{p}) = (\delta_{p}\gamma - \delta_{q}). \end{cases}$$

$N_{a\beta}$ are given by the relations

$N_{a\beta} = -N_{\beta a} = -N_{a\beta} = \pm 1$, for $\mathbf{r}(\alpha) + \mathbf{r}(\beta) = \mathbf{r}(\gamma)$ and otherwise $N_{a\beta} = 0$.

This algebra with 28 linearly independent generators, subject to the above standard commutation relations is the Lie algebra of type $D_4^{(2)}$ which is isomorphic to that of $O(8)$. An irreducible representation of $D_4$ can be characterized by a set of four non-negative integers $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ which gives the highest weight $\mathbf{M} = \sum_{\lambda_i} \lambda_i \mathbf{M}_i$, where the fundamental dominant weights $\mathbf{M}_i$ are\(^{23}\)

\(^{*}\) $Q(N, [\lambda], (\sigma)) = C[\lambda] - C[\sigma] + 2(N - \Sigma \nu_i)$ where $[\lambda]$ and $(\sigma)$ denotes the irreducible representation of $U(2l+1)$ and $O(2l+1)$, and $C[\lambda]$ and $[C[\sigma] - 2\Sigma \nu_i]$ are their Casimir operators, respectively.
\[ M_1 = (1, 0, 0, 0), \]
\[ M_2 = (1, 1, 0, 0), \]
\[ M_3 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) \]
\[ M_4 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}). \]

The Casimir operator of \( D_4 \) is
\[ \hat{C}(D_4) = \sum_p H_p^2 + \sum_{p,q} E(p, q) E(q, p) \]
\[ + \frac{1}{2} \sum_{p,q} \left\{ (E(p, q) E(\bar{p}, \bar{q}) + E(\bar{p}, \bar{q}) E(p, q)) \right\}, \]
and its eigenvalue is obtained as
\[ C(D_4) = v_1(v_1 + 6) + v_2(v_2 + 4) + v_3(v_3 + 2) + v_4^2, \]
where
\[ v_1 = \lambda_1 + \lambda_2 + \frac{\lambda_3 + \lambda_4}{2}, \]
\[ v_2 = \lambda_2 + \frac{\lambda_3 + \lambda_4}{2}, \]
\[ v_3 = \frac{(\lambda_3 + \lambda_4)}{2}, \]
\[ v_4 = \frac{(- \lambda_3 + \lambda_4)}{2}. \]

Now, let us return to the physical problem. From the meaning of \( H_i \), we get the restriction
\[ 0 \leq \lambda_1 + \lambda_2 + \frac{\lambda_3 + \lambda_4}{2} \leq \Omega. \]

The Hamiltonian (3.1) is rewritten as
\[ H = -G \left\{ \hat{C}(D_4) - \left( \sum_p H_p^2 + \sum_{p,q} E(p, q) E(q, p) \right) + \sum_{p,q} r_{pq} E(p, q) H_p \right\} \]
\[ = -G \left\{ \hat{C}(D_4) - \left( \sum_p H_p^2 + \sum_{p,q} E(p, q) E(q, p) \right) + 6 \left( \frac{N}{2} - 2 \Omega \right) \right\}. \]

The second term of this equation is the Casimir operator of the group \( U(4)^* \) which is used in Wigner's supermultiplet theory.\(^{10} \) The correspondence between

\(^{10} \) Strictly speaking, this is not a Casimir operator because the group \( U(4) \) is not semisimple. However, we conventionally use the terminology of semisimple groups for the corresponding quantities of unitary group, since unitary group shares many properties with the semisimple groups.
his operators and ours is as follows:

\[ \hat{\mathcal{H}} = \left( \frac{\hat{N}}{2} - 4\Omega \right) = \frac{1}{2} (H_1 + H_2 + H_3 + H_4), \]

\[ \hat{T}_b = \sum_t t_b^{(s)} = \frac{1}{2} (H_a - H_1 - H_3 - H_4), \]

\[ \hat{S}_b = \sum s_b^{(s)} = \frac{1}{2} (H_b - H_1 + H_3 + H_4), \]

\[ \hat{Y} = 2\sum s_b^{(s)} t_b^{(s)} = \frac{1}{2} (H_b - H_1 - H_3 + H_4), \]

\[ \hat{T}_+ = \sum t_+^{(s)} = \{E(1\bar{3}) + E(2\bar{4})\}, \]

\[ \hat{S}_+ = \sum s_+^{(s)} = \{E(1\bar{2}) + E(3\bar{4})\}, \]

\[ 2\sum s_0^{(s)} t_+^{(s)} = \{E(1\bar{3}) - E(2\bar{4})\}, \]

\[ 2\sum s_+^{(s)} t_0^{(s)} = \{E(1\bar{2}) - E(3\bar{4})\}, \]

\[ \sum s_+^{(s)} t_+^{(s)} = E(1\bar{4}), \]

\[ \sum s_+^{(s)} t_-^{(s)} = E(2\bar{3}), \]

etc.

