able; the error is about 10 percent. The comparison also shows, however, that the estimated values do not agree well with the calculated values, as was the case in subsonic flows; the discrepancy increases with decreasing cylinder diameter. The author believes that these estimated values are again in error, as in incompressible flows, and that the calculated values are probably not too very far from the "correct" values, with an estimated error of 10 percent.

Concluding Remarks

On the basis of comparisons presented in this paper, it can be said that the present eddy-viscosity formulation for thick axisymmetric boundary layers seems to agree satisfactorily with experiment for both incompressible and compressible flows.

References

1 Cebeci, T., and Smith, A. M. O., "A Finite-Difference Method for Calculating Compressible Laminar and Turbulent Boundary Layers," *Journal of Basic Engineering*, Trans. ASME, Series D, Vol. 92, No. 3, Sept. 1970, pp. 523–535.

2 Van Driest, E. R., "On Turbulent Flow Near a Wall," *Journal of Aerospace Science*, Vol. 23, No. 11, Nov. 1956.

3 Cebeci, T., "Calculation of Compressible Turbulent Boundary Layers with Heat and Mass Transfer," *AIAA Journal*, Vol. 9, No. 6, June 1971, pp. 1091–1098.

4 Cebeci, T., "Kinematic Eddy Viscosity at Low Reynolds Numbers," *AIAA Journal*, Vol. 11, No. 1, Jan. 1973, pp. 102–104.

5 Rao, G. N. V., "The Law of the Wall in Thick Axisymmetric Turbulent Boundary Layer," *Journal of Applied Mechanics*, Vol. 44, Trans. ASME, Vol. 89, Series E, No. 1, Mar. 1967, pp. 237–238.

6 Coles, D., "The Law of the Wake in the Turbulent Boundary Layer," *Journal of Fluid Mechanics*, Vol. 1, 1956, pp. 191–226.

7 Richmond, R., "Experimental Investigation of Thick Axially Symmetric Boundary Layers on Cylinders at Subsonic and Hypersonic Speeds," PhD thesis, California Institute of Technology, Pasadena, Calif., 1957.

8 White, F. M., "An Analysis of Axisymmetric Turbulent Flow Past a Long Cylinder," *Journal of Basic Engineering*, Trans. ASME, Series D, Vol. 94, No. 1, Mar. 1972, pp. 200–206.

9 Cebeci, T., "Laminar and Turbulent Incompressible Boundary Layers on Slender Bodies of Revolution in Axial Flow," *Journal of Basic Engineering*, Trans. ASME, Series D, Vol. 92, No. 3, Sept. 1970, pp. 545–554.

DISCUSSION

F. M. White[3]

This paper provides further input to the observed fact that thick axisymmetric boundary layers are different in character from two-dimensional boundary layers. I heartily agree with the eddy-viscosity formulation which Dr. Cebeci presents here. In essence, he has given an elegant derivation of a mixing length expression I suggested to him in the discussion portion of his earlier paper (p. 550 of reference [9]). He has also corrected my improper use of the velocity derivative $(\partial u/\partial y)$, which should be changed to $(\partial u/\partial Y)$ in the axisymmetric case. His final expressions, equations (35)–(36), are then in much better agreement with experiment and have the advantage of containing two-dimensional information as the special case of large r_0. My only criticism would be that I do not think thirty-six equations are necessary to establish this result. To my way of thinking, equation (36) simply follows directly from equation (20) and stands or falls upon this observation of Rao [5]. I believe it myself and have used it in my work [8]—but at that time the reviewers were unanimous in stating that equation (20) was not valid. I am pleased that Dr. Cebeci has not met with this erroneous criticism.

