Discussion

Integral Methods in Natural-Convection Flow[1]

E. M. Sparrow.[2] The analysis given by the author represents a significant advance over previous investigations of free-convection on nonvertical surfaces, in that the pressure gradient across the boundary layer for the first time is taken into consideration. In all previous analyses,[3] the modification, due to surface inclination, of the buoyancy term in the momentum equation for the x-direction has been made; but the pressure variation across the boundary layer has been ignored.

It is of interest to study the physical meaning of Equations [9] and [42] of the paper and to show their relationship to the conservation-of-momentum principle. To make this study, the full set of boundary-layer assumptions is applied to Equations [1] and [2]; with the result that $\partial^2 u/\partial x^2$ is deleted from Equation [1], and Equation [2] becomes

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = g\beta(T - T_1)\cos\phi \qquad [1]$$

From Equation [1], herewith, it follows that

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = g\beta\frac{\partial}{\partial x}\left[\cos\phi\int_y^\delta (T - T_1)dy\right] \qquad [2]$$

and, in particular

$$\frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_{y=0} = -g\beta\sin\phi\frac{\partial\phi}{\partial x}\int_0^\delta (T - T_1)dy + g\beta(\cos\phi)\int_0^\delta \frac{\partial(T - T_1)}{\partial x}dy \qquad [3]$$

Using this last equation, Equation [9] of the paper may be rewritten as follows

$$\nu\frac{\partial^2 u}{\partial y^2}\bigg|_{y=\delta} = g\beta(T_w - T_1)\sin\phi + \nu\frac{\partial^2 u}{\partial y^2}\bigg|_{y=0} - \frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_{y=0} \qquad [4]$$

Equation [4], and hence Equation [9] of the paper, state that the net shear force acting on an infinitesimal fluid element located at the edge of the boundary layer equals the net force acting on an infinitesimal fluid element located at the wall. Under the condition that

$$(\partial^2 u/\partial y^2)|_{y=\delta}$$

is zero, Equation [4] requires that the net force acting on a fluid element at the wall be zero. But these same conditions are obtained by imposing the x-momentum equation (Equation [1] of the paper), at the edge of the boundary layer and at the wall. So the use of Equation [4] of this discussion with

$$(\partial^2 u/\partial y^2)|_{y=\delta}$$

equal to zero is identical to satisfying the x-momentum equation at two places, at the wall and at the edge of the boundary layer. However, no restriction whatsoever is imposed on the momentum at all other points in the boundary layer by the use of Equation [4] of this discussion. Under the circumstance that

$$(\partial^2 u/\partial y^2)|_{y=\delta}$$

is not zero, Equation [4] actually is a contradiction of the x-momentum equation at the wall and at the edge of the boundary layer. So it seems plausible that velocity profiles with

$$(\partial^2 u/\partial y^2)|_{y=\delta}$$

different from zero should yield poor results.

Equation [42] is obtained from an integral equation into which the polynomial Expressions [11] and [12] have been introduced. This integral equation may be derived in a direct manner by integrating Equation [1] of the paper (with $(\partial^2 u/\partial x^2)$ deleted) from $y = 0$ to $y = \delta$, and by using Equations [2] of this discussion and Equation [4] of the paper. The result is

$$\frac{d}{dx}\int_0^\delta u^2 dy = g\beta\sin\phi\int_0^\delta (T - T_1)dy - \nu\left(\frac{\partial u}{\partial y}\right)\bigg|_{y=0} + \frac{d}{dx}\int_0^\delta dy\left[\int_\delta^y g\beta(T - T_1)\cos\phi dy\right] \qquad [5]$$

Inspection of foregoing Equation [5] shows that it is an expression of conservation of momentum for an element of boundary layer of dimensions δ by dx. So Equation [5] requires that momentum be conserved across the boundary layer as a whole, but does not require that momentum be conserved locally at any position within the boundary layer.

As is pointed out by the author, the superiority of Equation [5] of this discussion as compared with Equation [4] lies in the fact that the former takes account of momentum changes, while no momentum terms appear in the latter.