The irreducible representation of $U(4)$ is characterized by the four quantum number $\mathcal{H} (= N/2 - 2\Omega)$, $P$, $P'$ and $P''$ where $N$ is the nucleon number, $P$ is the maximum isospin among the states belonging to this representation, $P'$ is the maximum spin among $T = P$ states, and $P''$ is the maximum value of $Y$ among $T = P$, $S = P'$ states. It is well known that these quantum numbers determine the symmetry of the wavefunctions in charge-spin space. The Casimir operator of $U(4)$ has the eigenvalue,

\[ \left( \frac{N}{2} - 2\Omega \right)^2 + P(P + 4) + P'(P' + 2) + P''^2. \]  

(4.12)

Corresponding to Eq. (2.14), we introduce the quantum numbers, seniority $v$ and reduced symmetry $(\rho, \rho', \rho'')$ which are more physical than the purely group-theoretical $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$:

\[ v = 4\Omega - \sum_v v = 4\Omega - 2 \left( \lambda_1 + \lambda_2 + \frac{\lambda_3 + \lambda_4}{2} \right), \]

\[ \rho = \frac{1}{2} (v_1 + v_2 - v_3 - v_4) = \lambda_3 + \frac{\lambda_3 + \lambda_4}{2}, \]
Exact Treatment of a Charge Independent

\[ \rho' = \frac{1}{2}(v_1 - v_2 + v_3 - v_4) = (\lambda_3 + \lambda_4)/2, \] (4·13)

\[ \rho'' = \frac{1}{2}(v_1 - v_2 - v_3 + v_4) = (-\lambda_3 + \lambda_4)/2. \]

They are quantum numbers \((N, P, P', P'')\) of the state with the lowest weight of \(D_4\). Then the eigenvalue of the Hamiltonian is given by

\[ E = -G\left[ \left( \frac{N-v}{2} \right) \left( 4\Omega + 6 - \frac{v + N}{2} \right) \right. \]

\[ \left. + \{\rho(\rho + 4) + \rho'\rho'(\rho' + 2) + \rho''\rho''\} - \{P(P + 4) + P'(P' + 2) + P''P''\} \right]. \] (4·14)

A classification of the states in \(L-S\) coupling scheme can be carried out according to \(D_4\) and its subgroups as in § 3. It may be reasonable to specify the state as

\[ |(v - 4\Omega, \rho, \rho', \rho'') (N - 4\Omega, P, P', P'') ; (TS_T) T_0 S_0 Y \alpha L \rangle, \] (4·15)

where two additional quantum numbers \(\beta\) and \(\gamma\) are needed to distinguish all states of the same irreducible representation of \(D_4\), and the total orbital angular momentum \(L\) and \(\alpha\) stand for the labels which distinguish the equivalent but different irreducible representations of \(D_4\). To obtain the complete set of wavefunctions, we should carry out the reductions of the representations such as \(D_4 \supset U(4) \supset SU(2) \times SU(2)\) and \(O(2) \supset O(3)\) as in § 3. For this problem the reference can be found in the literature.\(^{24}\) Here, we only give the character and the dimension of the irreducible representation of \(D_4\).

The characters of the irreducible representation \(D(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) are given by

\[ \chi(\lambda_1, \lambda_2, \lambda_3, \lambda_4 ; \Phi) = \xi(\lambda_1, \lambda_2, \lambda_3, \lambda_4 ; \Phi)/\xi(0, 0, 0, 0), \]

\[ \xi(\lambda_1, \lambda_2, \lambda_3, \lambda_4 ; \Phi) = \frac{1}{2} \left| \left( \begin{array}{c} C((v_1 + 3)\Phi), C((v_2 + 2)\Phi), C((v_3 + 1)\Phi), C(v_4\Phi) \\ S((v_1 + 3)\Phi), S((v_2 + 2)\Phi), S((v_3 + 1)\Phi), S(v_4\Phi) \end{array} \right) \right|, \] (4·16)

where \(v_i\)'s are defined in Eq. (4·8), and

\[ C(l\Phi) = \begin{pmatrix} \exp(il\varphi_1) + \exp(-il\varphi_1) \\ \exp(il\varphi_3) + \exp(-il\varphi_3) \\ \exp(il\varphi_3) + \exp(-il\varphi_3) \\ \exp(il\varphi_4) + \exp(-il\varphi_4) \end{pmatrix}, \] (4·17)

and

\[ S(l\varphi) = \exp(il\varphi) - \exp(-il\varphi). \]