There has been another brief flurry of activity on this interesting subject, and I would like to add three new references to this paper: myself [10][4], Patel [11], and Chase [12]. All three papers substantiate the conclusions of Dr. Cebeci. The paper by Chase [12] deduces a nearly exact value for the experimental skin friction for the first case of Table 1 of the present paper (d = 0.024 in). Chase reports that C_f = 0.00788, in good agreement with both my calculation and Cebeci's present method. Thus Richmond's reported value is incorrect, as is Cebeci's earlier calculation [9]. The difference is even more striking if one considers Richmond's lowest Reynolds number, R_d = 188, where the transverse curvature effect is strongest. Thus I suggest adding the following case to Table 1:

		$C_f \times 10^3$			
d(in.)	R_θ(3-d)	Richmond [7]	White [8]	Cebeci [9]	Present method
0.024	954	5.92	9.87	14.6	?

[3]Professor of Mechanical and Ocean Engineering, University of Rhode Island, Kingston, R.I.

[4]Numbers 10–12 in brackets designate Additional References at end of discussion.

Again I am convinced that Richmond [7] is wrong, but unfortunately Chase [12] did not analyze this case. It would be very instructive if Dr. Cebeci would run his program for this case and report the computed skin friction. In the meantime, it is not clear to me how Dr. Cebeci performed his finite difference calculations for Tables 1 and 2. For example, in case 1 of Table 1, Richmond's measurement was made at x = 16 ft. Did Dr. Cebeci begin his calculation at x = 0 and integrate forward to x = 16? Fair enough. Or did he integrate forward until, as listed in the table, R_θ = 2100? Unfair, in my opinion. This may be unfair to say, but it often appears to me that workers who report finite difference calculations are really not interested in having their results verified or repeated.

The results in Table 1 marked (White [8]) are computed from our integral theory, which results in a simple formula for skin friction on a cylinder. Our latest paper [10] extends this result to compressible flow, including pressure gradient, and the cylinder formula then becomes:

$$C_f \simeq 0.455/\{A^2 \ln^2[bR_x(\mu_e/\mu_w)(T_e/T_w)^{1/2}/A]\}, \qquad (42)$$

where $b \simeq 0.06/[1 + 0.025(x/r_0)^{6/7}]$.

We believe this formula to be accurate to ± 10 percent in the range of R_x from 10^5 to 10^9, Mach numbers from zero to ten, and (x/r_0) from zero to 10^6, for hot or cold walls. The quantity A is van Driest's compressibility parameter [10] and has the value of 2.07 for a Mach number of 5.8 and adiabatic walls. Since R_x and (x/r_0) are known for Richmond's data, we may expand Table 2 to include values computed by this formula:

		$C_f \times 10^3$			
d(in.)	R_θ(3d)	Richmond [7]	Cebeci	Equation (42)	Clauser plot [10]
0.25	2140	2.03	2.30	1.74	1.70
0.064	5170	2.13	2.71	1.91	1.90
0.024	4390	2.34	3.97	2.36	2.40

The table shows that our own formula agrees roughly with Richmond's results and gives considerably smaller predictions than Dr. Cebeci's computations. Also included are the results of an axisymmetric compressible "Clauser plot" which we invented [10] and which may not stand the test of time. We feel that Cebeci's results are incorrect but cannot explain the discrepancy. Two comments are in order: 1) I feel that the agreement between theory and experiment in Figs. 3 and 4 is poor; and 2) we have no guarantee that the empirical parameters A, N, α, β, and Π which Cebeci uses are not strongly affected by transverse curvature. In fact, Cebeci himself points out that equation (7) for Π is invalid from Richmond's incompressible data for the smallest cylinder. There is also the unanswered question as to how Dr. Cebeci started and completed his finite-difference computations. In this respect we prefer a formula such as equation (42), which is keyed to the known parameters R_x and (x/r_0) rather than to R_θ, which is a priori unknown.

My final suggestion is that the two-dimensional wall/wake expression, equation (37), should be played down or even ignored when dealing with thick axisymmetric boundary layers. Under these conditions the parameter Π is much smaller and eventually vanishes altogether, regardless of the size of the traditional parameter R_θ. In my opinion, an axisymmetric integral theory should shy away from equation (37). For a finite-difference calculation, as Dr. Cebeci points out, the difficulty is resolved because the profile computation is completed *before* the outer eddy viscosity relation, equation (4), is called for. Thus, numerically, an outer wake never appears.