It may be noted that the pressure p, which is used throughout the paper, is not the static pressure. Rather, it is the static pressure minus the ambient pressure.

Carl Wagner.[4] In addition to the author's solution for heat transfer from a heated plate facing upward, a solution for heat transfer from a heated plate facing downward seems to be of interest. In accord with the author, the width of the plate is assumed to be small in comparison with its length. At the upper side of a heated plate, fluid flows from the edges to the middle and finally flows upward, whereas at the lower side, fluid approaches the plate from below and passes along the surface of the plate toward the edges. Thus the flow velocity u_1 of the fluid moving along the plate vanishes at the mid-line of the plate

$$u_1 = 0 \text{ at } x = 0 \qquad [6]$$

where x is the distance from the mid-line of the plate.

[1] By Salomon Levy, published in the December, 1955, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 77, 1955, pp. 515–522.

[2] NACA, Lewis Laboratory, Cleveland, Ohio.

[3] "Thermal Convection in Laminar Boundary Layers," by H. J. Merk and J. A. Prins, *App. Sci. Res.*, section A, vol. 4, 1953–1954, part I, pp. 11–24; part II, pp. 195–206; part III, pp. 206–221.

"Heat Transfer by Free Convection From Horizontal Cylinders in Diatomic Gases," by R. Hermann, NACA TM 1366, 1954.

[4] Department of Metallurgy, Massachusetts Institute of Technology, Cambridge, Mass.

Beyond the edges of the plate, the pressure is independent of x and the pressure distribution in the vertical direction is given by the bulk density of the liquid. In view of continuity, the same pressure distribution must be found underneath the plate in the vicinity of the edges. This is possible only if the thickness δ of the boundary layer decreases sharply at the edges

$$\delta \cong 0 \text{ at } x = \pm a \qquad [7]$$

where a is half the width of the plate.

Upon substituting Equation [13] of the paper for $\varphi = 0$ in his Equation [14] and introducing the auxiliary variables

$$z = \delta^4[\beta g(T_w - T_1)/1080\alpha\nu a^2]^{4/5} \qquad [8]$$

$$\xi = x/a \qquad [9]$$

where the symbols have the same meaning as in the paper, it follows that

$$\frac{1}{4} z^{1/4} d^2z/d\xi^2 + 1 = 0 \qquad [10]$$

In view of Equations [6] to [9] of this discussion the boundary conditions are

$$dz/d\xi = 0 \text{ at } \xi = 0 \qquad [11]$$

$$z = 0 \text{ at } \xi = 1 \qquad [12]$$

Integrating Equation [10] once and using Equation [11], herewith

$$dz/d\xi = -[32(z_0^{3/4} - z^{3/4})/3]^{1/2} \qquad [13]$$

where z_0 is the value of z at $\xi = 0$.

Substituting the auxiliary variable

$$t = (z/z_0)^{1/4} \qquad [14]$$

and integrating Equation [13]

$$\xi = \left(\frac{3}{2}\right)^{1/2} z_0^{5/8} \int_t^1 t^3(1 - t^3)^{-1/2} dt$$

$$= (6^{1/2}/5)[t(1 - t^3)^{1/2} + 3^{-1/4} F(k, \varphi)]z_0^{5/8} \qquad [15]$$

where $F(k, \varphi)$ is Legendre's standard form of the elliptic integral of first kind[5] and

$$k = \frac{1}{2} (2 + 3^{1/2})^{1/2} = 0.96593 = \sin 75 \text{ deg} \qquad [16]$$

$$\varphi = \cos^{-1}[(3^{1/2} - 1 + t)/(3^{1/2} + 1 - t)] \qquad [17]$$

Since $t = 0$ for $\xi = 1$ according to Equations [12] and [14] of this discussion, it follows from Equation [15] that

$$z_0 = 1.824 \qquad [18]$$

In view of the foregoing Equations [8] and [9] and Equation [17] of the paper, the local value of the Nusselt number is found to be