The dimension of this representation is
$N(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \mathcal{N}(\lambda_1, \lambda_2, \lambda_3, \lambda_4; \varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 0)

= (1 + \lambda_1)(1 + \lambda_2)(1 + \lambda_3)(1 + \lambda_4)
\times \left(1 + \frac{\lambda_1 + \lambda_2}{2}\right)\left(1 + \frac{\lambda_3 + \lambda_4}{2}\right)
\times \left(1 + \frac{\lambda_1 + \lambda_3 + \lambda_2 + \lambda_4}{3}\right)
\times \left(1 + \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_2}{4}\right)
\times \left(1 + \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_2}{5}\right).

(4.18)

Derivation of these formulas and the list of the lower dimensional representations are given in Appendix C.

§ 5. On the validity of the generalized Bogoliubov transformation

When we intend to include neutron-proton pairing interaction as well as proton-proton and neutron-neutron one, it is natural to generalize the Bogoliubov transformation as follows:

$\alpha_{j\mu\nu} = \sum_{\mu\nu} \{ u_{j\mu\nu} a_{j\mu\nu} - v_{j\mu\nu} ( -y_{j\mu\nu} a_{j\mu\nu}) \},

= U a_{j\mu\nu} U^{-1},

(5.1)

where $a_{j\mu\nu}$ is a quasi-particle annihilation operator and $\mu = 1, 2$ for $jj$ coupling scheme and $\mu = 1, 2, 3, 4$ for $L-S$ coupling scheme, which specify charge (spin) states. We call this unitary transformation $U$, the generalized Bogoliubov transformation. V. B. Belyaev, B. N. Zakhar'ev and V. G. Soloviev$^6$ used a special form of this type in $jj$ coupling scheme and Flowers$^7$ did so in $L-S$ coupling scheme. Flowers and Vujčić$^7$ criticized that these treatment is not a good approximation, since they cannot take account of four-body correlations. Here, we shall discuss this problem in connection with the soluble models treated above. For the present, we restrict ourselves to the $jj$ coupling model, since quite similar discussions can be applied to the $L-S$ coupling model.

The unitary transformation (5.1) is written in the form,

$U = \prod_j U_j$

$U_j = \exp i (\sum_{\lambda} t^{(j)}_{\lambda} H^{(j)}_{\lambda} + \sum_{a} t^{(j)}_a (E_a(j) + E_{-a}(j))
\times + \sum_a t^{(j)}_a (E_a(j) - E_{-a}(j))/i),

(5.2)

where the generators are those in Eq. (2.2b), and the real parameters $t^{(j)}$ are determined by using variational principle on the energy and subsidiary condition on nucleon number, etc. When we deal with the pair interacting system by this transformation in the strong coupling limit, $U$ becomes

$U_{sc} = \lim_{\varepsilon \to 0} \prod_j U_j
\exp i (\sum_{\lambda} t^{(j)}_\lambda H^{(j)}_{\lambda} + \sum_{a} t^{(j)}_a (E_a + E_{-a}) + \sum_a t^{(j)}_a (E_a - E_{-a})/i).

(5.3)
This is a certain element of the transformation group $C_2$. Since the vacuum state $|0\rangle$ and one particle state $a_{\mu \nu}^\dagger|0\rangle$ belong to the irreducible representations $D(0, Q)$ and $D(1, Q - 1)$ respectively, the ground and one quasi-particle states, $\Phi_0 = U_{\nu \mu}^\dagger|0\rangle$ and $\Phi_i = U_{\nu \mu} a_{\mu \nu}^\dagger|0\rangle$, also become eigenstates of the Casimir operator $\hat{C}$ of Eq. (2.8) which have the maximum eigenvalues of $\hat{C}$ of even and odd nucleon states respectively under the restriction (2.11). However, since the transformation $U$ does not conserve nucleon number and isospin, some errors still remain, even after the determination of appropriate $U$ under the condition $\langle 0|U^\dagger \hat{N} U|0\rangle = N$. The error in the ground state energy becomes

$$\Delta E_0 = G \left[ \langle 0|U^\dagger \left( \frac{\hat{N}^2 - N^2}{4} \right) U|0\rangle + \langle 0|U^\dagger (\hat{T}^2 - T(T+1)) U|0\rangle \right],$$

from Eqs. (2.8) and (2.13).