In conclusion, may I state that, in my opinion, Dr. Cebeci's approach to eddy-viscosity in axisymmetric flow is sound and useful. However, for compressible flow, I do not believe that the available data fully substantiate his formulation.

Additional References

10 White, F. M., Lessmann, R. C., and Christoph, G. H., "A Simplified Approach to the Analysis of Turbulent Boundary Layers in Two and Three Dimensions," Report F33615-71-C-1585, Dept. of Mechanical Engineering, Kingston, R.I., July 1972, *AIAA Journal*, May 1973.

11 Patel, V. C., "A Unified View of the Law of the Wall Using Mixing Length Theory," Report No. 137, Iowa Institute of Hydraulic Research, The University of Iowa, Iowa City, Apr. 1972.

12 Chase, D. M., "Mean Velocity Profile of a Thick Turbulent Boundary Along a Circular Cylinder," *AIAA Journal*, Vol. 10, No. 7, July 1972, pp. 849–850.

Author's Closure

I want to thank Professor White for his discussion. I agree with him that thirty-six equations are not necessary to establish the mixing-length distribution presented here. However, looking over the paper, I see that I used only eight equations (equations (23)–(29) to get (30)). Maybe I should have skipped the steps in between!

I have calculated the c_f-value for $d = 0.024$ in. and $R_\theta(3\text{-}d) = 954$. The new computed value is 9.30×10^{-3}, a value that agrees well with Dr. White's calculated value of 9.87×10^{-3}.

The calculations presented in my paper were started at $x = 0$ and were continued until the calculated R_θ matched the experimental value. I don't agree with Dr. White's method of integration or method of matching the experimental conditions. R_x is an ill-defined quantity for turbulent flows, contrary to R_θ which is a well-defined quantity. I believe that Dr. White integrates the momentum integral equation

$$\frac{d\theta}{dx} = \frac{c_f}{2} \tag{1}$$

by taking the lower limit of integration at $x = 0$. To me this is wrong because the flow cannot be turbulent at the leading edge. Minimum Reynolds number, R_x, necessary to have the flow to be turbulent, is approximately 3 to 4×10^5. Minimum

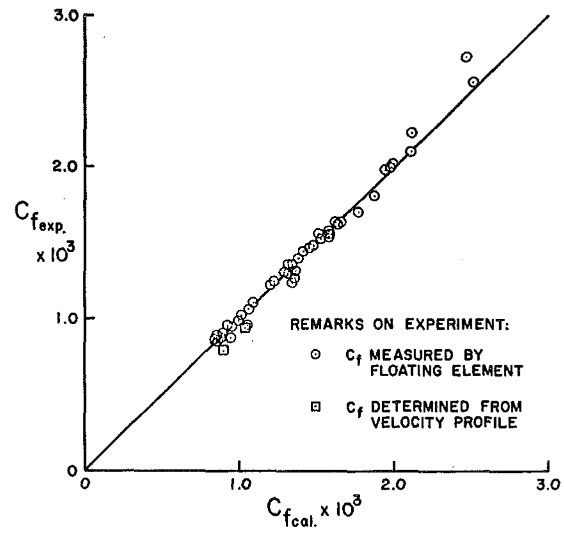

Fig. 5 Comparison of calculated and experimental local skin-friction values. The rms error based on 43 experimental values obtained by the floating technique is 3.5 percent.

$R_\theta = 400$ corresponds to $R_x = 3.6 \times 10^5$. Consequently, taking the lower limit equal to zero and matching the experimental data with that assumption will naturally lead to erroneous results at low Reynolds numbers. At high Reynolds numbers, that incorrect assumption will not be important. Since most of Richmond's data are for low Reynolds numbers, it is difficult for me to accept the manner in which Dr. White gets his R_x.