$$N_{\rm Nu} = 2a/\delta = 0.426(z_0/z)^{1/4}(N_{\rm Gr}N_{\rm Pr})^{1/5} \qquad [19]$$

where half the width of the plate a is used as the characteristic length for the definition of both the local Nusselt number and the Grashof number. The value of $(z_0/z)^{1/4}$ determined by Equations [14] and [15], herewith, varies only slightly except for the vicinity of the edges

x/a	0	0.2	0.4	0.6	0.8
$(z_0/z)^{1/4}$	1.00	1.01	1.03	1.11	1.28

[5] "Tables of Functions," by E. Jahnke and F. Emde, B. G. Teubner, Leipzig, Germany, fourth edition, 1938, p. 59.

The average value of the Nusselt number is found to be

$$N_{\rm Nu(avg)} = 0.50(N_{\rm Gr}N_{\rm Pr})^{1/5} \qquad [20]$$

Experimental data compiled by Fishenden and Saunders[6] are in accord with the general result that the rate of heat transfer at the lower side of a heated horizontal plate is proportional to the fifth root rather than the fourth root of the product $N_{\rm Gr}N_{\rm Pr}$. At the upper side of a heated plate, the rate of heat transfer increases somewhat more rapidly for, even at low values of the Grashof number, eddies are formed owing to the instability of a boundary layer having a lower density than the bulk fluid above. The occurrence of eddies under these conditions has been shown especially by Weise.[7]

Author's Closure

The author wishes to thank Mr. Sparrow and Dr. Wagner for their valuable comments. Mr. Sparrow's discussion leads to a better physical understanding of the fundamental equations and boundary conditions. Dr. Wagner gives another example of the value of integral methods by solving the case of a heated horizontal plate facing upward. His solution can be extended to low Prandtl number fluids by numerical methods.

As an additional closing comment the author wishes to point out that Equation [28] assumes that the surface heat flux varies as $N_{\rm Pr}^{-2/3}$ and its application to low Prandtl numbers may be opened to question.[8]

Some Solutions of the Timoshenko Beam Equations[1]

H. N. Abramson.[2] The authors have rendered a valuable service by providing "building blocks" which may be used to form a large number and variety of solutions of the Timoshenko beam equations. The availability of this information will be welcomed by all those who are working with such problems.

Because of the great interest currently being shown in the general problem of transverse impact on elastic beams, some comments of a general nature may be in order. Investigations concerned with this problem seem to be concentrated largely along the paths of (a) experimental investigations and (b) solutions of the Timoshenko beam equations, both approximate and exact. A crossing of these two paths was presented in a recent paper by Goland, Wickersham, and Dengler,[3] and the resulting discussions[4] brought forth many succulent comments. These discussions serve to point out that there is yet a third path to be followed; that is, investigations also must proceed from the general theory of elasticity to an exact analysis of the problem.

A critical appraisal of the Timoshenko theory, on this basis, was

[6] "An Introduction to Heat Transfer," by M. Fishenden and O. A. Saunders, Clarendon Press, Oxford, England, 1950, pp. 95–97.

[7] "Wärmübergang durch freie Konvektion an quadratischen Platten," by R. Weise, *Forschung auf dem Gebiete des Ingenieurwesens*, vol. 6, 1935, pp. 281–292.

[8] "Liquid-Metals Handbook, Sodium-NaK Supplement," Atomic Energy Commission, July, 1955, p. 54.

[1] By B. A. Boley and C. C. Chao, published in the December, 1955, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 77, pp. 579–586.

[2] Department of Engineering Mechanics, The University of Texas, Austin, Texas. Assoc. Mem. ASME.

[3] "Propagation of Elastic Impact in Beams in Bending," by M. Goland, P. D. Wickersham, and M. A. Dengler, Journal of Applied Mechanics, Trans. ASME, vol. 77, 1955, pp. 1–7.

[4] Ibid., discussion in Journal of Applied Mechanics, Trans. ASME, vol. 77, 1955, pp. 608–610.