Therefore, this limited consideration leads us to a modification of the generalized Bogoliubov transformation such as follows:

(I) After determining parameters of the generalized Bogoliubov transformation, to project quasi-particle states into definite number and definite isospin states and to renormalize them.

(II) To introduce new Lagrangian multipliers determined by the condition

$$\langle 0|U^\dagger \hat{N}^2 U|0\rangle = N^2, \text{ and } \langle 0|U^\dagger \hat{T}^2 U|0\rangle = T(T+1).$$

This is the same idea originated by Lipkin$^{25}$ and recently used by Y. Nogami.$^{26}$ According to this method, we obtain the lowest energy state for each isospin by different transformation separately.

However, it should be stressed that the energy gap calculated by the generalized Bogoliubov transformation is still meaningful. It represents the half of the energy difference between seniority 0 and 2 states in our generalized sense. Especially when we consider a system with a very large volume where $QG$ is finite but $Q \sim \infty$; many eigenstates with different $T$ or $t$ become approximately degenerate and the energy gap becomes conspicuous. This feature is illustrated in Fig. 3. Unfortunately, $Q$ is not so large in actual lighter nuclei where

\[ \text{Fig. 3. The low lying energy levels with the fixed number of nucleons. (a) Finite and (b) infinite nucleus.} \]
neutron-proton interaction is important.

These discussions can also be applied to the \( L-S \) coupling model with the corresponding modification.

The four-body correlation stressed by Flowers and Vujčič\(^9\) are taken into account in the projection procedure. This is also understood by the fact that the projection operators to the definite number and the isospin (symmetry) do not commute with the discriminant for four-body correlation proposed by the author.\(^{10}\) It would be expected that introduction of single particle energy difference does not change qualitative features discussed above, in analogy with the results of calculation for like-particle system by Kerman, Lawson and Macfarlane.\(^{11}\)

**Acknowledgements**

I should like to thank Prof. A. Arima for his guidance into the details of the shell model techniques and his helpful discussions. I am grateful to Prof. M. Nogami, Prof. F. Iwamoto and Prof. T. Terasawa for their keen interests and encouragement of this work.

**Appendix A**

Let us prove three theorems presented in § 3.

*Proof of (I).*

The characters of \( D(\lambda, 0) \) are given by

\[
x(\lambda, 0) = \frac{(x^{\lambda+2} - x^{-\lambda-2})}{(x-x^{-1})(y-y^{-1})} \left( y^{\lambda+2} - y^{-\lambda-2} \right) \]

\[
= \sum_{k} \left\{ (XY)^{2k+1} + (XY)^{-2k+1} - (XY^{-1})^{2k+1} - (XY)^{-2k+1} \right\} \frac{XY - XY^{-1} - X^{-1}Y + X^{-1}Y^{-1}}{XY - XY^{-1} - X^{-1}Y + X^{-1}Y^{-1}}
\]

\[
= \sum_{k=0}^{\lambda/2} \frac{X^{2k+1} - X^{-2k+1}}{X^{-1} - Y^{-1}} \frac{Y^{2k+1} - Y^{-2k+1}}{Y - Y^{-1}}
\]

\[
= \sum_{k=0}^{\lambda/2} \sum_{m=-k}^{k} \sum_{n=-k}^{k} X^{2m} Y^{2n}, \tag{A·1}
\]

where

\[
\begin{align*}
x &= \exp i\varphi_1 = \exp i\left( \frac{\psi_1 + \psi_2}{2} \right), \\
y &= \exp i\varphi_2 = \exp i\left( \frac{\psi_1 + \psi_2}{2} \right),
\end{align*}
\]
and

\[
\begin{align*}
X &= \exp i \left( \frac{\phi_1}{2} \right), \\
Y &= \exp i \left( \frac{\phi_2}{2} \right).
\end{align*}
\]

This gives a multiplicity of each weight.