It has been well established that at sufficiently high Reynolds numbers a turbulent boundary layer has a wake component. For two-dimensional flows the lower limit for this Reynolds number, R_θ, is approximately 5000. According to Coles' study [13],[5] the wake component varies with R_θ in the range 425 to 5000 becoming zero at $R_\theta = 425$. Recent studies (see references [14–16, 4]) confirm this result. Although a careful study similar to that of Coles' is lacking for axisymmetric flows, my guess is that the trend is the same and that the axisymmetric flows do have a wake component just like two-dimensional flows. The experimental data of Richmond for $d = 0.024$ in. (see Fig. 1) show this clearly.

I would like to point out that the "new" mixing-length distribution given in this paper does not change the results (velocity profiles, skin friction) from those obtained by the usual mixing-length distribution $l = 0.4y$. The real effect comes by the use of a variable α given by (6). Without the use of a variable α, the calculated profiles presented in Figs. 2–4 are quite poor. That again points out a need for low Reynolds number correction.

As for the compressible flow calculations, I don't know how good the calculated c_f-values are with the present method. I don't know how good the c_f-values based on Dr. White's invented Clauser plot are, either. Maybe we can get an idea by making calculations for a two-dimensional zero pressure gradient flow subject to the flow conditions, say, in Richmond's experimental setup. The following table shows the results for $d = 0.25$,

			Axisymmetric $c_f \times 10^3$		Two-dimensional
d(in.)	R_θ(3-d)	δ/r_0	Cebeci	White	$c_f \times 10^3$
0.25	2140	5	2.30	1.70	1.70

The calculations for $2d$-c_f were made by using the method of reference [1], which is a very accurate method for computing

[5] Numbers 13–17 in brackets designate Additional References at end of Closure.

compressible adiabatic turbulent boundary layers on a flat plate. To show the accuracy of the predictions of this method for c_f, we show a comparison of calculated and experimental local skin-friction values for compressible adiabatic turbulent boundary layers obtained by that method (see reference [17]). The calculated values in that figure cover a Mach number range of 0.41 to 5 and a momentum-thickness Reynolds number range of 1.6×10^3 to 702×10^3. The rms error based on 43 experimental values, all obtained by floating element technique is 3.5 percent, which is within the experimental scatter.

If we accept this value of c_f to be correct, the minimum c_f-value for Richmond's data is 1.70×10^{-3}. That means that with some transverse curvature effect, the experimental c_f-value must be higher than 1.70×10^{-3} since transverse curvature effect acts as a favorable pressure gradient and increases the skin friction. According to the calculations, the ratio of δ/r_0 is 5. Therefore, there must be some transverse-curvature effect and the c_f for that case must be higher than 1.70×10^{-3} a value predicted for two-dimensional flows and a value predicted by Dr. White.

Additional References

13 Coles, D., "The Turbulent Boundary Layer in a Compressible Fluid," Report R-403-PR, Sept. 162, Rand Corp., Santa Monica, Calif.

14 Cebeci, T., and Mosinskis, G. J., "Computation of Incompressible Turbulent Boundary Layers at Low Reynolds Numbers," *AIAA Journal*, Vol. 9, No. 8, Aug. 1971, pp. 1632-1634.

15 Bushnell, D. M., and Morris, D. J., "Shear-Stress, Eddy-Viscosity and Mixing-Length Distributions in Hypersonic Turbulent Boundary Layers," TM X-2310, Aug. 1971, NASA.

16 Huffman, D. G., and Bradshaw, P., "A Note on von Karman's Constant in Low Reynold's Number Turbulent Flows," *Journal of Fluid Mechanics*, Vol. 53, Part 1, May 1972, pp. 45-60.

17 Cebeci, T., Smith, A. M. O., and Mosinskis, G. J., "Calculation of Compressible Adiabatic Turbulent Boundary Layers," *AIAA Journal*, Vol. 8, No. 11, Nov. 1970, pp. 1974-1982.