From the last form of Eq. (A·1), we obtain the weight diagram as the sum of following diagrams with only simple points where simple means multiplicity one. Therefore the theorem is obtained.

\[\text{Fig. 4.}\]

**Proof of (II).**

The characters of \( D(0, \lambda) \) are given by

\[
x(0, \lambda) = \frac{(x^{\lambda+1} - x^{-(\lambda+1)}) (y^{(\lambda+1)} - y^{-(\lambda+1)}) - (x^{\lambda+1} - x^{-(\lambda+1)}) (y^{(\lambda+1)} - y^{-(\lambda+1)})}{(x - x^{-1}) (y - y^{-1}) (x - (y + y^{-1}) + x^{-1})}
\]

\[
= \sum_{k=0}^{\lambda} \frac{x^{\lambda+1-k} - x^{-(\lambda+1-k)}}{x - x^{-1}} \frac{y^{\lambda+1-k} - y^{-(\lambda+1-k)}}{y - y^{-1}}
\]

\[
= \sum_{k=0}^{\lambda} \sum_{l=0}^{\lambda-k} x^{-\lambda+k+l} y^{-\lambda+k+l}.
\]

(A·2)

This gives a multiplicity of each weight.

From the last form of (A·2), we obtain the weight diagram as the sum of the following diagrams with only simple points. Then we see that the points

\[\text{Fig. 5.}\]

on the fix number line has such a structure. Therefore the theorem II is
obtained.

**Proof of (III).**

The characters of $D(1, \lambda)$ are given by

$$
\chi(1, \lambda) = \frac{(x^{\lambda+1} - x^{-(\lambda+1)}) (y^{\lambda+1} - y^{-(\lambda+1)}) - (x^{\lambda+1} - x^{-\lambda+1}) (y^{\lambda+1} - y^{-\lambda+1})}{(x-x^{-1}) (y-y^{-1}) (x-(y+y^{-1})+x^{-1})}
$$

$$
= \sum_{k=0}^{\lambda} \frac{X^{(2k+1)} - X^{-(2k+1)}}{X-X^{-1}} \frac{Y^{2(\lambda+1-k)} - Y^{-2(\lambda+1-k)}}{Y-Y^{-1}}
$$

$$
= \sum_{k=0}^{\lambda} \sum_{m=\left(-\frac{\lambda+1}{2}\right)}^{\left(k+\frac{1}{2}\right)} X^{2m} \sum_{n=\left(-\frac{\lambda+1}{2}\right)}^{\left(k+\frac{1}{2}\right)} Y^{2n}. \quad (A \cdot 3)
$$

This gives a multiplicity of each weight.

From this form of characters, we obtain the weight diagram as the sum of the following diagrams with only simple points. Therefore we get the theorem III.

**Appendix B**

Reduction of the irreducible representation of $C_2$ through the two schemes of Eq. (3.5) is carried out by means of Eqs. (3.1) and (3.6) and the weight diagram which has very useful properties that it is symmetric with respect to $m_1, m_2, n$, and $T_0$ axes. The results are tabulated in the following tables up to $\lambda_1 + \lambda_2 \leq 6$. Then the classification through $C_2$ is completed for any levels with $N \leq 6$ such as $sd$-shell, $h_{11/2}, i_{11/2}$, etc. Usual quantities are given by the relation...
Exact Treatment of a Charge Independent

\[ N = \mathcal{J} + 2\Omega, \]

\[ \nu = 2\Omega - (\lambda_1 + 2\lambda_2) \quad (B \cdot 1) \]

and

\[ t = \lambda_1/2. \]

**Table I.**

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<tr>
<th>((\lambda_1, \lambda_2))</th>
<th>(N(\lambda))</th>
<th>((\mathcal{J}, T))</th>
<th>((S_\mu, S_\nu))</th>
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<td>(0, 0)</td>
</tr>
<tr>
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Table II.
Odd particle states.

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<td>$(\pm 3, 5/2)(\pm 5, 5/2)$</td>
<td>$(3/2, 2)(3/2, 1)(1, 5/2)(1, 3/2)(1/2, 3)$</td>
</tr>
<tr>
<td>(5, 1)</td>
<td>160</td>
<td>$(\pm 1, 1/2)(\pm 1, 3/2)^2(\pm 1, 5/2)^2(\pm 1, 7/2)^2(\pm 3, 3/2)^2(\pm 3, 5/2)$</td>
<td>$(1/2, 2)(0, 5/2)$</td>
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<tr>
<td></td>
<td></td>
<td>$(\pm 5, 3/2)(\pm 5, 5/2)(\pm 5, 7/2)(\pm 7, 5/2)$</td>
<td>$(1/2, 2)(0, 5/2)$</td>
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</table>

**Appendix C**

We derive the formulas (4·16) and (4·18) for the character and dimension of the irreducible representation of $D_6$. According to Weyl, $\xi(\lambda, \nu)$ in Eq. (4·16) are given by$^{[13]}$

$$\xi(\lambda, \nu) = \sum_s \delta_s \exp[i(\mathbf{S}K) \cdot \nu].$$  \hspace{1cm} (C·1)
Here \( S \) denotes the reflections and the products of reflections through a hyperplane perpendicular to the root \( r \), and \( \delta_\alpha = +1 ( -1) \) for the even (odd) number of reflections. And

\[
K = R + M(\lambda_1, \lambda_2, \lambda_3, \lambda_4),
\]
\[
R = \frac{1}{2} \sum_\alpha r(\alpha),
\]
where \( M \) is the highest weight and the sum is over the positive roots, i.e. those which have a positive first nonvanishing component, and

\[
K = \left( \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}{2}, \frac{\lambda_3 + \lambda_4}{2} + 1, -\frac{\lambda_3 + \lambda_4}{2} \right)
\]
\[
= (v_1 + 3, v_2 + 2, v_3 + 1, v_4).
\]

Noticing that the reflection interchanges the two components of \( K \) with or without change of their signs, we obtain the formulas

\[
\hat{\xi}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) + \hat{\xi}(\lambda_1, \lambda_2, \lambda_3, \lambda_3)
\]
\[
= |C((v_1 + 3)\varphi), C((v_2 + 2)\varphi), C((v_3 + 1)\varphi), C(v_4\varphi)|
\]

and

\[
\hat{\xi}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) - \hat{\xi}(\lambda_1, \lambda_2, \lambda_4, \lambda_3)
\]
\[
= |S((v_1 + 3)\varphi), S((v_2 + 2)\varphi), S((v_3 + 1)\varphi), S(v_4\varphi)|.
\]

Then the formula (4·16) is derived.

The dimension of the representation is determined\(^{50} \) by first setting

\[
\varphi_1 = 4\varphi, \varphi_2 = 3\varphi, \varphi_3 = 2\varphi, \varphi_4 = \varphi,
\]

and then letting \( \varphi \) tend to zero:

\[
N(\lambda_i) = \pi(\lambda_i; 0)
\]
\[
= \lim_{\varphi \to 0} \frac{\hat{\xi}(\lambda_i; 4\varphi, 3\varphi, 2\varphi, \varphi)}{\hat{\xi}(0; 4\varphi, 3\varphi, 2\varphi, \varphi)}
\]
\[
= \prod_{i<j} (\lambda_i - \lambda_j)(\lambda_i + \lambda_j)
\]
\[
= \prod_{i<j} (I^o - I_i^o)(I^o + I_i^o),
\]

where \( I_i = v_i + (4 - i) \),

and

\[
I^o = (3, 2, 1, 0).
\]

Inserting the relation (4·8) into (C·6), we obtain Eq. (4·18). Here we tabulate the lower dimensional representations of \( D_4 \).
Table III.

<table>
<thead>
<tr>
<th>( J, \rho, \rho' )</th>
<th>( \lambda_1, \lambda_2, \lambda_3, \lambda_4 )</th>
<th>( N(\lambda) )</th>
<th>( \nu = \lambda_1 + \lambda_2 + \frac{\lambda_3 + \lambda_4}{2} )</th>
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</table>

References

4) L. S. Kisslinger and R. A. Sorensen, Rev. Mod. Phys. 35 (1963), 853. Complete list of references is available in this paper.
Exact Treatment of a Charge Independent

16) According to correspondence from Professor G. E. Brown, Professor Lipkin and Professor
    Mottelson, and Parkin have handled a charge independent pairing interaction along much
    the same line as developed here. I should like to thank Professor Brown for informing
    this fact.
17) G. Racah, "Group Theory and Spectroscopy", Institute for Advanced Study, Lecture
    Details of group theory used here are explained in the excellent reviews 17) and 18).
19) A. de-Shalit and I. Talmi, Nuclear Shell Theory (Academic Press, New York and London,
    1963), §§ 33-36.
21) A. Arima, private communication.
23) E. Wigner, Phys. Rev. 51 (1937), 947.
    M. Hamermesh, Group Theory and Its Application to Physical Problem (Addison-Wesley

Note added in proof: After this paper was accepted for publication, Professor B. H. Flowers
and Dr. S. Szpikowski have very kindly sent me their reprint entitled: a generalized quasi-spin
LS-coupling, which are mostly equivalent to §§ 2 and 4 of this paper.